# Core Stability and Strategy-Proofness in Hedonic Coalition Formation Problems with Friend-Oriented Preferences

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# Core Stability and Strategy-Proofness in Hedonic Coalition Formation Problems with Friend-Oriented Preferences[*]

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## Abstract

We study hedonic coalition formation problems with friend-oriented preferences; that is, each agent has preferences over his coalitions based on a partition of the set of agents, except himself, into "friends" and "enemies" such that (E) adding an enemy makes him strictly worse off and (F) adding a friend together with a set of enemies makes him strictly better off. Friend-oriented preferences induce a so-called friendship graph where vertices are agents and directed edges point to friends.

We show that the partition associated with the strongly connected components (SCC) of the friendship graph is in the strict core. We then prove that the SCC mechanism, which assigns the SCC partition to each hedonic coalition formation problem with friend-oriented preferences, satisfies a strong group incentive compatibility property: *group strategy-proofness*. Our main result is that on any "rich" subdomain of friend-oriented preferences, the SCC mechanism is the only mechanism that satisfies *core stability* and *strategy-proofness*.

*JEL classification:* C71, C78, D71.

*Keywords:* hedonic coalition formation; (strict) core stability; (group) strategy-proofness; strongly connected components.

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# 1 Introduction

Hedonic coalition formation problems are used to model economic and political environments such as the provision of public goods in local communities or the formation of teams and organizations. Banerjee et al. (2001) and Bogomolnaia and Jackson (2002) introduce the formal model of hedonic coalition formation. A hedonic coalition formation problem consists of a finite set of agents, each of whom has preferences over the potential coalitions he can be a member of.[1] The "hedonic" aspect of preferences refers to the dependence of an agent's utility on the identity of members of his coalition (Drèze and Greenberg, 1980). An outcome of a hedonic coalition formation problem is a partition of the set of agents. The main concerns in the literature of hedonic coalition formation are the existence of stable partitions (for various stability concepts) and the existence and characterizations of mechanisms that assign stable partitions.[2]

In this paper, we consider hedonic coalition formation problems with friend-oriented preferences. To illustrate this class of hedonic coalition formation problems, consider a group of researchers that should divide themselves into research teams; e.g., at the workshops organized by the Leibniz Center for Informatics, working groups to initiate new research are formed (see, for instance, `https://www.dagstuhl.de/20301`). Each researcher, based on a partition of colleagues into those he would like to work with and those he would not want to work with, has preferences over the research teams he can be a member of. More generally, we suppose that at each problem and for each agent, the set of agents other than himself is partitioned into a set of "friends" and a set of "enemies." We say that an agent's preferences are friend-oriented if for each of his potential coalitions, (E) adding an enemy makes him strictly worse off and (F) adding a friend together with a set of enemies makes him strictly better off. Each profile of friend-oriented preferences induces a so-called friendship graph in which the vertices are the agents and the directed edges point to the agents' friends. The strongly connected components of the friendship graph induce a partition of the agents, the SCC partition, which plays a key role in our paper.

The terminology of "friends" and "enemies" has been used in various economic models before; e.g., Hiller (2017) considers network formation problems and Amorós (2019) considers problems where a group of jurors must choose a winner from a group of contestants. For hedonic coalition formation, Dimitrov and Sung (2004) introduce two preference domains; one based on the "appreciation" of friends and the other based on the "aversion" to enemies. Their appreciation of friends preference domain is a strict subset of our friend-oriented preference domain and their aversion to enemies preference domain is a strict subset of the enemy-oriented preference do-

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[1] Marriage problems and roommate problems (Gale and Shapley, 1962; Roth and Sotomayor, 1990) are special cases of hedonic coalition formation problems with coalitions of size at most two.

[2] We refer to Hajduková (2006) and Sung and Dimitrov (2007) for reviews of the literature on stability in hedonic coalition formation. Hajduková (2006) analyzes for which preference domains the existence of a stable partition is guaranteed (for various stability concepts). Sung and Dimitrov (2007) present a taxonomy of stability concepts and discuss the existence of stable partitions for various stability concepts.

main that we consider when discussing the robustness of our results (Appendix D). Dimitrov and Sung (2004) prove that there exists an individually stable partition in both their domains; a Nash stable partition exists when mutuality (of being friends or being enemies) is imposed. Moreover, they show that the corresponding algorithms to find individually stable and Nash stable partitions induce *strategy-proof* mechanisms. Dimitrov et al. (2006) show that if agents' preferences are based on the appreciation of friends, then a strictly core stable partition exists and can be found in polynomial time. They also show that if agents' preferences are based on the aversion to enemies, then there exists a core stable partition and the problem of finding a core stable partition is NP-hard.

We first consider core stable and strictly core stable partitions. A partition is *core stable* (and in the core) if there does not exist a coalition such that each member of the coalition strictly prefers it to his current coalition. A partition is *strictly core stable* (and in the strict core) if there does not exist a coalition such that each member of the coalition weakly prefers it to his current coalition and some member strictly prefers it to his current coalition.

We first prove a necessary condition for the core stability of partitions of hedonic coalition formation problems with friend-oriented preferences: any coalition in a core partition induces a strongly connected subgraph in the friendship graph (Proposition 1). This necessary condition for core stability implies that any core stable partition equals the SCC partition or is a refinement of the SCC partition (Corollary 1). We furthermore show that for each hedonic coalition formation problem with friend-oriented preferences, the SCC partition is in the strict core (Theorem 1).[3]

We then focus on mechanisms that assign a partition to each hedonic coalition formation problem with friend-oriented preferences. A mechanism is *(strictly) core stable* if it only assigns (strictly) core stable partitions. A mechanism is *group strategy-proof* if no coalition of agents can misreport their preferences so that all its members are weakly better off and at least one member is strictly better off. A mechanism is *strategy-proof* if no agent can misreport his preferences and be better off.

We first show that the SCC mechanism, which assigns the SCC partition to each hedonic coalition formation problem with friend-oriented preferences, is *group strategy-proof* (Proposition 3). Hence, the SCC mechanisms satisfies a strong stability notion (*strict core stability*) as well as a strong incentive compatibility property (*group strategy-proofness*). The main result of our paper is that no other mechanism satisfies these properties, or even the weaker properties of *core stability* and *strategy-proofness*: on any "rich" subdomain of friend-oriented problems,

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[3] We refer readers to Banerjee et al. (2001), Cechlárová and Romero-Medina (2001), Bogomolnaia and Jackson (2002), Burani and Zwicker (2003), Alcalde and Revilla (2004), Alcalde and Romero-Medina (2006), Dimitrov and Sung (2007), and Iehlé (2007) for studies investigating the existence of (strictly) core stable partitions in domains of preferences that are independent of the domain of friend-oriented preferences. Other stability concepts than (strict) core stability have been considered: Karakaya and Özbilen (2023) surveys stability concepts related to deviations of single agents, i.e., Nash stability, while Karakaya (2011) deals with more complex deviations, e.g., strong Nash stability.

the SCC mechanism is the only mechanism that satisfies *core stability* and *strategy-proofness* (Theorem 3).[4] Corollary 4 lists variations of this characterization result by strengthening *core stability* to *strict core stability* or *strategy-proofness* to *group strategy-proofness*.

Finally, we study an extension of friend-oriented problems where each agent partitions the set of other agents into a set of friends, enemies, and neutrals such that preferences still satisfy preference conditions (E) and (F) and, in addition, (N) adding or removing neutrals does not change the agent's welfare. For each friend-oriented problem with neutrals, we show that the SCC mechanism always yields a core stable partition (Theorem 2), but, in contrast with Theorem 1, the non-emptiness of the strict core is not guaranteed (Example 3). Next, we show that Proposition 3 and Corollary 4 (for *core stability* and *group strategy-proofness*) cannot be generalized to this setting: on any "very rich" subdomain of friend-oriented problems with neutrals, there is no mechanism that is *core stable* and *group strategy-proof* (Theorem 4). On the positive side, on each subdomain of friend-oriented problems with neutrals, the SCC mechanism satisfies *weak group strategy-proofness* (Proposition 5), and hence, *strategy-proofness*. However, the SCC mechanism is not the unique mechanism for friend-oriented problems with neutrals that is *core stable* and *weakly group strategy-proof* (Example 6). Thus, Theorem 3 cannot be generalized to this setting.

The papers that are most closely related to ours are Alcalde and Revilla (2004), Dimitrov et al. (2006), and Dimitrov and Sung (2007). Dimitrov et al. (2006) show that on a much smaller subdomain of friend-oriented preferences, the appreciation of friends preference domain (see Appendix A), "a strictly core stable coalition structure can be found in polynomial time." Alcalde and Revilla (2004) introduce the domain of top-responsive preferences for hedonic coalition formation problems. The domains of friend-oriented and top-responsive preferences are logically unrelated. Alcalde and Revilla (2004) show that on the domain of top-responsive preferences, their top covering mechanism is the only mechanism that satisfies *core stability* and *strategy-proofness*. Dimitrov and Sung (2007) show that the top covering mechanism even satisfies *strict core stability*. Appendix B recalls the definition of top-responsiveness and the two abovementioned results and shows the logical independence with friend-orientedness.

The rest of the paper is organized as follows. In Section 2, we present the hedonic coalition formation model, friend-oriented preferences, and the graph-theoretic definitions that are necessary for our analysis. In Section 3, we present our results on the (strict) core and its structure. In Section 4, we present our main result: on each "rich" subdomain of friend-oriented preferences, the SCC mechanism is the unique *(strictly) core stable* and *(group) strategy-proof* mechanism. At the end of Sections 3 and 4, we discuss how our results change for model variations, e.g., when allowing agents to partition the set of other agents into friends, enemies, and neutrals.

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[4] *Strategy-proof* mechanisms for hedonic coalition formation have been studied for various other preference domains, e.g., Alcalde and Revilla (2004), Pápai (2004), Rodríguez-Álvarez (2004), Barberà and Gerber (2007), Rodríguez-Álvarez (2009), Takamiya (2010), Takamiya (2013), and Leo et al. (2021).

# 2 Model

## Hedonic coalition formation problems, partitions, and the (strict) core

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents with $n \geq 3$. A *coalition* is a non-empty subset of agents $S \subseteq N$ ($S \neq \emptyset$). Each agent $i \in N$ has complete and transitive preferences $\succeq_i$ over the set of coalitions he belongs to, denoted by $\mathcal{C}_i \equiv \{S \subseteq N : i \in S\}$. Thus, for all coalitions $S, T \in \mathcal{C}_i$, if $S \succeq_i T$ then $i$ weakly prefers $S$ to $T$. Let $\mathcal{R}_i$ denote the set of (possible) preferences over $\mathcal{C}_i$. Let $\succ_i$ and $\sim_i$ denote the strict preference and indifference relation associated with $\succeq_i$. Let $\mathcal{R} \equiv \prod_{i \in N} \mathcal{R}_i$ denote the set of all preference profiles. A *(hedonic coalition formation) problem* is a pair $(N, \succeq)$ with $\succeq \in \mathcal{R}$. Since the set of agents $N$ is fixed throughout, we often denote a problem simply by its preference profile $\succeq$.

An outcome for a problem $\succeq \in \mathcal{R}$ is a *partition* of $N$. For each partition $\pi$ of $N$ and each $i \in N$, let $\pi_i$ denote the unique coalition in $\pi$ that contains agent $i$. To simplify notation, in examples, we denote a coalition using parentheses (instead of brackets) and removing commas, e.g., coalition $\{i, j, k\}$ is denoted by $(ijk)$. In preference tables we use a further simplification to $ijk$.

We assume that agents only care about the coalition they are a member of. Then, each agent $i$'s preferences over coalitions in $\mathcal{C}_i$ induces the following preferences over partitions: for each $i \in N$ and each pair of partitions $\pi, \pi'$,

$$\pi \succeq_i \pi' \text{ if and only if } \pi_i \succeq_i \pi'_i.$$

First, we introduce two well-known properties for partitions, a voluntary participation property and an efficiency property. A partition $\pi$ is *individually rational* if for each $i \in N$, $\pi_i \succeq_i \{i\}$. A partition $\pi$ is *Pareto-optimal* if there is no partition $\pi'$ such that for each $i \in N$, $\pi' \succeq_i \pi$ and for some $j \in N$, $\pi' \succ_j \pi$. Let $IR(\succeq)$ and $PO(\succeq)$ denote the *set of individually rational partitions* and the *set of Pareto-optimal partitions* of problem $\succeq$, respectively.

Second, we introduce two solutions that represent the idea of "stability" based on the absence of coalitions that can improve their situation by breaking up a partition to form a new coalition.

**Definition 1 ((Strict) core stability).** A partition is *weakly blocked* by coalition $S$ if for each $i \in S$, $S \succeq_i \pi_i$ and for some $j \in S$, $S \succ_j \pi_j$. A partition $\pi$ is *strictly core stable* if it is not weakly blocked by any coalition. Let $SC(\succeq)$ denote the *set of strictly core stable partitions* of problem $\succeq$, or the *strict core* for short. A partition $\pi$ is *blocked* by a coalition $S \subseteq N$ if for each $i \in S$, $S \succ_i \pi_i$. A partition $\pi$ is *core stable* if it is not blocked by any coalition. Let $C(\succeq)$ denote the *set of core stable partitions* of problem $\succeq$, or the *core* for short. ◇

Note that each *strictly core stable* partition is *individually rational*, *Pareto-optimal*, and *core stable*. For each $\succeq \in \mathcal{R}$, it holds that

$$SC(\succeq) \subseteq PO(\succeq) \text{ and } SC(\succeq) \subseteq C(\succeq) \subseteq IR(\succeq).$$

These set inclusions can be strict, even on the restricted domain of friend-oriented preferences, which we will discuss next.[5]

It is well-known that for the unrestricted domain of problems, the core may be empty (Banerjee et al., 2001). Here, we propose a new type of preference restriction that will guarantee a non-empty strict core and that is based on the ability of each agent to partition the set of agents (except himself) into "friends" and "enemies." Loosely speaking, friend-oriented preferences also express the "lexicographic principle" that being together with friends is more important than having enemies around as well.

## Friend-oriented preferences

Let $i \in N$ and $\succeq_i \in \mathcal{R}_i$. Then, agent $i$'s preferences are *friend-oriented* if the set $N \backslash \{i\}$ can be partitioned into a *set of friends* $F(\succeq_i)$ and a *set of enemies* $E(\succeq_i)$ such that for each coalition $S \in \mathcal{C}_i$, (E) adding an enemy makes agent $i$ strictly worse off and (F) adding a friend, possibly together with a set of enemies, makes agent $i$ strictly better off. Note that Condition (F) embeds the lexicographic principle that to improve a coalition, adding friends is strictly more important than removing enemies. Formally, let $i \in N$. Preferences $\succeq_i \in \mathcal{R}_i$ are *friend-oriented* if

**(E)** for each $S \in \mathcal{C}_i$ and each $e \in E(\succeq_i) \setminus S$,

$$S \succ_i S \cup \{e\};$$

and

**(F)** for each $S \in \mathcal{C}_i$, each $f \in F(\succeq_i) \setminus S$, and each $E \subseteq E(\succeq_i) \setminus S$,

$$S \cup \{f\} \cup E \succ_i S;$$

or equivalently, for each $S \in \mathcal{C}_i$, each $f \in F(\succeq_i) \setminus S$, and each $E \subseteq E(\succeq_i) \cap S$,

$$S \cup \{f\} \succ_i S \setminus E.$$

Clearly, at any friend-oriented problem, adding an enemy to a coalition is bad and adding a friend to a coalition is good. However, Condition (F) also expresses a lexicographic preference

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[5] Example 1 shows that $SC(\succeq) \subsetneq C(\succeq)$ is possible and Example 5 shows that $SC(\succeq) \subsetneq PO(\succeq)$ and $C(\succeq) \subsetneq IR(\succeq)$ are possible.

for friends over enemies: adding a friend is always beneficial, even if at the same time enemies are joining the coalition. At the end of Section 3, by means of an example (Example 8), we discuss why we cannot weaken Condition (F) to just require that adding friends is good (without the lexicographic aspect of friends being more important than enemies). In Appendix D, we discuss how our results change when preferences are "enemy-oriented" instead of friend-oriented (by switching the lexicographic roles of friends and enemies).

Note that if agent $i$'s preferences $\succeq_i \in \mathcal{R}_i$ are friend-oriented, then

- $F(\succeq_i) = \{j \in N : \{i,j\} \succ_i \{i\}\}$;
- $E(\succeq_i) = \{j \in N : \{i\} \succ_i \{i,j\}\}$; and
- for all $j \in N \setminus \{i\}$, $\{i,j\} \not\sim_i \{i\}$.

For each $i \in N$, when no confusion is possible, we write $F_i$ and $E_i$ instead of $F(\succeq_i)$ and $E(\succeq_i)$. Let $\mathcal{R}_i^f$ denote the set of preferences over $\mathcal{C}_i$ that are *friend-oriented.* Example 7 in Appendix A illustrates the possible friend-oriented preferences for an agent who has two friends and one enemy.

Let $\succeq_i \in \mathcal{R}_i^f$. Then, $F_i \cup \{i\}$ is the unique most preferred coalition for agent $i$ and $E_i \cup \{i\}$ is the unique least preferred coalition for agent $i$. Note that being friends does not need to be reciprocal, i.e., it is possible that for $\succeq_i \in \mathcal{R}_i^f$ and $\succeq_j \in \mathcal{R}_j^f$, $j \in F_i$ and $i \in E_j$.

We are using the terms "friends" and "enemies" to define our new preference restriction for two reasons. First, the polarizing terminology of friends and enemies clearly represents the desirability of one type of agent (friend) versus the preference for the absence of the other type of agent (enemy). However, while "friend" and "enemy" clearly represent the good versus the bad type of agent, we should keep in mind interpretations that are less emotional, such as "productive" and "unproductive" agents or "good" and "bad" agents according to other criteria.

In Appendix A, we also discuss two subdomains of friend-oriented preferences. First, we recall the definition of the *appreciation of friends* preference domain (e.g., Dimitrov et al., 2006) that is based on the number of friends versus the number of enemies in a coalition. Second, we introduce the domain of *lexicographically friend-oriented* preferences; an economically interesting subdomain of friend-oriented preferences that is based on lexicographically extending a strict ranking over all agents to a strict ranking over coalitions. Appendix B recalls the definition of top-responsiveness and provides a Venn-diagram that shows that the two above-mentioned subdomains of friend-oriented preferences are also subdomains of top-responsive preferences.

Next, we focus on the set of (strict) core partitions for friend-oriented problems and illustrate that the strict core can be a strict subset of the core (Example 1) and that there can be multiple (strict) core partitions (Example 2). In the next section, we furthermore show that for friend-oriented problems, the (strict) core is always non-empty.

**Example 1 (The strict core can be a strict subset of the core).**
Let $N = \{1,2,3\}$ and $\succeq \in \mathcal{R}^f$ such that $F_1 = \{2\}$, $F_2 = \{1,3\}$, and $F_3 = \{2\}$; furthermore, agent 2 is indifferent between friend 1 and friend 3, i.e., $(12) \sim_2 (23)$. The corresponding friend-oriented preferences are

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|
| 12 | 123 | 23 |
| 123 | $12 \sim 23$ | 123 |
| 1 | 2 | 3 |
| 13 | | 13 |

One easily verifies that $SC(\succeq) = \{\{(123)\}\} \subsetneq \{\{(12),(3)\},\{(1),(23)\},\{(123)\}\} = C(\succeq)$. $\diamond$

**Example 2 (The (strict) core needs not be a singleton).**
Let $N = \{1,2,3\}$ and $\succeq \in \mathcal{R}^f$ such that $F_1 = \{2\}$, $F_2 = \{1,3\}$, and $F_3 = \{1\}$. The corresponding friend-oriented preferences are

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|
| 12 | 123 | 13 |
| 123 | $\vdots$ | 123 |
| 1 | 2 | 3 |
| 13 | | 23 |

Here, $\vdots$ in the second column indicates the unspecified preferences $(12) \succ_2 (23)$, $(23) \succ_2 (12)$, or $(12) \sim_2 (23)$. One easily verifies that there are two (strictly) core stable partitions, namely $\{(123)\}$ and $\{(12),(3)\}$. $\diamond$

We are interested in the structure and non-emptiness of the (strict) core for friend-oriented problems. In the sequel, we use some graph theoretical tools that were first used for problems satisfying appreciation of friends (Dimitrov et al., 2006).

## Directed graphs induced by friend-oriented preferences

A *directed graph* is a pair $G = (V, A)$ where $V$ is a finite set of *vertices* and $A \subseteq \{(i,j) \in V \times V : i \neq j\}$ is a set of *directed edges*. For each $a = (v,w) \in A$, edge $a$ is an *outgoing edge* for $v$ and an *incoming edge* for $w$. A *subgraph* $G' = (V', A')$ of $G$ is a directed graph such that $V' \subseteq V$ and $A' \subseteq (V' \times V') \cap A$. An *induced graph* $G' = (V', A')$ of $G$ is a directed graph such that $V' \subseteq V$ and $A' = (V' \times V') \cap A$.

A *path* from a vertex $v_1$ to a vertex $v_m$ is an ordered sequence $(v_1, v_2, \ldots, v_m)$ of $m \geq 2$ vertices in $V$ such that for each $k$ with $1 \leq k \leq m-1$, $(v_k, v_{k+1}) \in A$. A path $(v_1, v_2, \ldots, v_{m-1}, v_1)$ is called a *cycle*. A cycle $(v_1, v_2, \ldots, v_{m-1}, v_1)$ is *simple* if for all $k, l \in \{1, \ldots, m-1\}$ with $k \neq l$,

$v_k \neq v_l$. In other words, a simple cycle is a cycle with no repeated vertices (except for the first and last vertex). An *acyclic* graph is a graph without cycles.

A graph $G = (V, A)$ is *strongly connected* if for each pair of vertices $v, w \in V$ there is a cycle that contains $v$ and $w$. A subgraph $G' = (V', A')$ of $G$ is a *strongly connected component* (SCC) of $G$ if it is strongly connected, and maximal with this property, i.e., there is no strongly connected subgraph $G'' = (V'', A'')$ of $G$ with $V' \subsetneq V''$. If a subgraph $G' = (V', A')$ of $G$ is a strongly connected component, then we refer to $V'$ as an *SCC coalition.*

For each directed graph $G = (V, A)$ and each $v \in V$ there exists a unique strongly connected component of $G$ that contains $v$ (Harary et al., 1965, Theorem 3.2). Thus,

- the strongly connected components (uniquely) partition the set $V$ and
- each strongly connected subgraph of $G$ is a subgraph of some strongly connected component of $G$.[6]

There exist linear-time algorithms that compute the strongly connected components of a given directed graph, e.g., Tarjan's Algorithm (Tarjan, 1972). Figure 1 provides an example of the partition of $V$ induced by the strongly connected components.

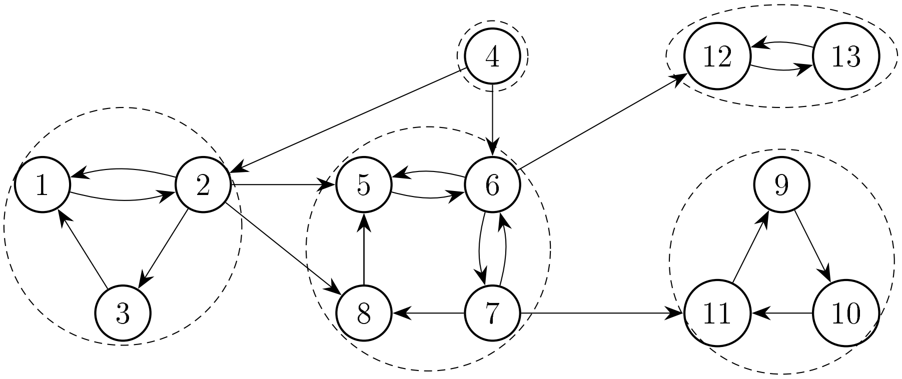

Figure 1: Strongly connected components are encircled and induce SCC coalitions $\{1, 2, 3\}$, $\{4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11\}$, and $\{12, 13\}$.

Let $G_1, \ldots, G_K$ be the strongly connected components of a directed graph $G = (V, A)$. For each $k$ with $1 \leq k \leq K$, let $G_k = (V_k, A_k)$. The *condensation graph* of $G$ is the directed graph $\bar{G} = (\bar{V}, \bar{A})$ with vertices $\bar{V} = \{G_1, \ldots, G_K\}$ and edges $\bar{A} = \{(G_k, G_l) \in \bar{V} \times \bar{V} :$ there exist $v_k \in$

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[6] Suppose to the contrary that a strongly connected subgraph $G' = (V', A')$ is not a subgraph of any strongly connected component of $G$. Then, there exists a collection of strongly connected components $\{G_1, \ldots, G_m\}$ (with $m \geq 2$) of $G$ such that for each $l$ with $1 \leq l \leq m$, $G_l = (V_l, A_l)$, $V'_l := V' \cap V_l \neq \emptyset$, and $V' = V'_1 \cup \cdots \cup V'_m$. Let $i_1 \in V'_1$. Then, for each $l$ with $2 \leq l \leq m$, there exists $i_l \in V'_l$ such that there exists a cycle in $G$ that contains both $i_1$ and $i_l$. Then, $G$ has a subgraph $\bar{G} = (V_1 \cup \cdots \cup V_m\ , \bar{A})$ that is strongly connected. This contradicts the fact that each $G_l = (V_l, A_l)$ is a strongly connected component.

$V_k$ and $v_l \in V_l$ with $(v_k, v_l) \in A\}$. Condensation graphs are acyclic (see, e.g., Harary et al., 1965, Theorem 3.6).[7] Figure 2 depicts the condensation graph induced by the graph in Figure 1.

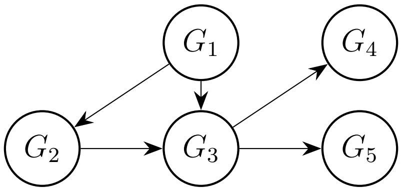

Figure 2: Condensation graph induced by the graph in Figure 1.

Each problem $\succeq \in \mathcal{R}^f$ and each coalition $S \subseteq N$ induce a directed *friendship graph* $\Gamma(\succeq^S) = (S, A^S)$ with $A^S = \{(i,j) \in S \times S : j \in F_i\}$. We write $\succeq$ for $\succeq^N$, i.e., when $S = N$.

Let $G_1, \ldots, G_K$ be the strongly connected components of $\Gamma(\succeq)$. For each $1 \le k \le K$, let $G_k = (V_k, A_k)$. Let $\pi^{SCC}(\succeq) = \{V_1, \ldots, V_K\}$ denote the *SCC partition* that consists of the SCC coalitions of $\Gamma(\succeq)$.

**Fact 1.** Since the condensation graph of $\Gamma(\succeq)$ is acyclic, without loss of generality, we can choose the labels of the SCC coalitions (or the associated strongly connected components) such that for all $l, l' \in \{1, \ldots, K\}$ with $l < l'$, there is no edge from any vertex in $V_{l'}$ to any vertex in $V_l$ in $\Gamma(\succeq)$. In particular, there is no edge from any vertex in $V_K$ to any vertex in any $V_l$ with $l < K$. See, e.g., Figure 2.

From now on, we assume that the labeling of strongly connected components $G_1 = (V_1, A_1), \ldots, G_K = (V_K, A_K)$ of $\Gamma(\succeq)$ complies with Fact 1. Note that this labeling is not necessarily unique.[8]

**Fact 2.** Fact 1 implies that for each $l \in \{1, \ldots, K\}$ and each $i \in V_l$,

$$\begin{aligned} F_i &= \underbrace{(F_i \cap (V_1 \cup \cdots \cup V_{l-1}))}_{=\emptyset} \cup \bigcup_{k=l}^{K} (F_i \cap V_k) = \bigcup_{k=l}^{K} (F_i \cap V_k). \\ E_i &= \underbrace{(E_i \cap (V_1 \cup \cdots \cup V_{l-1}))}_{=(V_1 \cup \cdots \cup V_{l-1})} \cup \bigcup_{k=l}^{K} (E_i \cap V_k) = \bigcup_{k=1}^{l-1} V_k \ \cup \ \bigcup_{k=l}^{K} (E_i \cap V_k). \end{aligned}$$

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[7] If $G' = (V', A')$ and $G'' = (V'', A'')$ with $V' \neq V''$ are two strongly connected components of $G$, then there can be edges from $V'$ to $V''$ or the other way around, but not both (otherwise, $G$ would have a subgraph $\bar{G} = (V' \cup V'', \bar{A})$ that is strongly connected, contradicting the fact that $G' = (V', A')$ and $G'' = (V'', A'')$ with $V' \neq V''$ are two strongly connected components of $G$).

[8] For instance, in Figure 2, we could swap the labels of $G_4$ and $G_5$.

# 3 The Structure of the Core

## 3.1 Friend-oriented preferences and core stability

Our first result establishes a necessary condition for any core partition: any coalition in a core partition induces a strongly connected subgraph in the friendship graph.

**Proposition 1.** *Let* $\succeq \in \mathcal{R}^f$ *and* $\pi \in C(\succeq)$*. Then, for each* $S \in \pi$*, the induced friendship graph* $\Gamma(\succeq^S)$ *is strongly connected.*

***Proof.*** Let $\succeq \in \mathcal{R}^f$, $\pi \in C(\succeq)$, and $S \in \pi$. Suppose to the contrary that the induced friendship graph $\Gamma(\succeq^S) = (S, A^S)$ is not strongly connected. Then, there exist $i, j \in S$ with $i \neq j$ such that there is no path from $j$ to $i$. Let $S'$ be the subset of vertices in $S$ from which $i$ is not reachable, i.e., $S' = \{k \in S : \text{there is no path from } k \text{ to } i \text{ in } \Gamma(\succeq^S)\}$. Note that $j \in S'$ and $i \in S \setminus S'$. So, $S' \neq \emptyset$ and $S \setminus S' \neq \emptyset$.

Next, we prove that for each $k \in S'$, $S \setminus S' \subseteq E_k$. Suppose that for some $k \in S'$, $S \setminus S' \nsubseteq E_k$. Then, $(S \setminus S') \cap F_k \neq \emptyset$. Let $l \in (S \setminus S') \cap F_k$. Since $l \in F_k$, there is an edge from $k$ to $l$ and since $l \in S \setminus S'$, there is a path from $l$ to $i$. Hence, there is a path from $k$ to $i$ in $\Gamma(\succeq^S)$, contradicting $k \in S'$. So, for each $k \in S'$, $S \setminus S' \subseteq E_k$.

It follows from Condition (E) of friend-oriented preferences that for each $k \in S'$, $S' \succ_k S' \cup (S \setminus S') = S = \pi_k$. Hence, $S' \neq \emptyset$ blocks $\pi$, which contradicts the fact that $\pi \in C(\succeq)$. □

Recall that Example 2 shows that there are friend-oriented problems with multiple core stable partitions. An important corollary to Proposition 1 is that any core stable partition equals the SCC partition or is a refinement of the SCC partition.

**Corollary 1.** *Let* $\succeq \in \mathcal{R}^f$*,* $\pi^{SCC}(\succeq) = \{V_1, \ldots, V_K\}$*, and* $\pi \in C(\succeq)$*. Then, for each* $S \in \pi$*, there is a* $V_k \in \pi^{SCC}(\succeq)$ *such that* $S \subseteq V_k$*.*

***Proof.*** Let $\succeq \in \mathcal{R}^f$, $\pi^{SCC}(\succeq) = \{V_1, \ldots, V_K\}$, $\pi \in C(\succeq)$, and $S \in \pi$. It follows from Proposition 1 that the induced graph $\Gamma(\succeq^S)$ is strongly connected. Hence, there exists a strongly connected component $G_k = (V_k, A_k)$ that contains $\Gamma(\succeq^S)$ (see Footnote 6). In particular, $S \subseteq V_k$. □

A result for problems with strongly connected components that equal simple cycles now follows easily.

**Proposition 2.** *Let* $\succeq \in \mathcal{R}^f$ *and consider a strongly connected component of* $\Gamma(\succeq)$ *denoted by* $\widetilde{G} = (\tilde{V}, \tilde{A})$*. Suppose* $\widetilde{G}$ *consists of a simple cycle. Then, for each* $\pi \in C(\succeq)$*,* $\widetilde{V} \in \pi$*.*

***Proof.*** Let $\pi \in C(\succeq)$. Let $S \in \pi$ such that $S \cap \widetilde{V} \neq \emptyset$. From Corollary 1 it follows that $S \subseteq \widetilde{V}$. Suppose $S \subsetneq \widetilde{V}$. Then, since $\widetilde{G}$ is a cycle, there exists $i \in S$ such that $S \backslash \{i\} \subseteq E_i$. It follows from Condition (E) of friend-oriented preferences that $\{i\} \succ_i S = \pi_i$, contradicting $\pi \in C(\succeq)$. Hence, $\widetilde{V} = S \in \pi$. □

Next, we state and prove our first main result: for any problem with friend-oriented preferences, the SCC partition is strictly core stable and hence, the strict core is non-empty.

**Theorem 1.** *For each* $\succeq \in \mathcal{R}^f$*,* $\pi^{SCC}(\succeq) \in SC(\succeq)$*. In particular,* $SC(\succeq) \neq \emptyset$*.*

Dimitrov et al. (2006, Theorem 5) show that on the smaller appreciation of friends preference domain, "a strictly core stable coalition structure can be found in polynomial time." Theorem 1 extends this result to the larger preference domain of friend-oriented preferences. Both results use the fact that the strongly connected components of the friendship graph determine a strict core partition that can be computed in polynomial time by, e.g., Tarjan's Algorithm (Tarjan, 1972).

***Proof***. Let $\succeq \in \mathcal{R}^f$. We prove that no coalition $S \subseteq N$ weakly blocks partition $\pi^{SCC}(\succeq)$. Using the labeling convention of Fact 1, let $\pi^{SCC}(\succeq) = \{V_1, \ldots, V_K\}$. The claim is immediate if for some $k$ with $1 \leq k \leq K$, $S = V_k$.

For some $k$ with $1 \leq k \leq K$, let $S \subsetneq V_k$. Since $S \neq V_k$, $V_k \setminus S \neq \emptyset$. Since $V_k$ is strongly connected in $\Gamma(\succeq) = (N, A)$, there is an edge $a = (i, j) \in A$ from some $i \in S$ to some $j \in V_k \setminus S$. Let $F = F_i \cap (V_k \setminus S)$ and $E = E_i \cap (V_k \setminus S)$. Then, $V_k = S \cup (E \cup F)$ and $S \cap (E \cup F) = \emptyset$. Note that $j \in F$. So, $F \neq \emptyset$. Hence, from Condition (F) of friend-oriented preferences it follows that $\pi_i^{SCC}(\succeq) = V_k = S \cup (E \cup F) \succ_i S$. Hence, $S$ does not weakly block $\pi^{SCC}(\succeq)$.

Next, we can assume that for each $k$ with $1 \leq k \leq K$, $S \not\subseteq V_k$. Then, there exists $m$ with $2 \leq m \leq K$ such that $S = S \cap (V_1 \cup \cdots \cup V_m)$, $S \cap (V_1 \cup \cdots \cup V_{m-1}) \neq \emptyset$, and

$$S \cap V_m \neq \emptyset. \tag{1}$$

Since we use the labeling convention of Fact 1, for any $l < m$, there is no edge from any vertex in $V_m$ to any vertex in $V_l$. Hence, Fact 2 implies that for each $i \in V_m$, $\emptyset \neq S \cap (V_1 \cup \cdots \cup V_{m-1}) \subseteq E_i$.

Suppose $V_m \subsetneq S$. Let $i \in V_m$. Since $\emptyset \neq S \cap (V_1 \cup \cdots \cup V_{m-1}) \subseteq E_i$, it follows from Condition (E) of friend-oriented preferences that $\pi_i^{SCC}(\succeq) = V_m \succ_i V_m \cup [S \cap (V_1 \cup \cdots \cup V_{m-1})] = [S \cap V_m] \cup [S \cap (V_1 \cup \cdots \cup V_{m-1})] = S$ (the penultimate equality follows from $V_m \subsetneq S$). Hence, $S$ does not weakly block $\pi^{SCC}(\succeq)$. Thus, $V_m \not\subseteq S$. So, $V_m \backslash S \neq \emptyset$.

We have established that $S \cap V_m \neq \emptyset$ (see (1)) and $V_m \backslash S \neq \emptyset$. Hence, since

$$V_m = (S \cap V_m) \cup (V_m \backslash S)$$

is strongly connected in $\Gamma(\succeq) = (N, A)$, there is an edge $a = (i, j) \in A$ from some $i \in S \cap V_m$ to some $j \in V_m \setminus S$. Let $F = F_i \cap (V_m \setminus S)$ and $E = E_i \cap (V_m \setminus S)$. Then,

$$V_m = (S \cap V_m) \cup (E \cup F) \text{ and } (S \cap V_m) \cap (E \cup F) = \emptyset. \tag{2}$$

Note that $j \in F$. So, $F \neq \emptyset$. From Condition (F) of friend-oriented preferences it follows that $\pi_i^{SCC}(\succeq) = V_m \succ_i V_m \backslash (E \cup F)$. Since $\emptyset \neq S \cap (V_1 \cup \cdots \cup V_{m-1}) \subseteq E_i$, it follows from Condition (E) of friend-oriented preferences that $V_m \setminus (E \cup F) \succ_i [V_m \setminus (E \cup F)] \cup [S \cap (V_1 \cup \cdots \cup V_{m-1})]$. From (2),

$$[V_m \setminus (E \cup F)] \cup [S \cap (V_1 \cup \cdots \cup V_{m-1})] = [S \cap V_m] \cup [S \cap (V_1 \cup \cdots \cup V_{m-1})] = S.$$

Thus, $\pi_i^{SCC}(\succeq) \succ_i S$. Hence, $S$ does not weakly block $\pi^{SCC}(\succeq)$. □

Theorem 1 and Proposition 2 imply the following.

**Corollary 2.** *Let* $\succeq \in \mathcal{R}^f$ *such that* $\Gamma(\succeq)$ *consists of a simple cycle. Then,* $\{N\}$ *is the unique (strictly) core stable partition.*

**Remark 1 (Model variations and core stability).** Recall that friend-oriented preferences are based on (1) the partition of other agents into friends and enemies, (2) the assumption that adding friends and removing enemies is good, and (3) the lexicographic aspect that adding friends is more important than removing enemies. In Appendix C, we remove the lexicographic aspect (3) and show that then, core stable partitions need not exist (Example 8).

In Appendix D we switch the lexicographic aspect (3) from adding friends being more important to removing enemies being more important. For a subclass of the thus defined *enemy-oriented preferences*, the class of preferences that satisfy *aversion to enemies*, Dimitrov et al. (2006, Example 4) show that the strict core can be empty and they establish the non-emptiness of the core (Dimitrov et al., 2006, Theorem 3). In Appendix D, we demonstrate that for the larger domain of enemy-oriented preferences, also the core can be empty (Example 10). ◇

## 3.2 Friend-oriented preferences with neutrals and core stability

Some of our results can be generalized to a setting where each agent partitions the set of other agents in a set of friends, a set of enemies, and a set of "neutrals:" adding / removing a neutral does not make any coalition better or worse. In particular, we drop the assumption that for all $i, j \in N$ with $i \neq j$, $\{i, j\} \not\sim_i \{i\}$.

Let $i \in N$ and $\succeq_i \in \mathcal{R}_i$. Then, agent $i$'s preferences are *friend-oriented with neutrals* if the set $N \backslash \{i\}$ can be partitioned into a *set of friends* $F(\succeq_i)$, a *set of enemies* $E(\succeq_i)$, and a *set of neutrals* $N(\succeq_i)$ such that for each coalition $S \in \mathcal{C}_i$, (E) adding an enemy makes agent $i$ strictly worse off, (F) adding a friend, possibly together with a set of enemies, makes agent $i$ strictly better off, and (N) adding a neutral does not change agent $i$'s welfare. Formally, let $i \in N$. Preferences $\succeq_i \in \mathcal{R}_i$ are *friend-oriented with neutrals* if they satisfy **(E)**, **(F)**, and

**(N)** for each $S \in \mathcal{C}_i$ and each $j \in N(\succeq_i) \setminus S$,

$$S \cup \{j\} \sim_i S.$$

Let $\mathcal{R}_i^{fn}$ denote the set of preferences over $\mathcal{C}_i$ that are *friend-oriented with neutrals.* Obviously, $\mathcal{R}_i^f \subsetneq \mathcal{R}_i^{fn}$. For each $i \in N$, when no confusion is possible, we write $F_i$, $E_i$, and $N_i$ instead of $F(\succeq_i)$, $E(\succeq_i)$, and $N(\succeq_i)$.

**Example 3 (Ota et al. (2017, Example 3), the strict core can be empty when preferences are friend-oriented with neutrals).** Let $N = \{1, 2, 3\}$ and $\succeq \in \mathcal{R}^{fn}$ such that $F_1 = \{2\}$, $E_1 = \{3\}$, $N_1 = \emptyset$, $F_2 = E_2 = \emptyset$, $N_2 = \{1, 3\}$, $F_3 = \{2\}$, $E_3 = \{1\}$, and $N_3 = \emptyset$. Then, from conditions (E), (F), and (N), the friend-oriented preferences with neutrals are as follows,

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|
| 12 | $2 \sim 12 \sim 23 \sim 123$ | 23 |
| 123 | | 123 |
| 1 | | 3 |
| 13 | | 13 |

One easily verifies that the core contains all partitions except $\{(13), (2)\}$. So, $C(\succeq) \neq \emptyset$. Furthermore, for each partition in the core, there exists a weak blocking coalition. Hence, $SC(\succeq) = \emptyset$. $\diamond$

Note that similarly to problems with friend-oriented preferences, each problem $\succeq \in \mathcal{R}^{fn}$ induces a directed *friendship graph* $\Gamma(\succeq)$. The only difference now is that if for any two distinct agents $i, j \in N$ we have $(i, j) \notin \Gamma(\succeq)$, then there are *two* possibilities: agent $j$ is either an enemy of or a neutral to agent $i$. While the above example shows that in the presence of neutrals the SCC partition needs not be strictly core stable, we show that it is always core stable.

**Theorem 2.** *For each $\succeq \in \mathcal{R}^{fn}$, $\pi^{SCC}(\succeq) \in C(\succeq)$. In particular, $C(\succeq) \neq \emptyset$.*

***Proof.*** Let $\succeq \in \mathcal{R}^{fn}$. We prove that no coalition blocks partition $\pi \equiv \pi^{SCC}(\succeq)$. Suppose to the contrary that $\pi$ is blocked by a coalition $S \subseteq N$. Then, for each $i \in S$, $S \succ_i \pi_i$. Let $\pi = \{V_1, \ldots, V_K\}$. Without loss of generality (here, we are not using the labeling convention of Fact 1), we now assume that for some $m$ with $1 \leq m \leq K$, $S = S \cap (V_1 \cup \cdots \cup V_m)$ and for each $\ell$ with $1 \leq \ell \leq m$, $S \cap V_\ell \neq \emptyset$.

**Claim.** *For each $\ell$ with $1 \leq \ell \leq m$, there is some agent $i \in S \cap V_\ell$ with a friend $f \in S \cap F_i$ that is not a member of $V_\ell$, i.e., $f \notin V_\ell$.*

*Proof of the claim.* Suppose to the contrary that the claim is not true for some $\ell$ with $1 \leq \ell \leq m$, that is, for each agent $j \in S \cap V_\ell$ all of his friends that are in $S$ are also members of $V_\ell$, i.e., $S \cap F_j \subseteq V_\ell$.

Let $i \in S \cap V_\ell$. Since $S \succ_i \pi_i = V_\ell$ and $S \cap F_i \subseteq V_\ell$, it follows from conditions (E), (F), and (N) that there is an enemy $e \in V_\ell \cap E_i$ that is not a member of $S$. Since $V_\ell$ is strongly connected and $i, e \in V_\ell$, there is a path $(i = i_1, i_2, \ldots, i_k = e)$ from $i$ to $e$ in $\Gamma(\succeq)$ that only uses edges between agents in $V_\ell$. Since $i \in S$ but $e \notin S$, the path contains an edge $(i_q, i_{q+1})$ with $i_q \in S$ and $i_{q+1} \notin S$. By definition of $\Gamma(\succeq)$, $i_{q+1} \in F_{i_q}$. Hence, agent $i_q \in S \cap V_\ell$ no longer has his friend

$i_{q+1} \in V_\ell$ in coalition $S$ (of which $i_q$ is a member). Since nonetheless $S \succ_{i_q} \pi_{i_q} = V_\ell$, it follows from conditions (E), (F), and (N) that agent $i_q \in S \cap V_\ell$ has another friend $f \in S \cap F_{i_q}$ that is not a member of $V_\ell$. This contradiction proves the claim. ■

The claim implies that for each $\ell$ with $1 \leq \ell \leq m$, there is some agent $i \in V_\ell$ with a friend $f \in V_{\ell'}$ for some $1 \leq \ell' \leq m$ with $\ell' \neq \ell$. But then there is a cycle in $\Gamma(\succeq)$ that traverses at least two components in $\{V_1, \ldots, V_m\}$, which contradicts the fact that the condensation graph of $\Gamma(\succeq)$ is acyclic. This contradiction completes the proof. □

Note that the above proof can also be used to show the non-emptiness of the core for problems with friend-oriented preferences. However, showing the non-emptiness of the strict core for problems with friend-oriented preferences (Theorem 1) requires some proof steps that do not work in the presence of neutrals.

# 4 Core Stability and Strategy-Proofness

## 4.1 Friend-oriented preferences: A characterization

For each $i \in N$, let $\widetilde{\mathcal{R}}_i \subseteq \mathcal{R}_i$. Let $\widetilde{\mathcal{R}} \equiv \prod_{i \in N} \widetilde{\mathcal{R}}_i$. A *mechanism* on $\widetilde{\mathcal{R}}$ is a function $\varphi$ that associates with each problem $\succeq \in \widetilde{\mathcal{R}}$ a partition $\varphi(\succeq)$. For each $i \in N$, let $\varphi_i(\succeq)$ denote agent $i$'s coalition at $\succeq \in \widetilde{\mathcal{R}}$ under mechanism $\varphi$. A mechanism $\varphi$ is individually rational / Pareto optimal / (strictly) core stable if for each $\succeq \in \widetilde{\mathcal{R}}$, $\varphi(\succeq)$ is individually rational / Pareto optimal / (strictly) core stable at $\succeq$.

The next two properties are incentive properties that model that no agent / coalition can benefit from misreporting his / their preferences. We use the standard notation $\succeq_{-i} = (\succeq_j)_{j \in N \setminus \{i\}}$ to denote the list of all agents' preferences, except for agent $i$'s preferences. Similarly, for each $S \subseteq N$ we define $\succeq_{-S} = (\succeq_j)_{j \in N \setminus S}$ to be the list of preferences of the members of $N \setminus S$.

**Definition 2 ((Group) strategy-proofness).** A mechanism is *strategy-proof* if no agent gets strictly better off by misreporting his preferences. Formally, mechanism $\varphi$ is *strategy-proof* if for each problem $\succeq \in \widetilde{\mathcal{R}}$, each $i \in N$, and each $\succeq_i' \in \widetilde{\mathcal{R}}_i$,

$$\varphi_i(\succeq) \succeq_i \varphi_i(\succeq_i', \succeq_{-i}).$$

A mechanism is *group strategy-proof* if there exists no problem where some coalition of agents can misreport their preferences so that all its members get weakly better off and at least one member gets strictly better off. Formally, a mechanism $\varphi$ is *group strategy-proof* if for each problem $\succeq \in \widetilde{\mathcal{R}}$, there do not exist $S \subseteq N$ and $\succeq_S' \in \prod_{i \in S} \widetilde{\mathcal{R}}_i$ such that

(g1) for each $i \in S$, $\varphi_i(\succeq_S', \succeq_{-S}) \succeq_i \varphi_i(\succeq)$ and (g2) for some $j \in S$, $\varphi_j(\succeq'_S, \succeq_{-S}) \succ_j \varphi_j(\succeq)$. ◇

It is easy to see that if a mechanism $\varphi$ is *group strategy-proof*, then it is *strategy-proof*.

For each $i \in N$, let $\widetilde{\mathcal{R}}^f_i \subseteq \mathcal{R}^f_i$ be a *generic subdomain of friend-oriented preferences*. Let $\widetilde{\mathcal{R}}^f \equiv \prod_{i \in N} \widetilde{\mathcal{R}}^f_i$ be the corresponding subdomain of friend-oriented problems. From now on, we assume that mechanisms are defined on subdomains of friend-oriented problems.

**Definition 3 (SCC mechanism).** The mechanism on $\widetilde{\mathcal{R}}^f$ that associates the partition $\pi^{SCC}(\succeq)$ with each $\succeq \in \widetilde{\mathcal{R}}^f$ is called the *SCC mechanism* (on $\widetilde{\mathcal{R}}^f$); we denote it by $\varphi^{SCC}$. ◇

Theorem 1 implies that the SCC mechanism is *strictly core stable*.

**Corollary 3.** *The SCC mechanism on* $\widetilde{\mathcal{R}}^f$ *is strictly core stable.*

We next show that the SCC mechanism is *group strategy-proof*.

**Proposition 3.** *The SCC mechanism on* $\widetilde{\mathcal{R}}^f$ *is group strategy-proof.*

We prove Proposition 3 in Appendix E.

We are now ready to state and prove our main result (Theorem 3), which shows that if agents' preferences are sufficiently "rich," then the SCC mechanism is, in fact, the only *core stable* and *strategy-proof* mechanism. We call an agent's friend-oriented preference domain *rich* if for each set of other agents, there is a preference relation that declares these agents as friends.

**Definition 4 (Rich preference domains).** Let $i \in N$. A subdomain of friend-oriented preferences $\widetilde{\mathcal{R}}^f_i \subseteq \mathcal{R}^f_i$ is *rich* if for each set $S \subseteq N \backslash \{i\}$, there are preferences $\succeq_i \in \widetilde{\mathcal{R}}^f_i$ such that $F(\succeq_i) = S$; thus, $E(\succeq_i) = N \backslash (S \cup \{i\})$. We say that the subdomain of friend-oriented problems $\widetilde{\mathcal{R}}^f \equiv \prod_{i \in N} \widetilde{\mathcal{R}}^f_i$ is *rich* if for each $i \in N$, $\widetilde{\mathcal{R}}^f_i$ is rich. ◇

Examples of rich subdomains of friend-oriented preferences for $i \in N$ include the set of friend-oriented preferences, the set of preferences that satisfy appreciation of friends, and the set of preferences that are lexicographically friend-oriented.

**Theorem 3.** *On each rich subdomain of friend-oriented problems* $\widetilde{\mathcal{R}}^f$*, a mechanism* $\varphi$ *is core stable and strategy-proof if and only if* $\varphi = \varphi^{SCC}$.

From Theorems 1 and 3 and Proposition 3 we immediately obtain the following corollary.

**Corollary 4.** *On each rich subdomain of friend-oriented problems* $\widetilde{\mathcal{R}}^f$*,*

- *a mechanism* $\varphi$ *is strictly core stable and strategy-proof if and only if* $\varphi = \varphi^{SCC}$*;*
- *a mechanism* $\varphi$ *is core stable and group strategy-proof if and only if* $\varphi = \varphi^{SCC}$*;*
- *a mechanism* $\varphi$ *is strictly core stable and group strategy-proof if and only if* $\varphi = \varphi^{SCC}$.

**Remark 2.** It follows immediately from Theorem 3 that in the class of mechanisms that only require each agent to state his set of friends (instead of the complete underlying friend-oriented preferences), there is a unique mechanism (namely $\varphi^{SCC}$) that is *core stable* and *strategy-proof.* $\diamond$

Let $\widetilde{\mathcal{R}}^f$ be a rich subdomain of friend-oriented problems. Then, by Theorem 1 and Proposition 3, the SCC mechanism is *(strictly) core stable* and *(group) strategy-proof.* To complete the proof of Theorem 3, we will prove Proposition 4 which states that the SCC mechanism on $\widetilde{\mathcal{R}}^f$ is the unique mechanism satisfying *core stability* and *strategy-proofness.*

**Proposition 4.** *On each rich subdomain of friend-oriented problems $\widetilde{\mathcal{R}}^f$, if $\varphi$ is core stable and strategy-proof, then $\varphi = \varphi^{SCC}$.*

## Proof of Proposition 4

Let $\widetilde{\mathcal{R}}^f$ be a rich subdomain of friend-oriented problems. Let $\varphi$ be a *core stable* and *strategy-proof* mechanism on $\widetilde{\mathcal{R}}^f$. We will show that for each $\succeq \in \widetilde{\mathcal{R}}^f$, $\varphi(\succeq) = \varphi^{SCC}(\succeq)$. The proof uses two lemmas: the friend-reduction lemma (Lemma 1) and the SCC-minimality lemma (Lemma 2), both of which are discussed below.

In order to state the friend-reduction lemma (Lemma 1) we first observe that if for some $\widetilde{\succeq} \in \widetilde{\mathcal{R}}^f$, $\varphi(\widetilde{\succeq}) \neq \varphi^{SCC}(\widetilde{\succeq})$, then, by Corollary 1, for each agent $i$ who is in different coalitions at $\varphi^{SCC}$ and $\varphi$, i.e., $\varphi_i(\widetilde{\succeq}) \neq \varphi_i^{SCC}(\widetilde{\succeq})$, we have that $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq})$. The friend-reduction lemma states that if such an agent $i$ at $\widetilde{\succeq}$ reduces his set of friends by one friend such that SCC coalitions do not change, then agent $i$ is still in different coalitions at $\varphi^{SCC}$ and $\varphi$; more precisely, after the change, agent $i$'s coalition at $\varphi$ is still a proper subset of his coalition at $\varphi^{SCC}$.

**Lemma 1 (Friend-reduction lemma).** *Let $\widetilde{\succeq} \in \widetilde{\mathcal{R}}^f$ and $i \in N$ such that $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq})$. Let $\succeq'_i \in \widetilde{\mathcal{R}}^f_i$ and $\succeq' \equiv (\succeq'_i, \widetilde{\succeq}_{-i})$ such that*

- *$\succeq'_i$ are preferences with one less friend than $\widetilde{\succeq}_i$, i.e., for some $f \in F(\widetilde{\succeq}_i)$, $F(\succeq'_i) = F(\widetilde{\succeq}_i) \backslash \{f\}$; and*
- *the SCC coalition of agent $i$ does not change, i.e., $\varphi_i^{SCC}(\succeq') = \varphi_i^{SCC}(\widetilde{\succeq})$ [note that then none of the other SCC coalitions changes either].*

*Then, $\varphi_i(\succeq') \subsetneq \varphi_i^{SCC}(\succeq')$.*

***Proof.*** Let $\widetilde{\succeq} \in \widetilde{\mathcal{R}}^f$ and $i \in N$ such that $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq})$. Let agent $i$ change his preferences at $\widetilde{\succeq}$ such that they have one less friend and his SCC coalition does not change, i.e., let $\succeq'_i \in \widetilde{\mathcal{R}}^f_i$ and $\succeq' \equiv (\succeq'_i, \widetilde{\succeq}_{-i})$ such that

- for some $f \in F(\widetilde{\succeq}_i)$, $F(\succeq'_i) = F(\widetilde{\succeq}_i) \backslash \{f\}$;[9] and

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[9] Here we use the richness of $\widetilde{\mathcal{R}}^f_i$.

- $\varphi_i^{SCC}(\succeq') = \varphi_i^{SCC}(\widetilde{\succeq})$.

By Corollary 1, we have $\varphi_i(\succeq') \subseteq \varphi_i^{SCC}(\succeq')$.

Suppose, by contradiction, that $\varphi_i(\succeq') = \varphi_i^{SCC}(\succeq')$. Then, together with $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq}) = \varphi_i^{SCC}(\succeq')$, it follows that

$$\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i(\succeq'). \tag{3}$$

Thus,

$$F(\widetilde{\succeq}_i) \cap \varphi_i(\widetilde{\succeq}) \subseteq F(\widetilde{\succeq}_i) \cap \varphi_i(\succeq').$$

However, $F(\widetilde{\succeq}_i) \cap \varphi_i(\widetilde{\succeq}) \subsetneq F(\widetilde{\succeq}_i) \cap \varphi_i(\succeq')$ would mean that by reporting $\succeq'_i$ instead of $\widetilde{\succeq}_i$, at mechanism $\varphi$, agent $i$ could be in a coalition with more friends and, by Condition (F) of friend-oriented preferences, be better off; contradicting *strategy-proofness* of $\varphi$. Hence, agent $i$ is in a coalition with the same set of friends he had at $\widetilde{\succeq}_i$, i.e.,

$$F(\widetilde{\succeq}_i) \cap \varphi_i(\widetilde{\succeq}) = F(\widetilde{\succeq}_i) \cap \varphi_i(\succeq'). \tag{4}$$

Equation (4) and $F(\succeq'_i) = F(\widetilde{\succeq}_i) \setminus \{f\}$ imply that then agent $i$ is also in a coalition with the same set of friends he has at $\succeq'_i$, i.e.,

$$F(\succeq'_i) \cap \varphi_i(\widetilde{\succeq}) = F(\succeq'_i) \cap \varphi_i(\succeq').$$

But then, (3) implies that when moving from $\succeq'_i$ to $\widetilde{\succeq}_i$, agent $i$ loses some enemies, i.e.,

$$E(\succeq'_i) \cap \varphi_i(\widetilde{\succeq}) \subsetneq E(\succeq'_i) \cap \varphi_i(\succeq')$$

and, by Condition (E) of friend-oriented preferences, is better off; contradicting *strategy-proofness* of $\varphi$. $\square$

Next, in order to discuss the SCC-minimality lemma (Lemma 2) we have to introduce a particular type of preference profiles. Specifically, starting from any $\succeq \in \widetilde{\mathcal{R}}^f$ and any SCC coalition $V$ at $\succeq$, we can (step by step) reduce the friend sets of agents in $V$ (as in Lemma 1) until any further reduction would break the SCC coalition $V$ apart. Note that by richness of $\widetilde{\mathcal{R}}^f$, there is a preference profile associated with the (final) reduced friend sets that is still in $\widetilde{\mathcal{R}}^f$. Such a preference profile is called *SCC-minimal* with respect to SCC coalition $V$.

**Definition 5 (SCC-minimal preference profile).** We call a preference profile $\succeq \in \widetilde{\mathcal{R}}^f$ *SCC-minimal* with respect to an SCC coalition $V \in \varphi^{SCC}(\succeq)$ if no agent in $V$ can delete a friend without changing the SCC coalition, i.e., for each $\succeq' \in \widetilde{\mathcal{R}}^f$ and for each $i \in V$, if

- $F(\succeq'_i) \subsetneq F(\succeq_i)$ and

- for each $j \in N\backslash\{i\}$, $F(\succeq'_j) = F(\succeq_j)$,

then $\varphi_i^{SCC}(\succeq') \neq V$. ◇

Let $\succeq \in \widetilde{\mathcal{R}}^f$. To *obtain a SCC-minimal profile* $\succeq' \in \widetilde{\mathcal{R}}^f$ *from* $\succeq$ with respect to an SCC coalition $V \in \varphi^{SCC}(\succeq)$, we consider the friendship graph $\Gamma(\succeq)$. For each $i \in V$, we delete, one at a time, edges $(i, k) \in \Gamma(\succeq)$ until removing any additional edge would break agent $i$'s strongly connected component into multiple components. At the end of this process, the obtained graph is *SCC-minimal* with respect to $V$ and, by the richness of $\widetilde{\mathcal{R}}^f$, an associated *SCC-minimal profile* that has the friend sets associated with the SCC-minimal graph can be selected. Figure 3 illustrates friendship graphs (zooming in on SCC coalition $V$) at the original preference profile $\succeq$ and at a preference profile $\succeq'$ that is SCC-minimal with respect to $V$. It is easy to see that, starting from $\succeq$, depending on the choice of friends that are deleted, one can obtain different graphs $\Gamma(\succeq')$ / profiles $\succeq'$ that are SCC-minimal with respect to $V$.

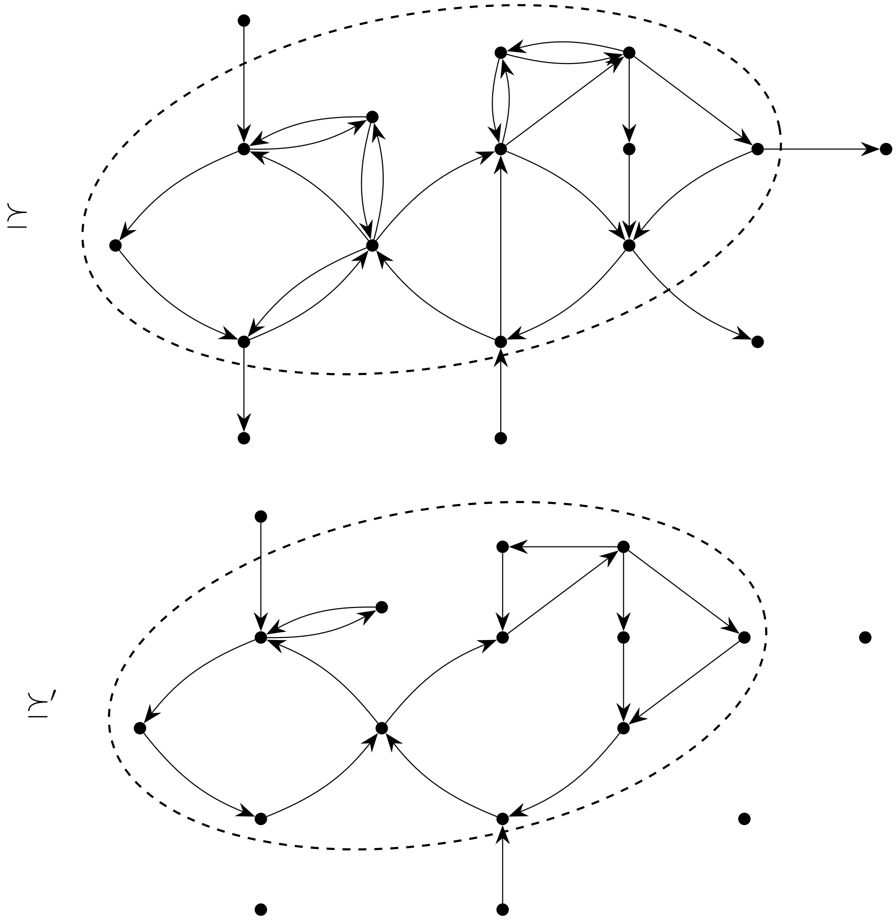

Figure 3: Incoming and outgoing edges of the vertices in the encircled SCC coalition $V$ at the original preference profile $\succeq$ (top) and at an SCC-minimal preference profile $\succeq'$ (bottom) that is obtained from $\succeq$ by trimming the set of edges within $V$ and removing all edges leaving $V$.

The following lemma is key. Its rather long proof is relegated to Appendix F.

**Lemma 2 (SCC-minimality lemma).** *Let* $\succeq \in \widetilde{\mathcal{R}}^f$ *and* $V \in \varphi^{SCC}(\succeq)$. *If* $\succeq$ *is SCC-minimal with respect to* $V$, *then for each* $i \in V$, $\varphi_i(\succeq) = \varphi_i^{SCC}(\succeq)$.

Equipped with the friend-reduction lemma (Lemma 1) and the SCC-minimality lemma (Lemma 2), the proof of Proposition 4 is straightforward. Assume, by contradiction, that for some $\widetilde{\succeq} \in \widetilde{\mathcal{R}}^f$, $\varphi(\widetilde{\succeq}) \neq \varphi^{SCC}(\widetilde{\succeq})$. Recall that by Corollary 1 for each $i \in N$ with $\varphi_i(\widetilde{\succeq}) \neq \varphi_i^{SCC}(\widetilde{\succeq})$, $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq})$. Now, let $V \in \varphi^{SCC}(\widetilde{\succeq})$ be an SCC coalition for which there exists an agent $i \in V$ with $\varphi_i(\widetilde{\succeq}) \subsetneq \varphi_i^{SCC}(\widetilde{\succeq}) = V$. Then, in fact, for each $j \in V$, $\varphi_j(\widetilde{\succeq}) \subsetneq \varphi_j^{SCC}(\widetilde{\succeq}) = V$. Starting from profile $\widetilde{\succeq}$, let agents in $V$ reduce their friend sets, step by step, as long as it does not break up their SCC coalition $V$. By the friend-reduction lemma (Lemma 1), at each reduction step (i.e., at each adjusted preference profile), the coalitions that $\varphi$ assigns to members of $V$ still constitute a proper refinement of the SCC coalition $V$ at $\varphi^{SCC}$. Thus, when the friend-reduction process stops at a preference profile $\widetilde{\succeq}'$ that is SCC-minimal with respect to $V$, for each $j \in V$, $\varphi_j(\widetilde{\succeq}') \subsetneq \varphi_j^{SCC}(\widetilde{\succeq}') = V$; contradicting Lemma 2. This contradiction completes the proof of Proposition 4. □

## Independence of the properties in Theorem 3 and Corollary 4

The following two examples show the logical independence of the two properties in Theorem 3 and Corollary 4. Let $\widetilde{\mathcal{R}}^f$ be a rich subdomain of friend-oriented problems.

The mechanism in our first independence example is *strictly core stable* but not *strategy-proof*.

**Example 4 (A mechanism that is *strictly core stable* but not *strategy-proof*).** Let $N = \{1,2,3\}$. Since $\widetilde{\mathcal{R}}^f$ is rich, there exists $\succeq \in \widetilde{\mathcal{R}}^f$ such that $F_1 = \{2\}$, $F_2 = \{1,3\}$, and $F_3 = \{1\}$ (see Example 2). One easily verifies that $\varphi^{SCC}(\succeq) = \{(123)\}$ and $\{(12),(3)\} \in SC(\succeq)$.

Let mechanism $\varphi^{\neg SP}$ assign $\{(12),(3)\}$ to problem $\succeq$ and the SCC partition to any other problem. Obviously, mechanism $\varphi^{\neg SP}$ is *strictly core stable*. To see that $\varphi^{\neg SP}$ is not *strategy-proof*, let agent 2 report preferences $\succ_2'$ with $F(\succ_2') = \{3\}$ and consider $\succeq' \equiv (\succeq_2', \succeq_{-2})$. Then, $\varphi_2^{\neg SP}(\succeq') = \varphi_2^{SCC}(\succeq') = (123) \succ_2 (12) = \varphi_2^{\neg SP}(\succeq)$. Therefore, $\varphi^{\neg SP}$ is not *strategy-proof*. ◇

The mechanism in our next example is *group strategy-proof* but not *core stable*. Moreover, the example shows that in the statements of Theorem 3 and Corollary 4, *(strict) core stability* cannot be replaced by [*individual rationality* and *Pareto-optimality*].

**Example 5 (A mechanism that is *individually rational*, *Pareto-optimal*, and *group strategy-proof* but not *core stable*).** For each $\succeq \in \widetilde{\mathcal{R}}^f$, let mechanism $\varphi^{\neg C}$ assign the partition that is obtained in three steps:

**Step 1.** if an agent has no friends, then he is left alone;

**Step 2.** each of the remaining agents that no longer has (available) friends is also left alone [repeat Step 2 until each remaining agent has some friend that is still present]; and

**Step 3.** each of the remaining agents is recursively gathered with his friends that are still present [the order of this recursive procedure is inconsequential].

The following example illustrates mechanism $\varphi^{\neg C}$. Let $\succeq \in \widetilde{\mathcal{R}}^f$ such that $N = \{1, 2, 3, 4, 5\}$, $F_1 = \{2\}$, $F_2 = \{3\}$, $F_3 = \{2, 4\}$, $F_4 = \{5\}$, and $F_5 = \emptyset$. The algorithm to compute $\varphi^{\neg C}(\succeq)$ proceeds as follows.

**Step 1.** Since agent 5 has no friends, he forms a singleton coalition;

**Step 2.** now agent 4 no longer has (available) friends and he also forms a singleton coalition; and

**Step 3.** the remaining agents 1, 2, and 3 still have available friends, and recursively gathering the friends of these three agents yields the unique coalition (123) (see right hand side of Figure 4).

Thus, $\varphi^{\neg C}$ assigns the partition $\{(123), (4), (5)\}$, which differs from the partition assigned by $\varphi^{SCC}$ (see left hand side of Figure 4).

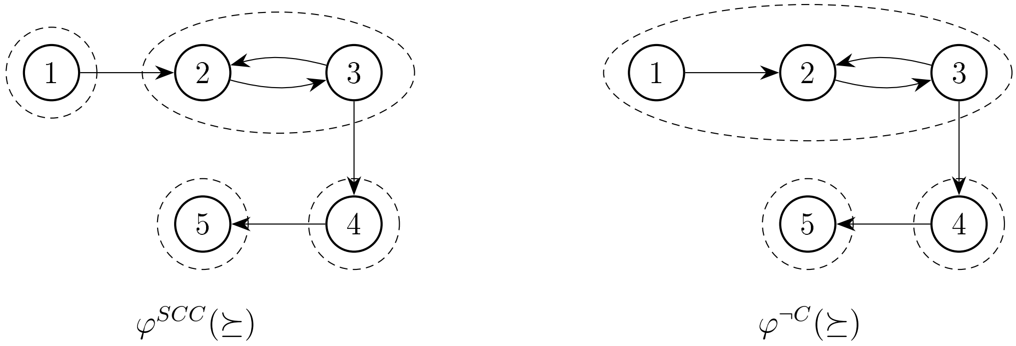

Figure 4: $\varphi^{SCC}$ and $\varphi^{\neg C}$ yield different partitions at $\succeq$, namely $\varphi^{SCC}(\succeq) = \{(1), (23), (4), (5)\}$ and $\varphi^{\neg C}(\succeq) = \{(123), (4), (5)\}$.

In order to discuss and prove the properties that mechanism $\varphi^{\neg C}$ satisfies, it is convenient to state the following three facts.

**Fact A.** Steps 1 and 2 to compute $\varphi^{\neg C}(\succeq)$ only assign singleton coalitions.

**Fact B.** Step 3 to compute $\varphi^{\neg C}(\succeq)$ assigns each agent to a non-singleton coalition that contains the non-empty set of still present friends.

**Fact C.** Let $T$ be a coalition assigned at Step 3 to compute $\varphi^{\neg C}(\succeq)$. Then, for each $T' \subsetneq T$ with $T' \neq \emptyset$, there is some agent $\ell' \in T'$ and some agent in $\ell \in T \setminus T'$ such that $\ell$ is a friend of $\ell'$ or $\ell'$ is a friend of $\ell$, i.e., $\ell \in F_{\ell'}$ or $\ell' \in F_{\ell}$.

Mechanism $\varphi^{\neg C}$ is *individually rational* because each agent's coalition is either a singleton (Fact A) or contains at least one friend (Fact B). We prove *Pareto-optimality* and *group strategy-proofness* of mechanism $\varphi^{\neg C}$ in Appendix G.

Finally, we show that $\varphi^{\neg C}$ is not *core stable*. Let $N = \{1, 2, 3\}$. Since $\widetilde{\mathcal{R}}^f$ is rich, there exists $\succeq \in \widetilde{\mathcal{R}}^f$ such that $F_1 = \{2\}$, $F_2 = \{3\}$, and $F_3 = \{2\}$. One easily verifies that $\varphi^{\neg C}(\succeq) = \{(123)\}$. Since coalition (23) blocks $\{(123)\}$, $\varphi^{\neg C}$ is not *core stable*. $\diamond$

## 4.2 Friend-oriented preferences with neutrals and (weak) group strategy-proofness

Remark 1 discussed the impact of model variations on the existence of core stable partitions. Specifically, we have seen that removing the lexicographic aspect of friend-oriented preferences may lead to an empty core (Appendix C). Similarly, we have seen that switching orientation, i.e., focusing on enemy-oriented preferences may lead to an empty core as well (Appendix D). The only model variation (generalization) that guarantees a non-empty core is that of adding neutrals (Subsection 3.2).[10] In particular, Theorem 2 implies that the straightforward extension of the SCC mechanism $\varphi^{SCC}$ to the domain of friend-oriented preferences with neutrals is *core stable*.

For each $i \in N$, let $\widetilde{\mathcal{R}}_i^{fn} \subseteq \mathcal{R}_i^{fn}$ be a *generic subdomain of friend-oriented preferences with neutrals*. Let $\widetilde{\mathcal{R}}^{fn} \equiv \prod_{i \in N} \widetilde{\mathcal{R}}_i^{fn}$ be the corresponding subdomain of friend-oriented problems with neutrals. From now on, we assume that mechanisms are defined on subdomains of friend-oriented problems with neutrals.

**Corollary 5.** *The SCC mechanism on $\widetilde{\mathcal{R}}^{fn}$ is core stable.*

Proposition 3 shows that the SCC mechanism is *group strategy-proof* on each subdomain of friend-oriented problems (without neutrals). Then, a natural question is whether the SCC mechanism is *group strategy-proof* on each subdomain of friend-oriented problems with neutrals. We show that this is not the case for "very rich" subdomains of friend-oriented problems with neutrals. In fact, we prove a stronger impossibility result (Theorem 4).

We call an agent's friend-oriented preference domain *very rich* if for each pair of disjoint sets of other agents, there is a preference relation that declares these two sets as friends and enemies (so that the (possibly empty) set of remaining agents are neutrals).

**Definition 6 (Very rich preference domains).** Let $i \in N$. A subdomain of friend-oriented preferences with neutrals $\widetilde{\mathcal{R}}_i^{fn} \subseteq \mathcal{R}_i^{fn}$ is *very rich* if for each pair of sets $S, T \subseteq N \backslash \{i\}$ with $S \cap T = \emptyset$, there are preferences $\succeq_i \in \widetilde{\mathcal{R}}_i^{fn}$ such that $F(\succeq_i) = S$ and $E(\succeq_i) = T$; thus, $N(\succeq_i) = N \backslash (S \cup T \cup \{i\})$. We say that the subdomain of friend-oriented problems with neutrals $\widetilde{\mathcal{R}}^{fn} \equiv \prod_{i \in N} \widetilde{\mathcal{R}}_i^{fn}$ is *very rich* if for each $i \in N$, $\widetilde{\mathcal{R}}_i^{fn}$ is very rich. $\diamond$

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[10] Example 3 shows that in the presence of neutrals, the strict core can be empty.

**Theorem 4.** *On each very rich subdomain of friend-oriented problems with neutrals* $\widetilde{\mathcal{R}}^{fn}$, *no mechanism is core stable and group strategy-proof.*

Theorem 4 also shows that the characterization of the SCC mechanism (in the absence of neutrals) by *core stability* and *group strategy-proofness* (Corollary 4) cannot be generalized to friend-oriented problems with neutrals.

***Proof.*** We prove the theorem for $N = \{1, 2, 3\}$.[11] Let $\widetilde{\mathcal{R}}^{fn}$ be a very rich subdomain of friend-oriented problems with neutrals. Suppose that there is a *core stable* and *group strategy-proof* mechanism $\varphi$ on $\widetilde{\mathcal{R}}^{fn}$.

By the very richness of $\widetilde{\mathcal{R}}^{fn}$, consider $\succeq \in \widetilde{\mathcal{R}}^{fn}$ such that $F_1 = \{2\}$, $E_1 = \{3\}$, $N_1 = \emptyset$, $F_2 = E_2 = \emptyset$, $N_2 = \{1, 3\}$, $F_3 = \{2\}$, $E_3 = \{1\}$, and $N_3 = \emptyset$ (as in Example 3). In Example 3 we have seen that $C(\succeq) = \{\{(1), (2), (3)\}, \{(1), (23)\}, \{(12), (3)\}, \{(123)\}\}$. We show that for each possible candidate partition $\varphi(\succeq) \in C(\succeq)$, there is a successful group manipulation by some coalition.

Suppose that $\varphi(\succeq) = \{(1), (2), (3)\}$, $\varphi(\succeq) = \{(1), (23)\}$, or $\varphi(\succeq) = \{(123)\}$. Then, by the very richness of $\widetilde{\mathcal{R}}^{fn}_2$, agent 2 can report preferences $\succeq'_2$ where agent 1 is his (unique) friend and agent 3 is his (unique) enemy so that for $\succeq' \equiv (\succeq_1, \succeq'_2, \succeq_3)$ the unique core partition is $\{(12), (3)\}$. Then, $\varphi(\succeq') = \{(12), (3)\}$ and $(\succeq_1, \succeq'_2)$ is a successful group manipulation by coalition (12).

Suppose that $\varphi(\succeq) = \{(12), (3)\}$. Then, by the very richness of $\widetilde{\mathcal{R}}^{fn}_2$, agent 2 can report preferences $\succeq''_2$ where agent 3 is his (unique) friend and agent 1 is his (unique) enemy so that for $\succeq'' \equiv (\succeq_1, \succeq''_2, \succeq_3)$ the unique core partition is $\{(1), (23)\}$. Then, $\varphi(\succeq'') = \{(1), (23)\}$ and $(\succeq''_2, \succeq_3)$ is a successful group manipulation by coalition (23).

We conclude that there is no *core stable* and *group strategy-proof* mechanism on any very rich subdomain of friend-oriented problems with neutrals. □

The assumption of preferences being very rich in Theorem 4 is rather strong because the proof only requires the existence of a problem where some agent has at least two neutrals (Example 3). In Appendix H we provide two stronger impossibility results and two examples that illustrate why it is difficult to clearly separate subdomains of friend-oriented problems with neutrals into those where *core stability* and *group strategy-proofness* are compatible and those where they are incompatible.

While Theorem 4 implies that the SCC mechanism is not *group strategy-proof* for very rich subdomains of friend-oriented problems with neutrals, we show that the SCC mechanism still satisfies the following weaker notion of *group strategy-proofness*.

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[11] To extend the proof to more agents, one can add agents with preferences where all other agents are enemies.

**Definition 7** (**Weak group strategy-proofness**)**.** A mechanism is *weakly group strategy-proof* if there exists no problem where some coalition of agents can misreport their preferences so that all its members get strictly better off. Formally, a mechanism $\varphi$ is *weakly group strategy-proof* if for each problem $\succsim \in \widetilde{\mathcal{R}}$, there do not exist $S \subseteq N$ and $\succsim'_S \in \prod_{i\in S} \widetilde{\mathcal{R}}_i$ such that for each $i \in S$,

$$\varphi_i(\succsim'_S, \succsim_{-S}) \succ_i \varphi_i(\succsim). \qquad \diamond$$

**Proposition 5.** *The SCC mechanism on $\widetilde{\mathcal{R}}^{fn}$ is weakly group strategy-proof.*

The proof of Proposition 5 is similar to that of Proposition 3; it is relegated to Appendix I.

Corollary 5 and Proposition 5 show that the SCC mechanism on $\widetilde{\mathcal{R}}^{fn}$ is *core stable* and *weakly group strategy-proof.*

> Can we characterize the SCC mechanism on $\widetilde{\mathcal{R}}^{fn}$ with these two properties (as in Theorem 3)?

The answer is in the negative: Example 6 for the domain of friend-oriented problems with neutrals $\mathcal{R}^{fn}$ shows that apart from the SCC mechanism there are other *core stable* and *weakly group strategy-proof* mechanisms.

**Example 6** (**Another mechanism that is *core stable* and *weakly group strategy-proof* when preferences are friend-oriented with neutrals**)**.** Let $\psi$ be the mechanism on $\mathcal{R}^{fn}$ obtained from $\varphi^{SCC}$ such that at each problem, if agent 1 does not have any enemies, then all his neutrals are turned into friends and mechanism $\varphi^{SCC}$ is applied to the adjusted sets of friends; otherwise $\psi$ yields the SCC partition directly. Formally, let $\bar{\succsim}_1 \in \mathcal{R}^{fn}_1$ such that $F(\bar{\succsim}_1) = N\backslash\{1\}$.[12] Let $\succsim \in \mathcal{R}^{fn}$. Then, $\psi(\succsim)$ is defined as follows. If $F(\succsim_1) \cup N(\succsim_1) = N\backslash\{1\}$, then $\psi(\succsim) \equiv \varphi^{SCC}(\bar{\succsim}_1, \succsim_{-1})$; otherwise, $\psi(\succsim) \equiv \varphi^{SCC}(\succsim)$. In Appendix J we show that mechanism $\psi$ satisfies *core stability* and *weak group strategy-proofness.* $\diamond$

# Appendices

## A Appendix: Subdomains of friend-oriented preferences

In this appendix, we discuss a subdomain of friend-oriented preferences that has been introduced and studied in the literature. We also introduce another economically interesting subdomain of friend-oriented preferences. We illustrate the subdomains through a simple example.

A smaller domain of friend-oriented preferences using the terminology of "friends" and "enemies" has been introduced by Dimitrov et al. (2006). Their preference domain is based on the

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[12] Note that the particular choice of $\bar{\succsim}_1$ is irrelevant because the only relevant input for $\varphi^{SCC}$ are the sets of friends.

number of friends versus the number of enemies in a coalition: agent $i$'s preferences $\succeq_i$ satisfy *appreciation of friends* if agent $i$, when comparing two coalitions, prefers the one with more friends. If two coalitions have the same number of friends, then agent $i$ prefers the one with fewer enemies. If the number of friends and the number of enemies in each of the two coalitions are the same, then agent $i$ is indifferent between the two coalitions. Let $\mathcal{R}_i^{af}$ denote the set of preferences over $\mathcal{C}_i$ that satisfy *appreciation of friends.* Formally, $\succeq_i \in \mathcal{R}_i^{af}$ if for all $S, T \in \mathcal{C}_i$,

- if $|S \cap F_i| > |T \cap F_i|$, then $S \succ_i T$;
- if $|S \cap F_i| = |T \cap F_i|$ and $|S \cap E_i| < |T \cap E_i|$, then $S \succ_i T$; and
- if $|S \cap F_i| = |T \cap F_i|$ and $|S \cap E_i| = |T \cap E_i|$, then $S \sim_i T$.

It is easy to see that if an agent's preferences $\succeq_i$ satisfy appreciation of friends, then they are friend-oriented, i.e., $\mathcal{R}_i^{af} \subsetneq \mathcal{R}_i^f$.[13]

The above subdomain of friend-oriented preferences allows for indifferences between coalitions. Our next subdomain of friend-oriented preferences extends a strict ranking over all agents lexicographically to a strict ranking over coalitions: agent $i$'s preferences $\succeq_i$ are *lexicographically friend-oriented* based on agent $i$'s strict ranking of all agents (including himself) if the following holds. Comparing two coalitions, agent $i$ first considers the highest ranked agent in each coalition and strictly prefers the coalition where the highest ranked agent is ranked higher (e.g., because the coalition contains a higher ranked best friend). If the highest ranked agent in both coalitions is the same, agent $i$ next considers the second highest ranked agent in each coalition and strictly prefers the coalition where the second highest ranked agent is ranked higher, etc.

Formally, let $i \in N$ and $\rhd_i$ be a strict ranking of all agents in $N$. To lexicographically compare sets based on ranking $\rhd_i$, even if they are of different cardinality, we introduce the following notion. Let $S \in \mathcal{C}_i$ such that $S = \{s_1^f, \dots, s_k^f, i, s_1^e, \dots, s_l^e\}$, $\{s_1^f, \dots, s_k^f\} \subseteq F(\succeq_i)$, $\{s_1^e, \dots, s_l^e\} \subseteq E(\succeq_i)$, $|S| = k + l + 1$, and $s_1^f \rhd_i \cdots \rhd_i s_k^f \rhd_i i \rhd_i s_1^e \rhd_i \cdots \rhd_i s_l^e$. Then, set $S$'s *(lexicographically) ordered representation* based on $\rhd_i$ equals

$$\ell(\rhd_i, S) \equiv (\overbrace{s_1^f, \dots, s_k^f}^{\text{friends}}, \overbrace{i, \dots\dots, i}^{n-k-l \text{ copies of } i}, \overbrace{s_1^e, \dots, s_l^e}^{\text{enemies}}).$$

Let $\ell_r(\rhd_i, S)$ denote the $r$-th coordinate of vector $\ell(\rhd_i, S)$.

Preferences $\succeq_i \in \mathcal{R}_i$ are *lexicographically friend-oriented* based on $\rhd_i$ if for all $S, T \in \mathcal{C}_i$,

- if $\ell_1(\rhd_i, S) \rhd_i \ell_1(\rhd_i, T)$, then $S \succ_i T$;
- if $\ell_1(\rhd_i, S) = \ell_1(\rhd_i, T)$ and $\ell_2(\rhd_i, S) \rhd_i \ell_2(\rhd_i, T)$, then $S \succ_i T$;

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[13] Hedonic coalition formation models related to the appreciation of friends preference domain have been the topic of various theoretical computer science papers, e.g., Nguyen et al. (2016), Ota et al. (2017), Rothe et al. (2018), Kerkmann et al. (2020), Flammini et al. (2022), and Chen et al. (2023).

- ...
- if $\ell_1(\rhd_i, S) = \ell_1(\rhd_i, T)$, $\ell_2(\rhd_i, S) = \ell_2(\rhd_i, T)$, ..., $\ell_{n-1}(\rhd_i, S) = \ell_{n-1}(\rhd_i, T)$, and $\ell_n(\rhd_i, S) \rhd_i \ell_n(\rhd_i, T)$, then $S \succ_i T$.

Note that for all $S, T \in \mathcal{C}_i$, we have $S \succ_i T$, $T \succ_i S$, or $S = T$.

Let $i \in N$ and $\succeq_i \in \mathcal{R}_i$. Then, agent $i$'s preferences are *lexicographically friend-oriented* if there exists a strict ranking $\rhd_i$ such that $\succeq_i$ are *lexicographically friend-oriented* based on $\rhd_i$. Note that each ranking of agents $\rhd_i$ produces a unique strict lexicographically friend-oriented preference relation over $\mathcal{C}_i$.

Let $\mathcal{R}_i^{lf}$ denote the set of preferences over $\mathcal{C}_i$ that are *lexicographically friend-oriented.* It is easy to see that if an agent's preferences $\succeq_i$ are lexicographically friend-oriented, then they are friend-oriented, i.e., $\mathcal{R}_i^{lf} \subsetneq \mathcal{R}_i^{f}$. Clearly, the subdomains of lexicographically friend-oriented and appreciation of friends preferences are disjoint, i.e.,

$$\mathcal{R}_i^{lf} \cap \mathcal{R}_i^{af} = \emptyset.$$

The next example illustrates the possible friend-oriented preferences for an agent who has two friends and one enemy.

**Example 7 (Examples of friend-oriented preferences).**
Let $N = \{1, 2, 3, 4\}$ and $\succeq \in \mathcal{R}^f$ such that $F_1 = \{2, 3\}$ and $E_1 = \{4\}$. Agent 1's 13 possible friend-oriented preference relations are depicted below.

| $\succeq_1^{af}$ | $\succeq_1^{w1}$ | $\succeq_1^{w2}$ | $\succeq_1^{w3}$ | $\succeq_1^{w4}$ | $\succeq_1^{w5}$ | $\succeq_1^{w6}$ | $\succeq_1^{lf1}$ | $\succeq_1^{lf2}$ | $\succeq_1^{s1}$ | $\succeq_1^{s2}$ | $\succeq_1^{s3}$ | $\succeq_1^{s4}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 | 123 |
| 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 | 1234 |
| $12 \sim 13$ | $12 \sim 13$ | $12 \sim 13$ | 12 | 13 | 12 | 13 | 12 | 13 | 12 | 12 | 13 | 13 |
| $124 \sim 134$ | 124 | 134 | 13 | 12 | $13 \sim 124$ | $12 \sim 134$ | 124 | 134 | 13 | 13 | 12 | 12 |
| 1 | 134 | 124 | $124 \sim 134$ | $124 \sim 134$ | 134 | 124 | 13 | 12 | 124 | 134 | 124 | 134 |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 134 | 124 | 134 | 124 | 134 | 124 |
|  | 14 | 14 | 14 | 14 | 14 | 14 | 1 | 1 | 1 | 1 | 1 | 1 |
|  |  |  |  |  |  |  | 14 | 14 | 14 | 14 | 14 | 14 |

It is easy to verify that all 13 preference relations satisfy friend-orientation. In order to see that there are no other friend-oriented preference relations, note that friend-orientation requires that

- the coalition with all friends, i.e., (123), is the unique most preferred coalition;
- the coalition with his unique enemy, i.e., (14), is the unique least preferred coalition;
- coalition (1234) is ranked below (123) by Condition (E) and is ranked above any other coalition by Condition (F);
- coalition (1) is ranked above (14) by Condition (E) and is ranked below any other coalition by Condition (F); and

- coalition (12) is ranked above (124) by Condition (E), and similarly, coalition (13) is ranked above (134).

The only preference relations that satisfy these five requirements are the 13 preference relations depicted above.

The first 7 of the 13 preference relations are not strict (i.e., they contain at least one indifference between two coalitions). The last 6 of the 13 preference relations are strict (i.e., they do not contain any indifferences).

One easily verifies that $\succeq_1^{af}$ is the unique preference relation that satisfies appreciation of friends and $\succeq_1^{lf1}$ and $\succeq_1^{lf2}$ are the only two preference relations that are lexicographically friend-oriented. ◇

# B Appendix: Relation to top-responsiveness

Alcalde and Revilla (2004) and Dimitrov and Sung (2007) obtain results similar to ours for the class of problems where agents' preferences satisfy top-responsiveness (Alcalde and Revilla, 2004). Preferences $\succeq_i$ satisfy *top-responsiveness* if

**a.** for each set $S \in \mathcal{C}_i$, there is a unique subset of $S$ that is most preferred according to $\succeq_i$, which we denote by $Ch_i(S)$ and

**b.** for any two sets $S, S' \in \mathcal{C}_i$, the following two conditions are fulfilled:

1. if $Ch_i(S) \succ_i Ch_i(S')$, then $S \succ_i S'$; and
2. if $Ch_i(S) = Ch_i(S')$ and $S \subsetneq S'$, then $S \succ_i S'$.

It is easy to verify that the appreciation of friends preference domain and the domain of lexicographically friend-oriented preferences (Appendix A) are both subdomains of the domain of top-responsive preferences. The Venn-diagram in Figure 5 also shows that top-responsiveness and friend-orientedness are logically independent. To see this note that in Example 7, the four friend-oriented preferences $\succeq_1^{w3}$, $\succeq_1^{w4}$, $\succeq_1^{s2}$, and $\succeq_1^{s3}$ are not top-responsive (because condition b.1 is violated). Conversely, it is easy to verify that when $N = \{1, 2, 3\}$, the preferences $\succ_3^*$ given by (13) $\succ_3^*$ (123) $\succ_3^*$ (23) $\succ_3^*$ (3) are top-responsive but not friend-oriented (Alcalde and Revilla, 2004, Example 5.3).

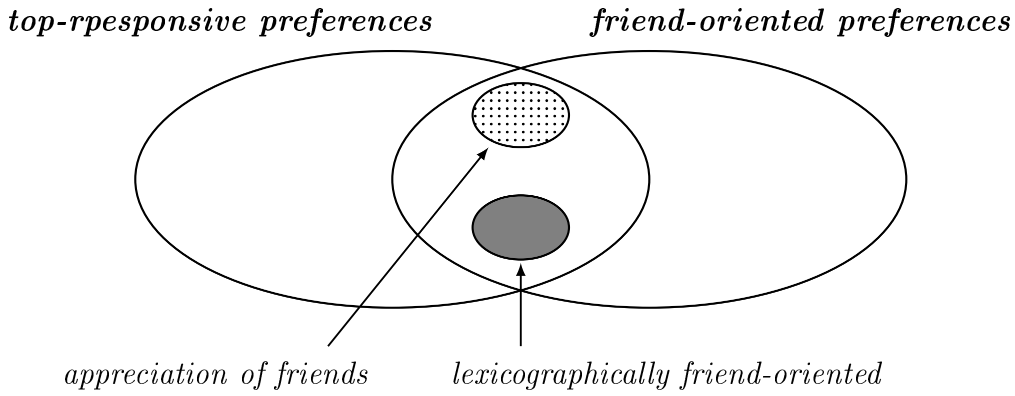

Figure 5: The relationship between top-responsive and friend-oriented preferences.

Alcalde and Revilla (2004, Definition 4.2) introduce a mechanism for the domain of top-responsive preferences, called the *top covering mechanism*.

**Theorem 5** (**Alcalde and Revilla, 2004, Theorem 5.4**). *On the full domain of top-responsive preferences, the top covering mechanism is the unique mechanism that is core stable and strategy-proof.*

Theorem 5 parallels our Theorem 3 for the SCC mechanism on each rich subdomain of friend-oriented preferences, i.e., including the full domain of friend-oriented preferences.

**Theorem 6** (**Dimitrov and Sung, 2007, Theorem 1**). *On the full domain of top-responsive preferences, the top covering mechanism is strictly core stable.*

Theorem 6 parallels our Corollary 3 for the SCC mechanism on the full domain of friend-oriented preferences.

# C Appendix: Removing the lexicographic aspect and core stability (Remark 1)

The following example illustrates that the core can be empty if we remove the lexicographic aspect incorporated in the definition of friend-oriented preferences; i.e., we only require that adding friends and removing enemies is good. Thus, we impose Condition **(E)** and weaken Condition **(F)** to only require

($\bar{\mathbf{F}}$) for each $S \in \mathcal{C}_i$ and each $f \in F(\succeq_i) \setminus S$, $S \cup \{f\} \succ_i S$.

Thus, (E) adding any enemy to any coalition yields a less preferred coalition and ($\bar{\text{F}}$) adding any friend to any coalition yields a more preferred coalition. In particular, additive preferences constitute a subdomain of preferences that satisfy (E) and ($\bar{\mathrm{F}}$). Formally, let $i \in N$. Agent $i$'s preferences $\succeq_i$ are *additive* if there exists a utility function $u_i : N \to \mathbb{R}\backslash\{0\}$ such that

$$\text{for all } S, T \in \mathcal{C}_i, \; \left[ S \succeq_i T \text{ if and only if } \sum_{j : j \in S} u_i(j) \geq \sum_{j : j \in T} u_i(j) \right]. \tag{5}$$

**Example 8** (**Removing the lexicographic aspect from friend-oriented preferences can lead to an empty core**)**.** Let $N = \{1, 2, 3, 4, 5\}$ with friend sets $F_1 = \{2, 5\}$, $F_2 = \{1, 3\}$, $F_3 = \{2, 4\}$, $F_4 = \{3, 5\}$, and $F_5 = \{1, 4\}$. Each agent has a best and a second-best friend such that the associated friendship graph $\Gamma(\succeq)$ (see Figure 6) has a particular circular structure: the edges induced by the best friends constitute a cycle, and the edges induced by the second-best friends yield the same cycle in the opposite direction. Furthermore, each agent has a strong aversion to his enemies and even in the presence of his friends, would like to be alone rather than in a coalition with an enemy. This aspect of agents' preferences violates the assumption that friends are lexicographically more important than enemies that was present in our friend-oriented preference assumptions (E) and (F).

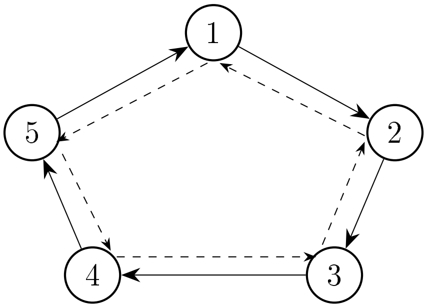

Figure 6: Friendship graph in Example 8. The continuous edges point to the best friends, while the discontinuous edges point to the second-best friends.

The agents' preferences $\succeq$ are listed in the following table (agents' enemies are underlined).

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ | $\succeq_4$ | $\succeq_5$ |
|---|---|---|---|---|
| 125 | 123 | 234 | 345 | 145 |
| 12 | 23 | 34 | 45 | 15 |
| 15 | 12 | 23 | 34 | 45 |
| 1 | 2 | 3 | 4 | 5 |
| $12\underline{3}5$ | $123\underline{4}$ | $234\underline{5}$ | $\underline{1}345$ | $1\underline{2}45$ |
| $12\underline{4}5$ | $123\underline{5}$ | $\underline{1}234$ | $\underline{2}345$ | $1\underline{3}45$ |
| $12\underline{3}$ | $23\underline{4}$ | $34\underline{5}$ | $\underline{1}45$ | $1\underline{2}5$ |
| $12\underline{4}$ | $23\underline{5}$ | $\underline{1}34$ | $\underline{2}45$ | $1\underline{3}5$ |
| $1\underline{3}5$ | $12\underline{4}$ | $23\underline{5}$ | $\underline{1}34$ | $\underline{2}45$ |
| $1\underline{4}5$ | $12\underline{5}$ | $\underline{1}23$ | $\underline{2}34$ | $\underline{3}45$ |
| $1\underline{3}$ | $2\underline{4}$ | $3\underline{5}$ | $\underline{1}4$ | $\underline{2}5$ |
| $1\underline{4}$ | $2\underline{5}$ | $\underline{1}3$ | $\underline{2}4$ | $\underline{3}5$ |
| $12\underline{34}5$ | $123\underline{45}$ | $\underline{1}234\underline{5}$ | $\underline{12}345$ | $1\underline{23}45$ |
| $12\underline{34}$ | $23\underline{45}$ | $\underline{1}34\underline{5}$ | $\underline{12}45$ | $1\underline{23}5$ |
| $1\underline{34}5$ | $12\underline{45}$ | $\underline{1}23\underline{5}$ | $\underline{12}34$ | $\underline{23}45$ |
| $1\underline{34}$ | $2\underline{45}$ | $\underline{1}3\underline{5}$ | $\underline{12}4$ | $\underline{23}5$ |

One can verify that preferences are additive. For instance, for each $i \in N$, let $u_i(i) = 0.25$, $u_i(i+1) = 2$, $u_i(i+2) = -3.5$, $u_i(i+3) = -4$, and $u_i(i+4) = 1$ (*modulo* 5). Then, for all $S, T \in \mathcal{C}_i$, $S \succeq_i T$ if and only if $\sum_{j:j\in S} u_i(j) \geq \sum_{j:j\in T} u_i(j)$. One can easily check that Conditions (E) and ($\bar{\text{F}}$) are satisfied.

Next, we show that the core is empty. Suppose to the contrary that the core is non-empty. Let $\pi$ be a core partition. By *individual rationality* of $\pi$, each coalition in $\pi$ is "fully connected" in the sense that there is an edge from each agent to each of the other agents in the same coalition. Hence, $\pi$ consists of singletons and / or pairs of neighbors. Since there is an odd number of agents, there is at least one singleton in $\pi$. Since the preference profile is "circular," we can assume, without loss of generality, that agent 1 is single at $\pi$, i.e., $(1) \in \pi$. Then, for agent 5, either $\pi_5 = (5)$ or $\pi_5 = (45)$. Thus, $(15) \succ_5 \pi_5$. Since also $(15) \succ_1 (1) = \pi_1$, coalition (15) blocks $\pi$. Hence, the core is empty. $\diamond$

# D Appendix: Enemy-oriented preferences and core stability (Remark 1)

Recall that friend-oriented preferences are based on the partition of other agents into friends and enemies, the assumption that adding friends and removing enemies is good, and the lexicographic aspect that adding friends is more important than removing enemies. By switching the lexicographic aspect from adding friends being more important to removing enemies being more important, we obtain the following preference restriction.

Agent $i$'s preferences are *enemy-oriented* if the set $N\backslash\{i\}$ can be partitioned into a *set of friends* $F(\succeq_i)$ and a *set of enemies* $E(\succeq_i)$ such that for each coalition $S \in \mathcal{C}_i$, (F') adding a friend, makes agent $i$ strictly better off and (E') adding an enemy, possibly together with a set of friends, makes agent $i$ strictly worse off. Note that Condition (E') now embeds the lexicographic principle that to improve a coalition, removing enemies is strictly more important than adding friends. Formally, let $i \in N$. Preferences $\succeq_i \in \mathcal{R}_i$ are *enemy-oriented* if

**(F')** for each $S \in \mathcal{C}_i$ and each $f \in F(\succeq_i) \setminus S$,

$$S \cup \{f\} \succ_i S;$$

and

**(E')** for each $S \in \mathcal{C}_i$, each $e \in E(\succeq_i) \setminus S$, and each $F \subseteq F(\succeq_i) \setminus S$,

$$S \succ_i S \cup \{e\} \cup F.$$

Let $\mathcal{R}_i^e$ denote the set of preferences over $\mathcal{C}_i$ that are *enemy-oriented.* For each $i \in N$, when no confusion is possible, we write $F_i$ and $E_i$ instead of $F(\succeq_i)$ and $E(\succeq_i)$.

A smaller domain of enemy-oriented preferences has been introduced by Dimitrov et al. (2006). Their preference domain is based on the number of friends versus the number of enemies in a coalition: agent $i$'s preferences $\succeq_i$ satisfy *aversion to enemies* if agent $i$, when comparing two coalitions, prefers the one with fewer enemies. If two coalitions have the same number of enemies, then agent $i$ prefers the one with more friends. If the number of enemies and the number of friends in each of the two coalitions are the same, then agent $i$ is indifferent between the two coalitions. Let $\mathcal{R}_i^{ae}$ denote the set of preferences over $\mathcal{C}_i$ that satisfy *aversion to enemies.* Formally, $\succeq_i \in \mathcal{R}_i^{ae}$ if for all $S, T \in \mathcal{C}_i$,

- if $|S \cap E_i| < |T \cap E_i|$, then $S \succ_i T$;
- if $|S \cap E_i| = |T \cap E_i|$ and $|S \cap F_i| > |T \cap F_i|$, then $S \succ_i T$; and
- if $|S \cap E_i| = |T \cap E_i|$ and $|S \cap F_i| = |T \cap F_i|$, then $S \sim_i T$.

It is easy to see that if an agent's preferences $\succeq_i$ satisfy aversion to enemies, then they are enemy-oriented, i.e., $\mathcal{R}_i^{ae} \subsetneq \mathcal{R}_i^e$.

Dimitrov et al. (2006, Example 4) showed that when preferences satisfy aversion to enemies, then a strictly core stable partition needs not exist.

**Example 9 (Dimitrov et al. (2006, Example 4), the strict core can be empty when preferences satisfy aversion to enemies).**
Let $N = \{1,2,3\}$ and $\succeq \in \mathcal{R}^{ae}$ such that $F_1 = \{2\}$, $F_2 = \{1,3\}$, and $F_3 = \{2\}$; furthermore, agent 2 is indifferent between friend 1 and friend 3, i.e., $(12) \sim_2 (23)$. Note that the sets of friends are exactly as those in Example 1 (where, in contrast, we assume that $\succeq \in \mathcal{R}^f$). The corresponding preferences that satisfy aversion to enemies are

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|
| 12 | 123 | 23 |
| 1 | $12 \sim 23$ | 3 |
| 123 | 2 | 123 |
| 13 | | 13 |

One easily verifies that $SC(\succeq) = \emptyset \subsetneq \{\{(12),(3)\},\{(1),(23)\}\} = C(\succeq)$. ◇

In Example 9, the core is non-empty. Dimitrov et al. (2006) showed how to find a core stable partition for any problem with preferences that satisfy aversion to enemies. We define a *clique* of the friendship graph $\Gamma(\succeq^S) = (S, A^S)$ as a coalition $T \subseteq S$ such that for all $i, j \in T$ with $i \neq j$, $(i,j) \in A^S$.

**Theorem 7 (Dimitrov et al., 2006, Theorem 3).** *Let $\succeq \in \mathcal{R}^{ae}$. Starting with the empty collection of coalitions, recursively adding a clique of maximal cardinality yields a core stable partition.*

Dimitrov et al. (2006, Lemma 6) showed that any core stable partition contains a clique of maximal cardinality in $\Gamma(\succeq)$. Since finding a clique of maximal cardinality is NP-hard, finding a core stable partition is also NP-hard (Dimitrov et al., 2006, Theorem 4).

In view of Theorem 7, a natural question is whether the existence of core stable partitions can be extended from the domain of preferences that satisfy aversion to enemies $\mathcal{R}^{ae}$ to the domain of preferences that are enemy-oriented $\mathcal{R}^e$. The next example answers this question in the negative.

**Example 10 (The core can be empty when preferences are enemy-oriented).**
Consider again the problem exhibited in Example 8. It is easy to check that each agent's preferences satisfy conditions (F') and (E'). Hence, preferences are enemy-oriented and the core is empty (see Example 8). ◇

# E Appendix: Proof of Proposition 3

We prove that the SCC mechanism on $\widetilde{\mathcal{R}}^f$ is *group strategy-proof* (Proposition 3).

***Proof.*** Suppose that $\varphi^{SCC}$ is not *group strategy-proof.* Then, there exists a problem $\succeq \in \widetilde{\mathcal{R}}^f$ and a coalition $S \subseteq N$ with preferences $\succeq'_S \in \widetilde{\mathcal{R}}^f_S$ such that

(a) for each $i \in S$, $\varphi_i^{SCC}(\succeq'_S, \succeq_{-S}) \succeq_i \varphi_i^{SCC}(\succeq)$ and

(b) for some $j \in S$, $\varphi_j^{SCC}(\succeq'_S, \succeq_{-S}) \succ_j \varphi_j^{SCC}(\succeq)$.

For each $i \in N$, let $F_i$ and $E_i$ denote the set of friends and enemies of agent $i$ at $\succeq$. Let $G_1, \ldots, G_K$ be the strongly connected components of graph $\Gamma(\succeq)$. For each $l \in \{1, \ldots, K\}$, the associated strongly connected component equals $G_l = (V_l, A_l)$. Based on the labeling of strongly connected components $G_1, \ldots, G_K$ according to Fact 1, for all $l, l' \in \{1, \ldots, K\}$ with $l < l'$, graph $\Gamma(\succeq)$ contains no edge from any vertex in $V_{l'}$ to any vertex in $V_l$. By definition of $\varphi^{SCC}$, for each $l \in \{1, \ldots, K\}$ and each $i \in V_l$, $\varphi_i^{SCC}(\succeq) = V_l$.

Let $\succeq' \equiv (\succeq'_S, \succeq_{-S})$. We will prove that for each $l \in \{1, \ldots, K\}$ and each $i \in V_l$, $\varphi_i^{SCC}(\succeq') = V_l$, which contradicts (b). First we consider $V_K$.

**Case *K*.1.** Suppose that $S \cap V_K = \emptyset$. Graph $\Gamma(\succeq)$ contains no edge from $V_K$ to $V_1 \cup \cdots \cup V_{K-1}$, see Figure 7.

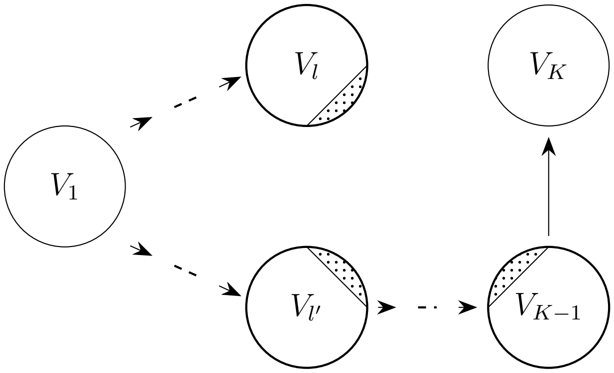

Figure 7: Case $K$.1 ($S \cap V_K = \emptyset$). Set $S$ is the union of dotted areas in sets $V_l$, $V_{l'}$, …, $V_{K-1}$.

Since $S \cap V_K = \emptyset$, $\succeq'_{V_K} = \succeq_{V_K}$. Thus, $V_K$ is an SCC coalition of graph $\Gamma(\succeq')$. Then, for each $i \in V_K$, $\varphi_i^{SCC}(\succeq') = V_K$.

**Case *K*.2.** Suppose that $S \cap V_K \neq \emptyset$. It follows from Fact 2 that each agent $i \in S \cap V_K$ is together with all his friends in coalition $\varphi_i^{SCC}(\succeq)$, i.e.,

$$F_i \subseteq V_K = \varphi_i^{SCC}(\succeq). \tag{6}$$

Then, from (a) and by Condition (F) of friend-oriented preferences, each agent $i \in S \cap V_K$ in coalition $\varphi_i^{SCC}(\succeq')$ is still together with all his friends, i.e.,

$$F_i \subseteq \varphi_i^{SCC}(\succeq'). \tag{7}$$

Next, we prove that for each agent $i \in S \cap V_K$, if agent $i$ in coalition $\varphi_i^{SCC}(\succeq)$ is together with an enemy $e$, then that enemy is also in his coalition $\varphi_i^{SCC}(\succeq')$, i.e.,

$$E_i \cap \varphi_i^{SCC}(\succeq) \subseteq E_i \cap \varphi_i^{SCC}(\succeq'). \tag{8}$$

Suppose, by contradiction, that some agent $i \in S \cap V_K$ in coalition $\varphi_i^{SCC}(\succeq)$ is together with an enemy $e$ who is not in his coalition $\varphi_i^{SCC}(\succeq')$, i.e., $e \in E_i \cap (\varphi_i^{SCC}(\succeq) \setminus \varphi_i^{SCC}(\succeq'))$. Since $\varphi_i^{SCC}(\succeq) = V_K$, $e \in V_K$. By definition of $\varphi^{SCC}(\succeq')$,

$$\text{agents } i \text{ and } e \text{ are in distinct SCC coalitions of } \Gamma(\succeq'). \tag{9}$$

For each $h \in V_K$, let $V'(h)$ denote the SCC coalition of $\Gamma(\succeq')$ that contains agent $h$. By definition of $V'(h)$, $V'(h) \cap V_K \neq \emptyset$. Moreover, from $i, e \in V_K$ and (9) it follows that $|\{V'(h)\}_{h \in V_K}| \geq 2$. Since the condensation graph of $\Gamma(\succeq')$ is acyclic, let $V' \in \{V'(h)\}_{h \in V_K}$ be an SCC coalition without an outgoing edge to any of the other SCC coalitions in $\{V'(h)\}_{h \in V_K} \setminus \{V'\}$.[14] Hence, there is no edge from any vertex in $V'$ to any vertex in $[\bigcup_{h \in V_K} V'(h)] \setminus V'$. In particular, in $\Gamma(\succeq')$, there is no edge from any vertex in $V' \cap V_K$ to any vertex in

$$\left[\bigcup_{h \in V_K} (V'(h) \cap V_K)\right] \setminus (V' \cap V_K) = V_K \setminus (V' \cap V_K).$$

However, since

$$[V' \cap V_K] \cup [V_K \setminus (V' \cap V_K)] = V_K$$

is an SCC coalition of $\Gamma(\succeq)$, there is an edge from some vertex in $V' \cap V_K$ to some vertex in $V_K \setminus (V' \cap V_K)$.[15] Let $(i^*, j^*)$ be an edge from $V' \cap V_K$ to $V_K \setminus (V' \cap V_K)$ in $\Gamma(\succeq)$, see left hand side of Figure 8.

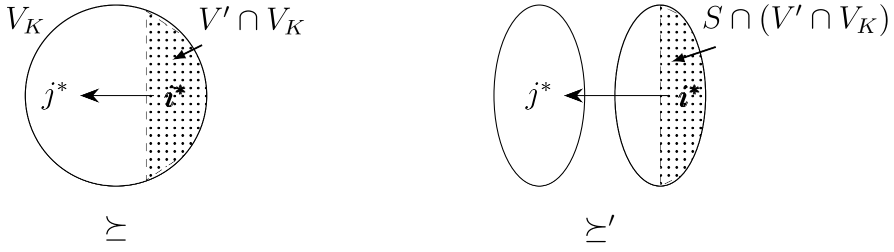

Figure 8: Case $K.2$ ($S \cap V_K \neq \emptyset$). Sets $V' \cap V_K$ (at $\succeq$, left) and $S \cap (V' \cap V_K)$ (at $\succeq'$, right) indicated as dotted areas.

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[14] Note that since $|\{V'(h)\}_{h \in V_K}| \geq 2$, $|\{V'(h)\}_{h \in V_K} \setminus \{V'\}| \geq 1$.

[15] If $\widetilde{V}$ is an SCC coalition of a graph, then for each $T \subsetneq \widetilde{V}$ with $T \neq \emptyset$, there is an edge from some vertex in $T$ to some vertex in $\widetilde{V} \setminus T$.

In particular,

$$j^* \notin V'. \tag{10}$$

Since $(i^*, j^*)$ is an edge in $\Gamma(\succeq)$,

$$j^* \in F_{i^*}. \tag{11}$$

Since there is no edge from any vertex in $V' \cap V_K$ to any vertex in $V_K \setminus (V' \cap V_K)$ in $\Gamma(\succeq')$, $(i^*, j^*)$ is not an edge in $\Gamma(\succeq')$. Then, since the only agents that possibly change preferences from $\succeq$ to $\succeq'$ are in $S$, we conclude that $i^* \in S$. Thus,

$$i^* \in S \cap V_K, \tag{12}$$

see right hand side of Figure 8. From (7), (11), and (12), $j^* \in \varphi^{SCC}_{i^*}(\succeq')$. However, since by definition of $V'$, $\varphi^{SCC}_{i^*}(\succeq') = V'$, we obtain $j^* \in V'$; contradicting (10). This proves (8).

Hence, through (6), (7), and (8), we have now shown that each agent $i \in S \cap V_K$ in coalition $\varphi^{SCC}_i(\succeq')$ is still together with all his friends and together with the same enemies as before. Thus, (a), together with Condition (E) of friend-oriented preferences, implies that for each $i \in S \cap V_K$, $\varphi^{SCC}_i(\succeq') = \varphi^{SCC}_i(\succeq) = V_K$. Hence, for each $i \in V_K$, $\varphi^{SCC}_i(\succeq') = V_K$. In particular, $V_K$ is an SCC coalition of $\Gamma(\succeq')$.

Next, let $l \in \{1, \ldots, K-1\}$. Previous Cases $K, K-1, \ldots, l+1$ imply that

$$V_K, V_{K-1}, \ldots, V_{l+1} \text{ are SCC coalitions of graph } \Gamma(\succeq'). \tag{13}$$

Consider $V_l$.

**Case *l*.1.** Suppose that $S \cap V_l = \emptyset$. Graph $\Gamma(\succeq)$ contains no edge from $V_l$ to $V_1 \cup \cdots \cup V_{l-1}$. Since $S \cap V_l = \emptyset$, $\succeq'_{V_l} = \succeq_{V_l}$. Thus, from (13), $V_l$ is an SCC coalition of graph $\Gamma(\succeq')$. Then, for each $i \in V_l$, $\varphi^{SCC}_i(\succeq') = V_l$.

**Case *l*.2.** Suppose that $S \cap V_l \neq \emptyset$. It follows from Fact 2 that each agent $i \in S \cap V_l$ in coalition $\varphi^{SCC}_i(\succeq)$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$, i.e.,

$$\varphi^{SCC}_i(\succeq) \cap \bigcup_{l' \in \{l+1,\ldots,K\}} (F_i \cap V_{l'}) = \emptyset \text{ and } \bigcup_{l' \in \{1,\ldots,l\}} (F_i \cap V_{l'}) = F_i \cap V_l \subseteq \varphi^{SCC}_i(\succeq). \tag{14}$$

Then, from (13), (a), and by Condition (F) of friend-oriented preferences, each agent $i \in S \cap V_l$ in coalition $\varphi^{SCC}_i(\succeq')$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$, i.e.,

$$\varphi^{SCC}_i(\succeq') \cap \bigcup_{l' \in \{l+1,\ldots,K\}} (F_i \cap V_{l'}) = \emptyset \text{ and } \bigcup_{l' \in \{1,\ldots,l\}} (F_i \cap V_{l'}) = F_i \cap V_l \subseteq \varphi^{SCC}_i(\succeq'). \tag{15}$$

Next, we prove that for each agent $i \in S \cap V_l$, if agent $i$ in coalition $\varphi_i^{SCC}(\succeq)$ is together with an enemy $e$, then that enemy is also in his coalition $\varphi_i^{SCC}(\succeq')$, i.e.,

$$E_i \cap \varphi_i^{SCC}(\succeq) \subseteq E_i \cap \varphi_i^{SCC}(\succeq'). \tag{16}$$

Suppose, by contradiction, that some agent $i \in S \cap V_l$ in coalition $\varphi_i^{SCC}(\succeq)$ is together with an enemy $e$ who is not in his coalition $\varphi_i^{SCC}(\succeq')$, i.e., $e \in E_i \cap (\varphi_i^{SCC}(\succeq) \setminus \varphi_i^{SCC}(\succeq'))$. Since $\varphi_i^{SCC}(\succeq) = V_l$, $e \in V_l$. By definition of $\varphi^{SCC}(\succeq')$,

$$\text{agents } i \text{ and } e \text{ are in distinct SCC coalitions of } \Gamma(\succeq'). \tag{17}$$

For each $h \in V_l$, let $V'(h)$ denote the SCC coalition of $\Gamma(\succeq')$ that contains agent $h$. By definition of $V'(h)$, $V'(h) \cap V_l \neq \emptyset$. Moreover, from $i, e \in V_l$ and (17) it follows that $|\{V'(h)\}_{h \in V_l}| \geq 2$. Since the condensation graph of $\Gamma(\succeq')$ is acyclic, let $V' \in \{V'(h)\}_{h \in V_l}$ be an SCC coalition without an outgoing edge to any of the other SCC coalitions in $\{V'(h)\}_{h \in V_l} \setminus \{V'\}$.[16] Hence, there is no edge from any vertex in $V'$ to any vertex in $[\bigcup_{h \in V_l} V'(h)] \setminus V'$. In particular, in $\Gamma(\succeq')$, there is no edge from any vertex in $V' \cap V_l$ to any vertex in

$$\left[\bigcup_{h \in V_l} (V'(h) \cap V_l)\right] \setminus (V' \cap V_l) = V_l \setminus (V' \cap V_l).$$

However, since

$$[V' \cap V_l] \cup [V_l \setminus (V' \cap V_l)] = V_l$$

is an SCC coalition of $\Gamma(\succeq)$, there is an edge from some vertex in $V' \cap V_l$ to some vertex in $V_l \setminus (V' \cap V_l)$. Let $(i^*, j^*)$ be an edge from $V' \cap V_l$ to $V_l \setminus (V' \cap V_l)$ in $\Gamma(\succeq)$. In particular,

$$j^* \notin V'. \tag{18}$$

Since $(i^*, j^*)$ is an edge in $\Gamma(\succeq)$,

$$j^* \in F_{i^*}. \tag{19}$$

Since there is no edge from any vertex in $V' \cap V_l$ to any vertex in $V_l \setminus (V' \cap V_l)$ in $\Gamma(\succeq')$, $(i^*, j^*)$ is not an edge in $\Gamma(\succeq')$. Then, since the only agents that possibly change preferences from $\succeq$ to $\succeq'$ are in $S$, we conclude that $i^* \in S$. Thus,

$$i^* \in S \cap V_l. \tag{20}$$

From (15), (19), and (20), $j^* \in \varphi_{i^*}^{SCC}(\succeq')$. However, since by definition of $V'$, $\varphi_{i^*}^{SCC}(\succeq') = V'$, we obtain $j^* \in V'$; contradicting (18). This proves (16).

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[16] Note that since $|\{V'(h)\}_{h \in V_l}| \geq 2$, $|\{V'(h)\}_{h \in V_l} \setminus \{V'\}| \geq 1$.

Hence, through (14), (15), and (16), we have now shown that each agent $i \in S \cap V_l$ in coalition $\varphi_i^{SCC}(\succeq')$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$ and together with the same enemies as before. Thus, (a), together with Condition (E) of friend-oriented preferences, implies that for each $i \in S \cap V_l$, $\varphi_i^{SCC}(\succeq') = \varphi_i^{SCC}(\succeq) = V_l$. Hence, for each $i \in V_l$, $\varphi_i^{SCC}(\succeq') = V_l$. In particular, $V_l$ is an SCC coalition of $\Gamma(\succeq')$.

We have recursively shown that for each $l \in \{1, \ldots, K\}$ and each $i \in V_l$, $\varphi_i^{SCC}(\succeq') = V_l$. Hence, for each $i \in N$, $\varphi_i^{SCC}(\succeq) = \varphi_i^{SCC}(\succeq')$, which contradicts (b). Therefore, $\varphi^{SCC}$ is *group strategy-proof.* □

# F Appendix: Proof of Lemma 2

Let $\widetilde{\mathcal{R}}^f$ be a rich subdomain of friend-oriented problems and mechanism $\varphi$ be *core stable* and *strategy-proof.* Let $\succeq \in \widetilde{\mathcal{R}}^f$ and $i \in N$ such that $\succeq$ is SCC-minimal with respect to an SCC coalition $V \equiv \varphi_i^{SCC}(\succeq)$. Then, showing that $\varphi_i(\succeq) = V$ proves the SCC-minimality lemma (Lemma 2).

***Proof.*** We prove that $\varphi_i(\succeq) = V$ by induction on $|V|$, i.e., the number of agents in $V$.

***Vertex-induction basis.*** Let $|V| = 1$, i.e., $\varphi_i^{SCC}(\succeq) = V = \{i\}$. Since $\succeq$ is SCC-minimal with respect to $V$, agent $i$ at $\succeq$ has no friends. Thus, by *core stability* of $\varphi$, $\varphi_i(\succeq) = \{i\} = V$.

***Vertex-induction hypothesis.*** For each $\succeq^* \in \widetilde{\mathcal{R}}^f$ and each $i^* \in N$ such that $\succeq^*$ is SCC-minimal with respect to $V^* \equiv \varphi_{i^*}^{SCC}(\succeq^*)$ with $|V^*| \leq \ell - 1$, we have $\varphi_{i^*}(\succeq^*) = V^*$.

Since $\varphi$ refines $\varphi^{SCC}$ (Corollary 1), it follows from the friend-reduction lemma (Lemma 1) that the vertex-induction hypothesis also applies without requiring SCC-minimality of $\succeq^*$ with respect to $V^*$.

***Vertex-induction hypothesis*.*** For each $\succeq^* \in \widetilde{\mathcal{R}}^f$ and each $i^* \in N$ with $V^* \equiv \varphi_{i^*}^{SCC}(\succeq^*)$ and $|V^*| \leq \ell - 1$, we have $\varphi_{i^*}(\succeq^*) = V^*$.

***Vertex-induction step.*** In this step we will prove that, when $|V| = \ell$, then $\varphi_i(\succeq) = V$.

The proof is by induction on the number of edges in the induced friendship graph $\Gamma(\succeq^V) = (V, A^V)$ with $A^V = \{(j,k) \in V \times V : k \in F(\succeq_j)\}$. Note that by SCC-minimality of $\succeq$ with respect to $V$, agents in $V$ do not have any friends outside $V$. Formally, let $\epsilon(\succeq^V) \geq |V| = \ell$ denote the number of edges in $\Gamma(\succeq^V)$.[17]

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[17] Note that, by definition, $\Gamma(\succeq^V)$ only contains edges that start from some vertex in $V$ and end in some other vertex in $V$. Since $\Gamma(\succeq^V)$ is strongly connected, each vertex in $V$ has at least one outgoing edge. Hence, $\epsilon(\succeq^V) \geq |V|$. Note that $\epsilon(\succeq^V) = |V|$ if $\Gamma(\succeq^V)$ consists of a simple cycle.

***Edge-induction basis.*** Let $\epsilon(\succeq^V) = \ell$. Then, $\Gamma(\succeq^V)$ consists of a simple cycle.[18] It follows immediately from Proposition 2 that $\varphi_i(\succeq) = V$.

***Edge-induction hypothesis.*** Let $\epsilon \geq \ell + 1$. Then, for each $\succeq^* \in \widetilde{\mathcal{R}}^f$ and each $i^* \in N$ such that $\succeq^*$ is SCC-minimal with respect to $V^* \equiv \varphi_{i^*}^{SCC}(\succeq^*)$ with $|V^*| = \ell$ and $\epsilon((\succeq^*)^{V^*}) \leq \epsilon - 1$, we have that $\varphi_{i^*}(\succeq^*) = V^*$.

Since $\varphi$ refines $\varphi^{SCC}$ (Corollary 1), it follows from the friend-reduction lemma (Lemma 1) that the edge-induction hypothesis also applies without requiring SCC-minimality of $\succeq^*$ with respect to $V^*$.[19]

***Edge-induction hypothesis*.*** Let $\epsilon \geq \ell + 1$. Then, for each $\succeq^* \in \widetilde{\mathcal{R}}^f$ and each $i^* \in N$ with $V^* \equiv \varphi_{i^*}^{SCC}(\succeq^*)$, $|V^*| = \ell$, and $\epsilon((\succeq^*)^{V^*}) \leq \epsilon - 1$, we have that $\varphi_{i^*}(\succeq^*) = V^*$.

***Edge-induction step.*** In this step we will prove that, when $\epsilon(\succeq^V) = \epsilon$, then $\varphi_i(\succeq) = V$. Assume, by contradiction, that $\varphi_i(\succeq) \neq V$. Then, by Corollary 1,

$$\varphi_i(\succeq) \subsetneq V. \tag{21}$$

We distinguish between two cases.

**CASE 1.** $\Gamma(\succeq^V)$ contains two distinct simple cycles that have exactly one vertex in common, i.e., there are simple cycles $c_1 = (v_1, v_2, \ldots, v_m, v_1)$ and $c_2 = (w_1, w_2, \ldots, w_p, w_1)$ with $c_1 \neq c_2$ and $|\{v_1, v_2, \ldots, v_m\} \cap \{w_1, w_2, \ldots, w_p\}| = 1$.

Denote the set of vertices in the two simple cycles $c_1$ and $c_2$ by

$$Z \equiv \{v_1, \ldots, v_m, w_1, \ldots, w_p\}.$$

Note that $|Z| = m+p-1 \leq \ell$ and $Z \subseteq V$. If $Z = V$, then by SCC-minimality of $\succeq$ with respect to $V$, $\Gamma(\succeq^V)$ is composed of cycles $c_1$ and $c_2$. Otherwise, $Z \subsetneq V$ and cycles $c_1$ and $c_2$ constitute a strict subgraph of $\Gamma(\succeq^V)$ and additional vertices and edges are contained in $\Gamma(\succeq^V)$.

Without loss of generality we can assume that $\{i\} = \{v_1, v_2, \ldots, v_m\} \cap \{w_1, w_2, \ldots, w_p\}$ and $i = v_1 = w_1$. Let $j \equiv v_m$ and $b \equiv v_2$ be the predecessor and successor, respectively, of $i$ in cycle $c_1$. Let $k \equiv w_p$ and $a \equiv w_2$ be the predecessor and successor, respectively, of $i$ in cycle $c_2$. Then, agent $i$ has agents $a$ and $b$ as friends, i.e., $\{a, b\} \subseteq F(\succeq_i)$. See Figure 9 for an illustration. Note

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[18] Since $\Gamma(\succeq^V)$ is strongly connected, any two vertices in $V$ are connected by a cycle. Hence, each of the $\ell$ vertices in $V$ has at least one incoming edge and at least one outgoing edge. Since there are only $\ell$ edges, each of the $\ell$ vertices in $V$ has in fact exactly one incoming edge; and similarly, each of the $\ell$ vertices in $V$ has in fact exactly one outgoing edge. But then there is a simple cycle $c$ that traverses all vertices in $V$ and $\Gamma(\succeq^V)$ consists of $c$.

[19] Note that in particular the definition of the number of edges $\epsilon$ does not require SCC-minimality of $\succeq^*$ with respect to $V^*$. More precisely, $\epsilon((\succeq^*)^{V^*})$ in the edge-induction hypothesis* is the number of edges in $\Gamma((\succeq^*)^{V^*})$, i.e., ignoring any edges that leave $V^*$ or enter $V^*$.

that $a \neq b$,[20] but $b = j$ or $a = k$ is possible.

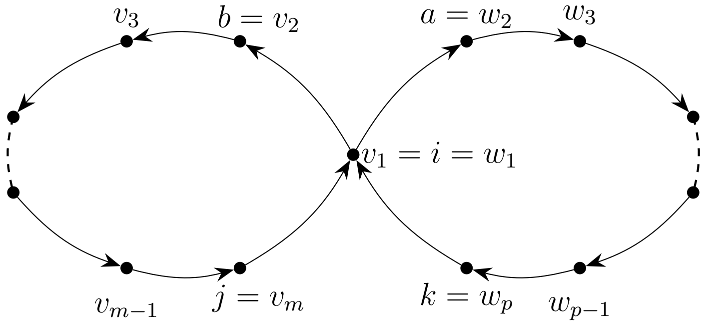

Figure 9: The set of vertices $Z \subseteq V$ and the two simple cycles $c_1$ and $c_2$ which are part of $\Gamma(\succeq^V)$. By SCC-minimality of $\succeq$ with respect to $V$, the displayed edges are the only edges between vertices in $Z$ at $\Gamma(\succeq^V)$.

Next, using the richness of $\widetilde{\mathcal{R}}^f$, let

- $\succeq'_k \in \widetilde{\mathcal{R}}^f_k$ such that $F(\succeq'_k) = (F(\succeq_k)\backslash\{i\}) \cup \{b\}$ [note that $b \in F(\succeq'_k)$];
- $\overline{\succeq}_j \in \widetilde{\mathcal{R}}^f_j$ such that $F(\overline{\succeq}_j) = (F(\succeq_j)\backslash\{i\}) \cup \{a\}$ [note that $a \in F(\overline{\succeq}_j)$];
- $\succeq'_i \in \widetilde{\mathcal{R}}^f_i$ such that $F(\succeq'_i) = F(\succeq_i)\backslash\{b\}$ [note that $a \in F(\succeq'_i)$]; and
- $\overline{\succeq}_i \in \widetilde{\mathcal{R}}^f_i$ such that $F(\overline{\succeq}_i) = F(\succeq_i)\backslash\{a\}$ [note that $b \in F(\overline{\succeq}_i)$].

Using these four individual preferences together with preference profile $\succeq$, we introduce the following four preference profiles in $\widetilde{\mathcal{R}}^f$:

- $\succeq' \equiv (\succeq'_k, \succeq_{-k})$;
- $\overline{\succeq} \equiv (\overline{\succeq}_j, \succeq_{-j})$;
- $\succeq'' \equiv (\succeq'_i, \succeq'_k, \succeq_{-\{i,k\}})$; and
- $\overline{\overline{\succeq}} \equiv (\overline{\succeq}_i, \overline{\succeq}_j, \succeq_{-\{i,j\}})$.

Figure 10 shows how we can move between the above defined profiles and $\succeq$ by changing one preference relation at a time (the agent above the transition arrow $\longleftrightarrow$ is the one who changes his preferences).

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[20] If $a = b$, then, since $a \neq i$ and $b \neq i$, we would have that $c_1$ and $c_2$ have more than one vertex in common, which contradicts the assumption of Case 1.

$$\begin{tikzcd}
\succeq'' \arrow[r, leftrightarrow, "i"] & \succeq' \arrow[r, leftrightarrow, "k"] & \succeq \arrow[r, leftrightarrow, "j"] & \overline{\succeq} \arrow[r, leftrightarrow, "i"] & \overline{\overline{\succeq}}
\end{tikzcd}$$

Figure 10: Unilateral preference transitions between five preference profiles.

Using Figure 9 and the definition of the four new preference profiles, one easily verifies that Figures 11 and 12 depict all edges between vertices in $Z$ at $\succeq'$ and $\overline{\succeq}$ (Figure 11) as well as $\succeq''$ and $\overline{\overline{\succeq}}$ (Figure 12).

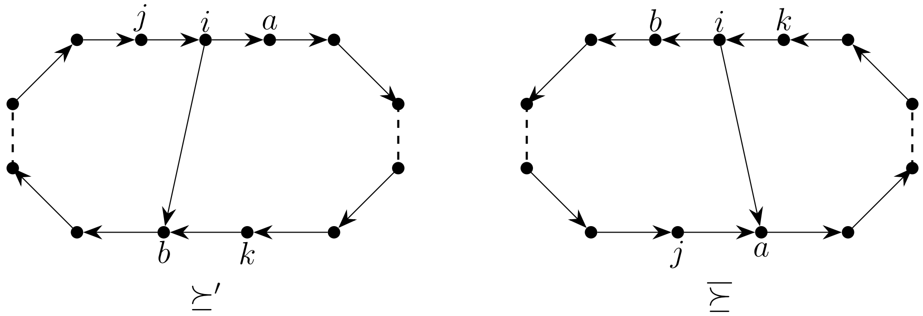


Figure 11: The set of vertices $Z$ and all edges between vertices in $Z$ at $\Gamma(\succeq')$ and $\Gamma(\overline{\succeq})$.

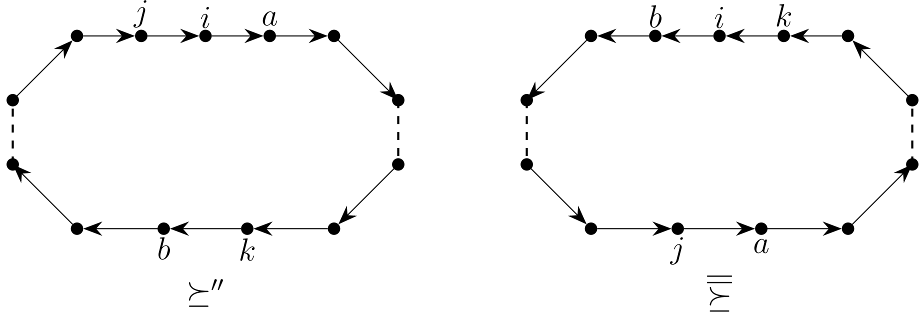


Figure 12: The set of vertices $Z$ and all edges between vertices in $Z$ at $\Gamma(\succeq'')$ and $\Gamma(\overline{\overline{\succeq}})$.

**Claim 1.** *$V$ is an SCC coalition of $\Gamma(\succeq'')$ and $\Gamma(\overline{\overline{\succeq}})$. Moreover, $\varphi_i(\succeq'') = V = \varphi_i(\overline{\overline{\succeq}})$.*

*Proof of Claim 1.* (a) If $Z = V$, then the agents in $V$ at each of the graphs $\Gamma(\succeq'')$ and $\Gamma(\overline{\overline{\succeq}})$ form a simple cycle (see Figure 12). Moreover, by SCC-minimality of $\succeq$ with respect to $V$, none of the vertices in $V$ has an outgoing edge to a vertex outside $V$. Thus, $V$ is an SCC coalition of $\Gamma(\succeq'')$ and $\Gamma(\overline{\overline{\succeq}})$ and, by Proposition 2, $\varphi_i(\succeq'') = V = \varphi_i(\overline{\overline{\succeq}})$.

(b) If $Z \subsetneq V$, then Proposition 2 cannot be applied and it is in this part of the proof that we will use the edge-induction hypothesis*.

We first prove that $V$ is an SCC coalition of $\Gamma(\succeq'')$. To see this, first recall that $V$ is an SCC coalition of $\Gamma(\succeq)$. Transforming $\Gamma(\succeq)$ into $\Gamma(\succeq'')$ consists of the removal of the edges $(i, b)$

and $(k,i)$ and the addition of the edge $(k,b)$. Since $i,b,k \in Z = \{v_1,\ldots,v_m,w_1,\ldots,w_p\}$ and since $(w_1,\ldots,w_p,v_2,\ldots,w_m,v_1)$ is a cycle in $\Gamma(\succeq'')$ that contains all vertices in $Z$, it follows that for each pair of vertices $x,y \in V$ there is a cycle in $\Gamma(\succeq'')$ that contains $x$ and $y$. Since $\succeq$ is SCC-minimal with respect to $V$, there are no edges that leave $V$ in $\Gamma(\succeq)$. Then, no edges leave $V$ in $\Gamma(\succeq'')$. Hence, $V$ is an SCC coalition of $\Gamma(\succeq'')$. Similar arguments show that $V$ is an SCC coalition of $\Gamma(\overline{\overline{\succeq}})$.

Since $\epsilon((\succeq'')^V) = \epsilon((\overline{\overline{\succeq}})^V) = \epsilon - 1$, the edge-induction hypothesis* implies that $\varphi_i(\succeq'') = V = \varphi_i(\overline{\overline{\succeq}})$. ■

**Claim 2.** *$V$ is an SCC coalition of $\Gamma(\succeq')$ and $\Gamma(\overline{\succeq})$. Moreover, $\varphi_i(\succeq') = V = \varphi_i(\overline{\succeq})$.*

*Proof of Claim 2.* Since $\Gamma(\succeq')$ is obtained from $\Gamma(\succeq'')$ by adding the edge $(i,b)$ and since $i,b \in V$, it follows from Claim 1 that $V$ is an SCC coalition of $\Gamma(\succeq')$. Similarly, $V$ is an SCC coalition of $\Gamma(\overline{\succeq})$.

Suppose, by contradiction, $\varphi_i(\succeq') \neq V$. Then, since $i \in V$ and $V \in \varphi^{SCC}(\succeq')$, it follows from Corollary 1 that $\varphi_i(\succeq') \subsetneq V$.

Recall that $\succeq' = (\succeq_i, \succeq'_k, \succeq_{-i,k})$ and $\succeq'' = (\succeq'_i, \succeq'_k, \succeq_{-\{i,k\}})$. Hence, when moving from $\succeq'$ to $\succeq''$, agent $i$ has one less friend but his SCC coalition does not change. Then, by the friend-reduction lemma (Lemma 1), $\varphi_i(\succeq'') \subsetneq V$; contradicting Claim 1. Thus, $\varphi_i(\succeq') = V$. Similar arguments show that $\varphi_i(\overline{\succeq}) = V$. ■

Recall that, by (21), $\varphi_i(\succeq) \subsetneq V = \varphi_i^{SCC}(\succeq)$. Thus, $\varphi(\succeq)$ is a refinement of $\varphi^{SCC}(\succeq)$ such that there are (non-empty) coalitions $U_1,\ldots,U_L \in \varphi(\succeq)$ with $L \geq 2$ and $U_1 \cup \cdots \cup U_L = V$. Let $U \in \{U_1,\ldots,U_L\}$ such that $i \in U = \varphi_i(\succeq)$.

**Claim 3.** *$a,b \notin U = \varphi_i(\succeq) \subsetneq V$.*

*Proof of Claim 3.* Since $U = \varphi_i(\succeq) \subsetneq V$ follows immediately, we only have to prove that $a,b \notin U$.

Since $k \in V$, it follows from Claim 2 that $\varphi_k(\succeq') = V$. Since $\succeq$ is SCC-minimal with respect to $V$, $F(\succeq_k) \subseteq V$. So, $F(\succeq_k) \subseteq \varphi_k(\succeq')$.

Then, it follows from *strategy-proofness* of $\varphi$ and Condition (F) of friend-oriented preferences that $F(\succeq_k) \subseteq \varphi_k(\succeq)$ (because otherwise, agent $k$, at profile $\succeq$, could report $\succeq'_k$ to move to profile $\succeq'$, at which he is matched with all his friends). Since $i \in F(\succeq_k)$, $i \in \varphi_k(\succeq)$. Then, $\varphi_k(\succeq) = \varphi_i(\succeq) = U$. So, $k \in U$ and $F(\succeq_k) \subseteq \varphi_k(\succeq) = U \subsetneq V = \varphi_k(\succeq')$.

Suppose $b \in \varphi_k(\succeq)$. Then, since $F(\succeq'_k) = (F(\succeq_k) \backslash \{i\}) \cup \{b\}$ we obtain a contradiction with *strategy-proofness* of $\varphi$ and Condition (E) of friend-oriented preferences (because otherwise, agent $k$, at profile $\succeq'$, could report $\succeq_k$ to move to profile $\succeq$, at which he is still matched with all his friends but with fewer enemies). Hence, $b \notin \varphi_k(\succeq)$.

Using Claim 2 for profile $\succeq'$, we have shown that $k \in U$ and $b \notin \varphi_k(\succeq)$. Using Claim 2 for profile $\overline{\succeq}$, and applying similar arguments, yields $j \in U$ and $a \notin \varphi_j(\succeq)$. Thus, $U = \varphi_k(\succeq) = \varphi_j(\succeq)$ and $a, b \notin U$. ■

Next, using the richness of $\widetilde{\mathcal{R}}^f$, let

- $\widetilde{\succeq}_i \in \widetilde{\mathcal{R}}^f_i$ such that $F(\widetilde{\succeq}_i) = F(\succeq_i)\backslash\{a\}$

and define the following preference profile in $\widetilde{\mathcal{R}}^f$:

- $\widetilde{\succeq} \equiv (\widetilde{\succeq}_i, \succeq_{-i})$.

Furthermore, let $\widetilde{V} \equiv \varphi_i^{SCC}(\widetilde{\succeq})$.

**Claim 4.** *$\varphi_i(\widetilde{\succeq}) = \widetilde{V}$.*

*Proof of Claim 4.* Recall that $i \in \varphi_i^{SCC}(\succeq) = V$ and $\succeq$ is SCC-minimal with respect to $V$. Hence, by removing the edge $(i, a)$ to obtain $\Gamma(\widetilde{\succeq})$ from $\Gamma(\succeq)$, $V$ is broken into multiple SCC coalitions. In other words, $V$ is not an SCC coalition of $\Gamma(\widetilde{\succeq})$. Moreover, at $\Gamma(\widetilde{\succeq})$, agents $i$ and $a$ are in different SCC coalitions, i.e., $\varphi_i^{SCC}(\widetilde{\succeq}) \neq \varphi_a^{SCC}(\widetilde{\succeq})$ (otherwise $\succeq$ would not be SCC-minimal with respect to $V$ because the edge $(i, a)$ would be "redundant"). Hence, $a \in V\backslash\widetilde{V}$ and $|\tilde{V}| < |V| = \ell$. Thus, the vertex-induction hypothesis* implies that $\varphi_i(\widetilde{\succeq}) = \widetilde{V}$. ■

**Claim 5.** *$F(\widetilde{\succeq}_i) \subseteq \widetilde{V}$.*

*Proof of Claim 5.* Note that $F(\succeq_i) = F(\widetilde{\succeq}_i) \cup \{a\}$. We first show that for each friend $f \in F(\succeq_i)\backslash\{a\}$, there is a cycle $c_f$ in $\Gamma(\succeq)$ that

(1) only contains vertices in $V = \varphi_i^{SCC}(\succeq)$;

(2) contains both $i$ and $f$; and

(3) does not contain the edge $(i, a)$.

To see this, first note that there is a cycle in $\Gamma(\succeq)$ that satisfies (1) and (2) because $V$ is an SCC coalition in $\Gamma(\succeq)$ and $F(\succeq_i) \subseteq V$ (because $\succeq$ is SCC-minimal with respect to $V$). Next, each such cycle must contain edge $(i, f)$; otherwise $\succeq$ would not be SCC-minimal with respect to $V$ (the edge $(i, f)$ would be "redundant"). But then each such cycle consists of $(i, f)$ and a path back to $i$. Cutting the path when it returns to $i$ for the first time yields a cycle $c_f$ that satisfies (1), (2), and (3).

Thus, for each friend $f \in F(\succeq_i)\backslash\{a\}$, $c_f$ is a cycle in $\Gamma(\widetilde{\succeq})$. Hence, by definition of $\varphi^{SCC}$, $F(\widetilde{\succeq}_i) = F(\succeq_i)\backslash\{a\} \subseteq \varphi_i^{SCC}(\widetilde{\succeq}) = \widetilde{V}$. ■

By Claim 3, $a, b \notin \varphi_i(\succeq)$, i.e., at profile $\succeq$, agent $i$ is not matched with his friends $a$ and $b$. By Claims 4 and 5, $F(\tilde{\succeq}_i) \subseteq \varphi_i(\tilde{\succeq})$. Since $b \in F(\succeq_i) = F(\tilde{\succeq}_i) \cup \{a\}$ and $b \neq a \notin F(\tilde{\succeq}_i)$, it follows that agent $i$ by moving from $\succeq$ to $\tilde{\succeq}$ will become member of a coalition that includes *all* his $F(\succeq_i)$–friends (in particular friend $b$), except possibly for friend $a$. Thus, by moving from $\succeq$ to $\tilde{\succeq}$, agent $i$ is matched with a superset of friends, contradicting *strategy-proofness* of $\varphi$ and Condition (F) of friend-oriented preferences. This completes the proof of Case 1.

**Fact 3.** Let $\succeq^* \in \widetilde{\mathcal{R}}^f$ and $i^* \in N$. Let $V^* \equiv \varphi_{i^*}^{SCC}(\succeq^*)$. From the friend-reduction lemma (Lemma 1), the edge-induction hypothesis*, and Case 1, it follows that if $\Gamma((\succeq^*)^{V^*})$ contains two distinct simple cycles that have exactly one vertex in common, $|V^*| = \ell$, and $\epsilon((\succeq^*)^{V^*}) \leq \epsilon$, then $\varphi_{i^*}(\succeq^*) = V^*$. $\diamond$

**Case 2.** $\Gamma(\succeq^V)$ does *not* contain two distinct simple cycles that have exactly one vertex in common.

Since $\epsilon(\succeq^V) = \epsilon > \ell = |V|$, there is some agent in $V$ whose outdegree in $\Gamma(\succeq^V)$ is larger or equal to 2. Without loss of generality, we can assume that agent $i$ has outdegree $\geq 2$. Thus, there are at least two other, distinct agents in $V$, say $v_2$ and $w_2$ with $v_2 \neq w_2$, so that $(i, v_2)$ and $(i, w_2)$ are edges in $\Gamma(\succeq^V)$. Since $V$ is an SCC coalition, there is a simple cycle $c_1 \equiv (v_1, v_2, \ldots, v_m, v_1)$ with $v_1 = i$. Moreover, $w_2 \notin \{v_1, v_2, \ldots, v_m\}$; otherwise $\succeq$ would not be SCC-minimal with respect to $V$ (because the edge $(i, w_2)$ would be "redundant"). Since $V$ is an SCC coalition, there exists a path from $w_2$ to $i$ in $\Gamma(\succeq^V)$ that, without loss of generality, does not have repeated vertices. Let $(w_2, w_3, \ldots, w_p, i)$ be such a path. It contains some vertex in $\{v_2, \ldots, v_m\}$; otherwise, $c_1$ and $(i, w_2, w_3, \ldots, w_p, i)$ would be two distinct simple cycles that have exactly one vertex in common (namely $i$), which contradicts the assumption of Case 2. Let $w_{s+1}$ with $s \geq 1$ be the first vertex in $(w_2, w_3, \ldots, w_p)$ that is contained in $\{v_2, \ldots, v_m\}$, say $v_r$. Let $c_2 \equiv (w_1 = i, w_2, \ldots, w_s, v_r, v_{r+1}, \ldots, v_m, v_1 = i)$. Denote the set of vertices in the two simple cycles $c_1$ and $c_2$ by

$$Z \equiv \{v_1, \ldots, v_m, w_2, \ldots, w_s\}.$$

Note that $Z \subseteq V$. If $Z = V$, then by SCC-minimality of $\succeq$ with respect to $V$, $\Gamma(\succeq^V)$ is composed of cycles $c_1$ and $c_2$. Otherwise, $Z \subsetneq V$ and cycles $c_1$ and $c_2$ constitute a strict subgraph of $\Gamma(\succeq^V)$ and additional vertices and edges are contained in $\Gamma(\succeq^V)$.

Note that $v_2$ and $w_2$ are the successors of agent $i$ in cycles $c_1$ and $c_2$, respectively. Let $j \equiv v_r$ and let $a \equiv v_{r-1}$ and $b \equiv w_s$ be the predecessors of agent $j$ in cycles $c_1$ and $c_2$, respectively. See Figure 13 for an illustration.

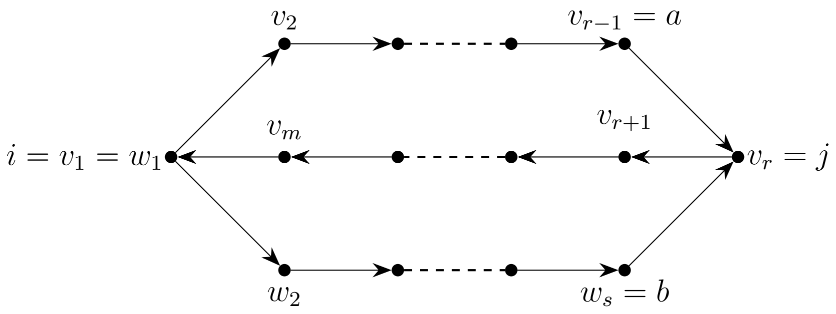


Figure 13: The set of vertices $Z \subseteq V$ and the two simple cycles $c_1$ and $c_2$ which have vertices $j, v_{r+1}, \ldots, v_m, i$ in common and are part of $\Gamma(\succeq^V)$. By SCC-minimality of $\succeq$ with respect to $V$, the displayed edges are the only edges between vertices in $Z$ at $\Gamma(\succeq^V)$.

Note that agents $i$, $v_2$, and $w_2$ are three distinct agents (agent $i$ has outdegree $\geq 2$). If $v_2 = j$, then we obtain a contradiction with $\succeq$ being SCC-minimal with respect to $V$ (because the direct edge $(i, j)$ would be "redundant"). Hence, $v_2 \neq j$. Similarly it follows that $w_2 \neq j$. Furthermore, $i \neq j$ ($i = j$ is in CASE 1). Thus, $i, j, v_2, w_2$ are four distinct agents. This implies that $i, j, a, b$ are four distinct agents as well.

Next, note that agent $i$ is an enemy of agent $a$, i.e., $i \in E(\succeq_a)$. To see this, suppose $i \notin E(\succeq_a)$. Then, since $a \neq i$, $i \in F(\succeq_a)$, which yields a contradiction with $\succeq$ being SCC-minimal with respect to $V$ (because the edge $(a, j)$ would be "redundant"). Hence, $i \in E(\succeq_a)$. Similarly it follows that agent $i$ is an enemy of agent $b$, i.e., $i \in E(\succeq_b)$.

Next, using $i \in E(\succeq_a)$, $i \in E(\succeq_b)$, and the richness of $\widetilde{\mathcal{R}}^f$, let

- $\succeq_a' \in \widetilde{\mathcal{R}}_a^f$ such that $F(\succeq_a') = (F(\succeq_a) \backslash \{j\}) \cup \{i\}$ and
- $\succeq_b'' \in \widetilde{\mathcal{R}}_b^f$ such that $F(\succeq_b'') = (F(\succeq_b) \backslash \{j\}) \cup \{i\}$.

Using these two preferences together with preference profile $\succeq$, we introduce the following two preference profiles in $\widetilde{\mathcal{R}}^f$:

- $\succeq' \equiv (\succeq_a', \succeq_{-a})$ and
- $\succeq'' \equiv (\succeq_b'', \succeq_{-b})$.

Using Figure 13 and the definition of the two new preference profiles, one easily verifies that Figure 14 depicts all edges between vertices in $Z$ at $\succeq'$ and $\succeq''$.

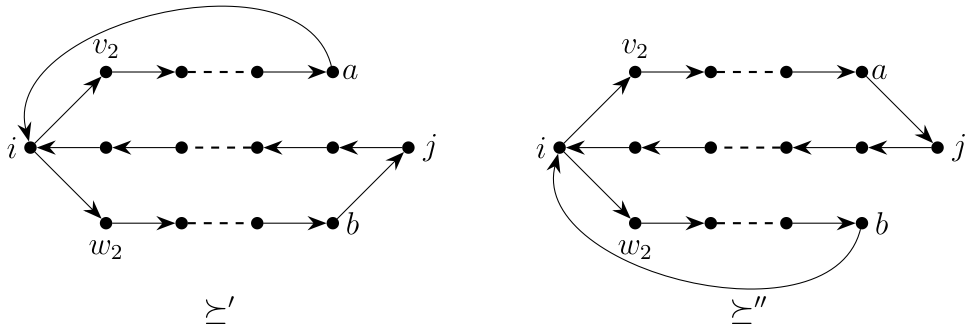


Figure 14: The set of vertices $Z$ and all edges between vertices in $Z$ at $\Gamma(\succeq')$ and $\Gamma(\succeq'')$.

Since $V$ is an SCC coalition of $\Gamma(\succeq)$, it follows that $V$ is also an SCC coalition of both $\Gamma(\succeq')$ and $\Gamma(\succeq'')$. Also, note that $\Gamma((\succeq')^V)$ and $\Gamma((\succeq'')^V)$ each contain two distinct simple cycles that have vertex $i$ in common (Case 1 applies). Furthermore, $|V| = \ell$ and $\epsilon((\succeq')^V) = \epsilon = \epsilon((\succeq'')^V)$. Hence, by Fact 3, we know that $\varphi_i(\succeq') = V = \varphi_i(\succeq'')$. Note that since $a, b \in V$, $\varphi_a(\succeq') = \varphi_b(\succeq') = V$.

Since $F(\succeq_a) \subseteq V$ (because $\succeq$ is SCC-minimal with respect to $V$), it follows from $\varphi_a(\succeq') = V$, *strategy-proofness* of $\varphi$, and Condition (F) of friend-oriented preferences that $F(\succeq_a) \subseteq \varphi_a(\succeq)$ (because otherwise, agent $a$, at profile $\succeq$, could report $\succeq'_a$ to move to profile $\succeq'$, at which he is matched with all his friends). It follows similarly that $F(\succeq_b) \subseteq \varphi_b(\succeq)$.

We now show that $i \notin \varphi_a(\succeq)$. Suppose $i \in \varphi_a(\succeq)$. Then, since $F(\succeq'_a) = (F(\succeq_a)\backslash\{j\}) \cup \{i\}$ and $F(\succeq_a) \subseteq \varphi_a(\succeq)$, $F(\succeq'_a) \subseteq \varphi_a(\succeq)$. Recall that by (21), $\varphi_a(\succeq) \subsetneq V = \varphi_a(\succeq')$, which contradicts *strategy-proofness* of $\varphi$ and Condition (E) of friend-oriented preferences (because agent $a$, at profile $\succeq'$, can report $\succeq_a$ to move to profile $\succeq$, at which he is still matched with all his friends but with fewer enemies). Hence, $i \notin \varphi_a(\succeq)$. Similar arguments for agent $b$ and profile $\succeq''$ show that $i \notin \varphi_b(\succeq)$.

Recall that, by (21), $\varphi_i(\succeq) \subsetneq V = \varphi_i^{SCC}(\succeq)$. Thus, $\varphi(\succeq)$ is a refinement of $\varphi^{SCC}(\succeq)$ such that there are (non-empty) coalitions $U_1, \ldots, U_L \in \varphi(\succeq)$ with $L \geq 2$ and $U_1 \cup \cdots \cup U_L = V$. Let $U \in \{U_1, \ldots, U_L\}$ such that $i \in U = \varphi_i(\succeq)$. Then,

$$a, b \notin \varphi_i(\succeq) = U. \tag{22}$$

Since $j \in F(\succeq_a) \subseteq \varphi_a(\succeq)$ and $j \in F(\succeq_b) \subseteq \varphi_b(\succeq)$, it follows that

$$a, b \in \varphi_j(\succeq). \tag{23}$$

Consider the path $(v_1 = i, v_2, \ldots, v_{r-1} = a, v_r = j)$ in $\Gamma(\succeq)$; see Figure 15 for an illustration. Recall that $i \in U$ and $a \notin U$. Let $q$, $1 \leq q \leq r-2$, be the smallest integer such that $v_q \in U$ and

$v_{q+1} \notin U$. Let $v \equiv v_q$ and $v' \equiv v_{q+1}$. Note that $v' \in F(\succeq_v)$ and $v' \notin U = \varphi_i(\succeq) = \varphi_v(\succeq)$.

$$\underbrace{i = v_1 \longrightarrow v_2 \longrightarrow \dashrightarrow \underset{= v}{v_q}}_{\in U} \longrightarrow \underbrace{\underset{= v'}{v_{q+1}} \longrightarrow \dashrightarrow \underset{= a}{v_{r-1}}}_{\notin U} \longrightarrow v_r = j$$

Figure 15: Path $(v_1 = i, v_2, \ldots, v_{r-1} = a, v_r = j)$ in $\Gamma(\succeq)$.

Using the richness of $\widetilde{\mathcal{R}}^f$, let

- $\widetilde{\succeq}_v \in \widetilde{\mathcal{R}}^f_v$ such that

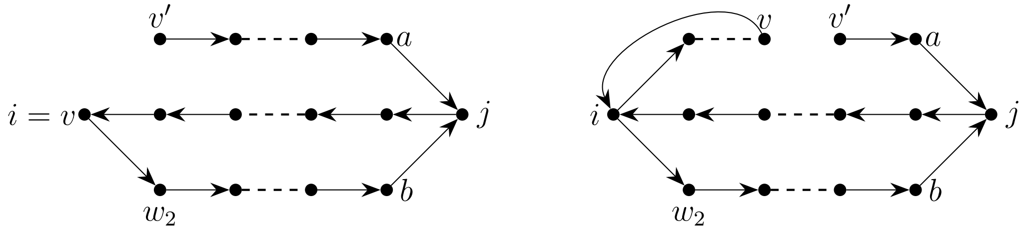
$$F(\widetilde{\succeq}_v) = \begin{cases} F(\succeq_v)\backslash\{v'\} & \text{if } v = i; \\ (F(\succeq_v)\backslash\{v'\}) \cup \{i\} & \text{if } v \neq i. \end{cases}$$

When $v = i$, $\widetilde{\succeq}_v$ are preferences with one less friend (namely $v'$) than preferences $\succeq_v$. When $v \neq i$, note that $i \notin F(\succeq_v)$; otherwise $\succeq$ would not be SCC-minimal with respect to $V$ (because the edge $(v, i)$ would be "redundant"). In this case, $\widetilde{\succeq}_v$ are preferences with the same number of friends.

Using these preferences together with preference profile $\succeq$, we introduce the following preference profile in $\widetilde{\mathcal{R}}^f$:

- $\widetilde{\succeq} \equiv (\widetilde{\succeq}_v, \succeq_{-v})$.

Using Figure 13 and the definition of the new preference profiles, one easily verifies that Figure 16 depicts all edges between vertices in $Z$ at $\widetilde{\succeq}$.



$\widetilde{\succeq}$ when $v = i$ $\qquad$ $\widetilde{\succeq}$ when $v \neq i$

Figure 16: The set of vertices $Z$ and all edges between vertices in $Z$ at $\Gamma(\widetilde{\succeq})$.

**Claim 6.** $\varphi_v(\widetilde{\succeq}) = \varphi_v^{SCC}(\widetilde{\succeq}) \subsetneq V$.

*Proof of Claim 6.* Note that $v' \notin \varphi_i^{SCC}(\widetilde{\succeq})$; otherwise, $\succeq$ would not be SCC-minimal with respect to $V$ (because the edge $(v, v')$ would be "redundant"). Hence, $\varphi_i^{SCC}(\widetilde{\succeq}) \subsetneq V = \varphi_i^{SCC}(\succeq)$ and $|\varphi_i^{SCC}(\widetilde{\succeq})| < |\varphi_i^{SCC}(\succeq)| = \ell$. Then, the vertex-induction hypothesis* implies that $\varphi_i(\widetilde{\succeq}) = \varphi_i^{SCC}(\widetilde{\succeq})$. Since $v \in \varphi_i^{SCC}(\widetilde{\succeq})$, $\varphi_v(\widetilde{\succeq}) = \varphi_v^{SCC}(\widetilde{\succeq}) \subsetneq V$. ■

**Claim 7.** *$i \in \varphi_v(\succeq) \subseteq \varphi_v(\widetilde{\succeq})$.*

*Proof of Claim 7.* Note that $i \in \varphi_v(\succeq)$ follows immediately from $v \in U = \varphi_i(\succeq)$.

Suppose $\varphi_v(\succeq) \nsubseteq \varphi_v(\widetilde{\succeq})$. Let $y \in \varphi_v(\succeq) \backslash \varphi_v(\widetilde{\succeq}) \neq \emptyset$. Note that $y \neq v$. Since $v \in U = \varphi_i(\succeq)$, $y \in \varphi_v(\succeq) = U$.

By Proposition 1, the graph $\Gamma(\succeq^U)$ is strongly connected. Since $v, y \in U$ and $v' \notin U$, there is a cycle $\widetilde{c}$ in $\Gamma(\succeq^U)$ that contains $v$ and $y$ but not $v'$. So, cycle $\widetilde{c}$ does not use the edge $(v, v')$. Hence, $\widetilde{c}$ is also a cycle in $\Gamma(\widetilde{\succeq})$. Hence, $y \in \varphi_v^{SCC}(\widetilde{\succeq})$. Then, from Claim 6, $y \in \varphi_v(\widetilde{\succeq})$, which contradicts $y \in \varphi_v(\succeq) \backslash \varphi_v(\widetilde{\succeq})$. Hence, $\varphi_v(\succeq) \subseteq \varphi_v(\widetilde{\succeq})$. ■

**Claim 8.** *$F(\succeq_v) \backslash \{v'\} \subseteq \varphi_v(\widetilde{\succeq})$ and $F(\succeq_v) \backslash \{v'\} \subseteq \varphi_v(\succeq)$.*

*Proof of Claim 8.* We first show that for each friend $f \in F(\succeq_v) \backslash \{v'\}$, there is a cycle $c_f$ in $\Gamma(\succeq)$ that

(1) only contains vertices in $V = \varphi_i^{SCC}(\succeq) = \varphi_v^{SCC}(\succeq)$;

(2) contains both $v$ and $f$; and

(3) does not contain the edge $(v, v')$.

To see this, note first that there is a cycle in $\Gamma(\succeq)$ that satisfies (1) and (2) because $V$ is an SCC coalition in $\Gamma(\succeq)$ and $F(\succeq_v) \subseteq V$ (because $\succeq$ is SCC-minimal with respect to $V$). Next, each such cycle must contain edge $(v, f)$; otherwise $\succeq$ would not be SCC-minimal with respect to $V$ (the edge $(v, f)$ would be "redundant"). But then each such cycle consists of $(v, f)$ and a path back to $v$. Cutting the path when it returns to $v$ for the first time yields a cycle $c_f$ that satisfies (1), (2), and (3).

For each friend $f \in F(\succeq_v) \backslash \{v'\}$, $c_f$ is a cycle in $\Gamma(\widetilde{\succeq})$. Hence, by definition of $\varphi^{SCC}$, $(F(\succeq_v) \backslash \{v'\}) \subseteq \varphi_v^{SCC}(\widetilde{\succeq})$.

From Claim 6, $\varphi_v^{SCC}(\widetilde{\succeq}) = \varphi_v(\widetilde{\succeq})$. Hence, $(F(\succeq_v) \backslash \{v'\}) \subseteq \varphi_v(\widetilde{\succeq})$. Since $v' \notin U = \varphi_v(\succeq)$, it follows from *strategy-proofness* of $\varphi$ and Condition (F) of friend-oriented preferences that $(F(\succeq_v) \backslash \{v'\}) \subseteq \varphi_v(\succeq)$ (because otherwise, agent $v$, at profile $\succeq$, could report $\widetilde{\succeq}_v$ to move to profile $\widetilde{\succeq}$, at which he is matched with all his friends, except friend $v'$ whom he neither is matched with at $\varphi_v(\succeq)$). ■

**Claim 9.** *$\varphi_v(\succeq) = \varphi_v(\widetilde{\succeq})$.*

*Proof of Claim 9.* From Claim 7, $\varphi_v(\succeq) \subseteq \varphi_v(\widetilde{\succeq})$. Suppose $\varphi_v(\succeq) \subsetneq \varphi_v(\widetilde{\succeq})$. Then, Claim 8, together with $i \in \varphi_v(\succeq) \subseteq \varphi_v(\widetilde{\succeq})$ (Claim 7), yields a violation of *strategy-proofness* of $\varphi$ and Condition (E) of friend-oriented preferences (because agent $v$, at profile $\widetilde{\succeq}$, can report $\succeq_v$ to move to profile $\succeq$, at which he is still matched with all his $\widetilde{\succeq}_v$ friends but with fewer enemies). Hence, $\varphi_v(\succeq) = \varphi_v(\widetilde{\succeq})$. ■

Recall that $c_2 = (w_1 = i, w_2, \ldots, w_s, v_r = j, v_{r+1}, \ldots, v_m, v_1 = i)$ is a cycle in $\Gamma(\succeq^V)$. Since $1 \leq q \leq r-2$, $c_2$ does not contain the edge $(v, v') = (v_q, v_{q+1})$. Hence, $c_2$ is a cycle in $\Gamma(\widetilde{\succeq})$. So, $j \in \varphi_i^{SCC}(\widetilde{\succeq})$.

Next, note that either $v = i$ or $[v \neq i$ and $(v_1 = i, v_2, \ldots, v_q = v, v_1 = i)$ is a cycle in $\Gamma(\widetilde{\succeq})]$. Hence, $v \in \varphi_i^{SCC}(\widetilde{\succeq})$. Then, from $j \in \varphi_i^{SCC}(\widetilde{\succeq})$ it follows that $j \in \varphi_v^{SCC}(\widetilde{\succeq})$. Hence, from Claim 6 ($\varphi_v^{SCC}(\widetilde{\succeq}) = \varphi_v(\widetilde{\succeq})$) and Claim 9 ($\varphi_v(\widetilde{\succeq}) = \varphi_v(\succeq)$), we conclude that $j \in \varphi_v(\succeq)$. So, $v \in \varphi_j(\succeq)$.

By definition, $v = v_q \in U = \varphi_i(\succeq)$. Since $v \in \varphi_i(\succeq)$ and $v \in \varphi_j(\succeq)$, it follows that $\varphi_i(\succeq) = \varphi_j(\succeq)$. From (23), we have that $a, b \in \varphi_j(\succeq)$. Hence, $a, b \in \varphi_i(\succeq)$, which contradicts (22). This contradiction completes the proof. □

# G Appendix: $\varphi^{\neg C}$ is *Pareto-optimal* and *group strategy-proof* (Example 5)

***Proof of Pareto-optimality of $\varphi^{\neg C}$.*** Suppose, by contradiction, that for some $\succeq \in \widetilde{\mathcal{R}}^f$, $\pi^{\neg C} \equiv \varphi^{\neg C}(\succeq)$ is Pareto dominated by a partition $\pi$. Then, there is some $j \in N$ such that $\pi_j \succ_j \pi_j^{\neg C}$. The next claim (taking $S = \pi_j$) shows that there is an enemy $e \in E_j$ in agent $j$'s coalition $\pi_j^{\neg C}$ that is no longer present in his coalition $\pi_j$ at the Pareto dominating partition $\pi$. Since the claim is applied a second time, it is formulated slightly more generally.

Let $\{N^1, N^2, N^3\}$ be the partition of $N$ such that for each $k = 1, 2, 3$, $N^k$ is the (possibly empty) set of agents in $N$ that are assigned at Step $k$ to compute $\pi^{\neg C}$.

**Claim.** *Let $S \subseteq N$ be a non-empty coalition such that*

$$\text{for each } i \in S,\ S \succeq_i \pi_i^{\neg C} \text{ and } S \neq \pi_i^{\neg C}. \tag{24}$$

*Then, $S \cap (N^1 \cup N^2) = \emptyset$. Moreover, for each $j \in S \cap N^3$ with $S \succ_j \pi_j^{\neg C}$, there is an enemy $e \in E_j$ such that $e \in \pi_j^{\neg C}$ and $e \notin S$.*

*Proof of the claim.* We first show that $S \cap N^1 = \emptyset$. Suppose, by contradiction, that $S \cap N^1 \neq \emptyset$. Since being alone is the most preferred coalition for each of the agents in $N^1$, *individual rationality* of $\varphi^{\neg C}$ implies that for each $i \in S \cap N^1$, $\pi_i^{\neg C} = \{i\}$. Hence, from (24), $|S| = 1$. Let $S = S \cap N^1 = \{i\}$. Then, $S = \pi_i^{\neg C}$, which contradicts (24). This proves that $S \cap N^1 = \emptyset$.

Next, we show that $S \cap N^2 = \emptyset$. Suppose, by contradiction, that $S \cap N^2 \neq \emptyset$. Let $i \in S \cap N^2$. From Fact A and (24) it follows that $S \succeq_i \pi_i^{\neg C} = \{i\}$ and $S \neq \pi_i^{\neg C}$. Thus, since agent $i$'s preferences are friend-oriented, coalition $S$ contains at least one friend of agent $i$, say $\ell \in F_i \cap S$. Since agent $i$ was removed at some iteration of Step 2 to compute $\pi^{\neg C}$, his friend $\ell$ was removed earlier: either at Step 1 or at some earlier iteration of Step 2.[21] In particular, by Fact A, $\pi_\ell^{\neg C} = \{\ell\}$. Since $\ell \in S$, (24) implies that $S \succeq_\ell \pi_\ell^{\neg C} = \{\ell\}$. By $i \in S$, $S \neq \{\ell\}$. Thus, since agent $\ell$'s preferences are friend-oriented, coalition $S$ contains at least one friend of agent $\ell$, say $\ell' \in F_\ell \cap S$. Since agent $\ell$ was removed at (i) Step 1 or (ii) some iteration of Step 2, his friend $\ell'$ was removed at Step 1 (possible in both cases (i) and (ii)) or at an even earlier iteration of Step 2 (only possible in case (ii)). Therefore, in either case, $\ell' \neq i$. Since the number of agents is finite, there is a finite number of iterations of Step 2. Hence, repeating the previous arguments eventually identifies a "Step 1 agent" in coalition $S$; contradicting $S \cap N^1 = \emptyset$. This proves that $S \cap N^2 = \emptyset$ and thus the first part of the claim ($S \cap (N^1 \cup N^2) = \emptyset$).

To prove the second part of the claim, let $j \in S \cap N^3$ with $S \succ_j \pi_j^{\neg C}$. It follows from Fact B that Step 3 to compute $\pi^{\neg C}$ assigns agent $j$ to a non-singleton coalition that contains the non-empty set of all still present friends (but not any friends that were assigned at Steps 1 and 2), i.e.,

$$F_j \cap (N^1 \cup N^2) \cap \pi_j^{\neg C} = \emptyset \text{ and } F_j \cap N^3 \subseteq \pi_j^{\neg C}.$$

Then, since $S \succ_j \pi_j^{\neg C}$ and agent $j$ has friend-oriented preferences,

(1) there is a friend $f \in F_j \cap N^1$ such that $f \in S$; or

(2) there is a friend $f \in F_j \cap N^2$ such that $f \in S$; or

(3) there is an enemy $e \in E_j$ such that $e \in \pi_j^{\neg C}$ and $e \notin S$.

From the first part of the Claim, $S \cap N^1 = \emptyset$ and $S \cap N^2 = \emptyset$. Hence, (1) and (2) do not hold. So, (3) holds. This completes the proof of the second part of the claim. ■

Applying the claim to $S \equiv \pi_j$, it follows that there exists an enemy $e \in E_j$ such that $e \in \pi_j^{\neg C}$ and $e \notin S$.

Let $T \equiv \pi_j^{\neg C}$ and $T' \equiv \pi_j^{\neg C} \cap S$. Then, since $e \notin S$, $T' \subsetneq T$. Since $j \in T'$, $T' \neq \emptyset$. Since $e, j \in T$, $|T| > 1$, so that $T = \pi_j^{\neg C}$ is a coalition assigned at Step 3 to compute $\pi^{\neg C}$. From Fact C, it follows that there is some agent $\ell' \in T'$ and some agent in $\ell \in T \backslash T'$ such that $\ell \in F_{\ell'}$ or $\ell' \in F_\ell$. Note that since $\ell, \ell' \in T$ and $T$ is assigned at Step 3 to compute $\pi^{\neg C}$, $\ell, \ell' \in N^3$. Thus,

$$\ell' \in F_\ell \cap N^3 \text{ or } \ell \in F_{\ell'} \cap N^3. \tag{25}$$

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[21] By definition of $\varphi^{\neg C}$, at the moment that an agent is removed at Step 2, none of his friends is present.

Since $\ell' \in T' \subseteq S = \pi_j$ and $\ell \notin S = \pi_j$, agents $\ell$ and $\ell'$ are in different coalitions of partition $\pi$. Hence,

$$\ell' \notin \pi_\ell \text{ and } \ell \notin \pi_{\ell'}. \tag{26}$$

From (25) and (26),

$$F_\ell \cap N^3 \nsubseteq \pi_\ell \text{ or } F_{\ell'} \cap N^3 \nsubseteq \pi_{\ell'}. \tag{27}$$

Since $\ell, \ell' \in T = \pi_j^{\neg C}$, $\pi_\ell^{\neg C} = T = \pi_{\ell'}^{\neg C}$. But then, since also $\pi_\ell \neq \pi_{\ell'}$, it follows that $\pi_\ell \neq \pi_\ell^{\neg C}$ and $\pi_{\ell'} \neq \pi_{\ell'}^{\neg C}$. Thus, from the claim (applied to coalitions $\pi_\ell$ and $\pi_{\ell'}$),

$$\pi_\ell \cap (N^1 \cup N^2) = \emptyset \text{ and } \pi_{\ell'} \cap (N^1 \cup N^2) = \emptyset.$$

In particular,

$$F_\ell \cap \pi_\ell \cap (N^1 \cup N^2) = \emptyset \text{ and } F_{\ell'} \cap \pi_{\ell'} \cap (N^1 \cup N^2) = \emptyset. \tag{28}$$

It follows from Fact B that Step 3 to compute $\pi^{\neg C}$ assigns agent $\ell$ (agent $\ell'$, respectively) to a non-singleton coalition that contains the non-empty set of all still present friends (but not any friends that were assigned at Steps 1 and 2), i.e.,

$$F_\ell \cap N^3 \subseteq \pi_\ell^{\neg C} \text{ and } F_{\ell'} \cap N^3 \subseteq \pi_{\ell'}^{\neg C}. \tag{29}$$

Assume that in (27), $F_\ell \cap N^3 \nsubseteq \pi_\ell$. Since $N^1$, $N^2$, and $N^3$ partition the set of agents $N$, $F_\ell \subseteq N^1 \cup N^2 \cup N^3$. By (27), $\pi_\ell$ does not contain all of $\ell$'s friends in $N^3$. Furthermore, by (28), $\pi_\ell$ does not contain any of $\ell$'s friends in $N^1 \cup N^2$. Next, by (29), $\pi_\ell^{\neg C}$ contains all of $\ell$'s friends in $N^3$ and by definition of $\varphi^{\neg C}$, $\pi_\ell^{\neg C}$ does not contain any of $\ell$'s friends in $N^1 \cup N^2$. Thus, the set of friends assigned to $\ell$ at $\pi_\ell$ is a strict subset of the set of friends assigned to $\ell$ at $\pi_\ell^{\neg C}$. Then, Condition (F) of friend-orientated preferences implies that $\pi_\ell^{\neg C} \succ_\ell \pi_\ell$. If in (27), $F_{\ell'} \cap N^3 \nsubseteq \pi'_\ell$, then it similarly follows that $\pi_{\ell'}^{\neg C} \succ_{\ell'} \pi_{\ell'}$. Hence,

$$\pi_\ell^{\neg C} \succ_\ell \pi_\ell \text{ or } \pi_{\ell'}^{\neg C} \succ_{\ell'} \pi_{\ell'};$$

contradicting that $\pi^{\neg C}$ is Pareto dominated by partition $\pi$. This completes the proof of *Pareto-optimality* of mechanism $\varphi^{\neg C}$. □

***Proof of group strategy-proofness of $\varphi^{\neg C}$.*** Suppose, by contradiction, that there exists a problem $\succeq \in \widetilde{\mathcal{R}}^f$ and a coalition $S \subseteq N$ with preferences $\succeq'_S \in \prod_{i \in S} \widetilde{\mathcal{R}}_i^f$ such that

(g1). for each $i \in S$, $\varphi_i^{\neg C}(\succeq'_S, \succeq_{-S}) \succeq_i \varphi_i^{\neg C}(\succeq)$ and

(g2). for some $j \in S$, $\varphi_j^{\neg C}(\succeq'_S, \succeq_{-S}) \succ_j \varphi_j^{\neg C}(\succeq)$.

Let $\succeq' \equiv (\succeq'_S, \succeq_{-S})$. Let $\pi \equiv \varphi^{\neg C}(\succeq)$ and $\pi' \equiv \varphi^{\neg C}(\succeq')$.

We will show that $\varphi^{\neg C}(\succeq) = \varphi^{\neg C}(\succeq')$, which contradicts (g2) and hence completes the proof.

Let $\{N^1, N^2, N^3\}$ be the partition of $N$ such that for each $k = 1, 2, 3$, $N^k$ is the (possibly empty) set of agents in $N$ that are assigned at Step $k$ to compute $\pi$.

For each $k = 1, 2, 3$, let $S^k \subseteq S$ be the agents in $S$ that are assigned to a coalition at Step $k$ to compute $\pi$. Similarly, for each $k = 1, 2, 3$, let $\bar{S}^k \subseteq N \backslash S$ be the agents in $N \backslash S$ that are assigned to a coalition at Step $k$ to compute $\pi$. Note that for each $k = 1, 2, 3$, $N^k = S^k \cup \bar{S}^k$.

We first show that

$$\text{for each } i \in N^1 = S^1 \cup \bar{S}^1,\ \pi'_i = \pi_i = \{i\}. \tag{30}$$

By definition of $N^1$, for each $i \in N^1$, agent $i$'s set of friends at $\succeq$ is empty and $\pi_i = \{i\}$. Hence, for each agent $i \in N^1$, the (unique) most preferred coalition at $\succeq$ is the singleton $\{i\}$. Thus, it follows from (g1) that for each $i \in S^1$, $\pi'_i = \{i\}$. For each $i \in \bar{S}^1$, $\succeq'_i = \succeq_i$, so that from Step 1 to compute $\pi'$, $\pi'_i = \{i\}$. This completes the proof of (30).

Next, we show that

$$\text{for each } i \in N^2 = S^2 \cup \bar{S}^2,\ \pi'_i = \pi_i = \{i\}. \tag{31}$$

Suppose to the contrary that (31) does not hold. Then, at $\succeq'$, $\varphi^{\neg C}$ assigns some agent in $N^2$ to a non-singleton coalition. Step 2 to compute $\pi$ consists of possibly multiple iterations where exactly all agents in $N^2$ are assigned (specifically, they become singleton coalitions, see Fact A). Consider the first iteration $t^*$ of Step 2 to compute $\pi$ where some agent $j \in N^2$ is assigned to the singleton coalition $\pi_j = \{j\}$ but $\pi'_j \neq \{j\}$. By definition of $\varphi^{\neg C}$, each (true) friend $h \in F_j$ of agent $j$ is "removed from problem $\succeq$," i.e., not assigned to agent $j$, either (a) at Step 1 to compute $\pi$ or (b) at some iteration $t < t^*$ of Step 2 to compute $\pi$. In Case (a), $h \in N^1$ and (30) implies that $\pi'_h = \{h\}$. In Case (b), $h \in N^2$ and, by definition of iteration $t^*$, $\pi'_h = \{h\}$. Hence, $\pi'_j \cap F_j = \emptyset$. Since $\pi'_j \neq \{j\}$, $\varphi^{\neg C}_j(\succeq) = \pi_j = \{j\} \succ_j \pi'_j = \varphi^{\neg C}_j(\succeq')$; which contradicts (g1) if $j \in S^2$ and *individual rationality* of $\pi' = \varphi^{\neg C}(\succeq')$ if $j \in \bar{S}^2$. This contradiction completes the proof of (31).

Finally, we show that

$$\text{for each } i \in N^3 = S^3 \cup \bar{S}^3,\ \pi'_i = \pi_i. \tag{32}$$

We first consider $S^3$. It follows from Fact B that at $\pi$ each agent $i \in S^3$ is in a coalition that contains all friends that are still present at Step 3 to compute $\pi$, i.e.,

$$\text{for each } i \in S^3,\ \emptyset \neq F_i \cap N^3 \subseteq \pi_i. \tag{33}$$

Then, from (30), (31), (g1), and by Condition (F) of friend-oriented preferences, at $\pi'$ each agent $i \in S^3$ is still in a coalition with all of these friends, i.e.,

$$\text{for each } i \in S^3,\ \emptyset \neq F_i \cap N^3 \subseteq \pi'_i. \tag{34}$$

In particular, for each $i \in S^3$, since $F_i \cap N^3 \neq \emptyset$, coalition $\pi'_i$ contains at least one agent different from $i$, which by Fact B implies that agent $i$ is assigned to a (non-singleton) coalition at Step 3 to compute $\pi'$. Furthermore, for each $i \in S^3$, (30) and (31) imply that $\pi_i$ and $\pi'_i$ do not contain any of $i$'s true friends in $N^1 \cup N^2$ while (33) and (34) imply that both $\pi_i$ and $\pi'_i$ contain all of $i$'s true friends in $N^3$.

Next, we consider $\bar{S}^3$. It follows from Fact B that at $\pi$ each agent $i \in \bar{S}^3$ is in a coalition that contains all friends that are still present at Step 3 to compute $\pi$, i.e.,

$$\text{for each } i \in \bar{S}^3,\ \emptyset \neq F_i \cap N^3 \subseteq \pi_i. \tag{35}$$

Since $\bar{S}^3 \subseteq N \backslash S$,

$$\text{for each } i \in \bar{S}^3,\ \succeq'_i = \succeq_i . \tag{36}$$

From (30), (31), (34), (35), and (36), it follows that each agent $i \in \bar{S}^3$ is assigned to a coalition at Step 3 to compute $\pi'$ and

$$\text{for each } i \in \bar{S}^3,\ \emptyset \neq F_i \cap N^3 \subseteq \pi'_i. \tag{37}$$

Note that for each $i \in \bar{S}^3$, (30) and (31) imply that $\pi_i$ and $\pi'_i$ do not contain any of $i$'s true friends in $N^1 \cup N^2$ while (35) and (37) imply that both $\pi_i$ and $\pi'_i$ contain all of $i$'s true friends in $N^3$.

Equations (34) and (37) imply that at Step 3 to compute $\pi'$, each of the remaining agents (which by (30) and (31) is the set $N^3 = S^3 \cup \bar{S}^3$) is assigned to a coalition that contains his coalition at $\pi$. In other words,

$$\text{for each } i \in N^3 = S^3 \cup \bar{S}^3,\ \pi_i \subseteq \pi'_i. \tag{38}$$

Let $i \in S^3$ and assume that $\pi_i \subsetneq \pi'_i$. As mentioned after equation (34), both $\pi_i$ and $\pi'_i$ contain all of $i$'s true friends in $N^3$. Hence, by (38), the only difference between $\pi_i$ and $\pi'_i$ is that $\pi_i$ contains fewer enemies than $\pi'_i$ and by Condition (E) of friend-oriented preferences, agent $i$ strictly prefers $\pi_i$ to $\pi'_i$; contradicting (g1). Hence,

$$\text{for each agent } i \in S^3,\ \pi_i = \pi'_i. \tag{39}$$

Let $i \in \bar{S}^3$. If $\pi_i \cap S^3 \neq \emptyset$, then (39) implies $\pi_i = \pi'_i$. If $\pi_i \cap S^3 = \emptyset$, then (36) together with (39) imply $\pi_i = \pi'_i$. Hence,

$$\text{for each agent } i \in \bar{S}^3,\ \pi_i = \pi'_i. \tag{40}$$

Equations (39) and (40) complete the proof of (32).

From (30), (31), and (32), $\varphi^{\neg C}(\succeq) = \pi = \pi' = \varphi^{\neg C}(\succeq')$, which contradicts (g2). This completes the proof of *group strategy-proofness* of mechanism $\varphi^{\neg C}$. □

# H Appendix: (Non-)existence of a [*core stable* and *group strategy-proof*] mechanism on subdomains of friend-oriented preferences with neutrals

We already observed that the assumption of preferences being very rich in Theorem 4 is rather strong because the proof only requires the existence of a problem where some agent has at least two neutrals (Example 3). Formally, the proof of Theorem 4 implies the following proposition.

**Proposition 6.** *Let* $\widetilde{\mathcal{R}}^{fn}$ *be a domain of friend-oriented problems with neutrals such that*

(i) $\widetilde{\mathcal{R}}^{fn} \cap \mathcal{R}^{f}$ *is rich and*

(ii) *it contains a problem where some agent has at least two neutrals.*

*Then, no mechanism on* $\widetilde{\mathcal{R}}^{fn}$ *is* core stable *and* group strategy-proof.

The following proposition shows that condition (ii) in Proposition 6 can be replaced by the requirement that the domain of friend-oriented problems with neutrals contains a problem where some agent has exactly one neutral and at least one friend.

**Proposition 7.** *Let* $\widetilde{\mathcal{R}}^{fn}$ *be a domain of friend-oriented problems with neutrals such that*

(i) $\widetilde{\mathcal{R}}^{fn} \cap \mathcal{R}^{f}$ *is rich and*

(ii)' *it contains a problem where some agent has exactly one neutral and at least one friend.*

*Then, no mechanism on* $\widetilde{\mathcal{R}}^{fn}$ *is* core stable *and* group strategy-proof.

***Proof***. We prove the proposition for $N = \{1, 2, 3\}$.[22] Suppose that there is a *core stable* and *group strategy-proof* mechanism $\varphi$ on $\widetilde{\mathcal{R}}^{fn}$.

From conditions (i) and (ii)', it follows without loss of generality that there exists $\succeq \in \widetilde{\mathcal{R}}^{fn}$ such that $F_1 = \{2\}$, $E_1 = \{3\}$, $N_1 = \emptyset$, $F_2 = 1$, $E_2 = \emptyset$, $N_2 = \{3\}$, $F_3 = \{1\}$, $E_3 = \{2\}$, and $N_3 = \emptyset$. Then, from conditions (E), (F), and (N), the friend-oriented preferences with neutrals are as follows,

| $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|
| 12 | $12 \sim 123$ | 13 |
| 123 | $2 \sim 23$ | 123 |
| 1 | | 3 |
| 13 | | 23 |

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[22] To extend the proof to more agents, one can add agents with preferences where all other agents are enemies.

One easily verifies that $SC(\succeq) = \{\{(12),(3)\}\}$ and $C(\succeq) = \{\{(12),(3)\},\{(123)\}\}$.

We show that for each possible candidate partition $\varphi(\succeq) \in C(\succeq)$, there is a successful group manipulation by some coalition.

Suppose that $\varphi(\succeq) = \{(12),(3)\}$. Then, by condition (i), agent 2 can report preferences $\succeq_2'$ where agent 3 is his (unique) friend and agent 1 is his (unique) enemy so that for $\succeq' \equiv (\succeq_1, \succeq_2', \succeq_3)$ the unique core partition is $\{(123)\}$. Then, $\varphi(\succeq') = \{(123)\}$ and $(\succeq_2', \succeq_3)$ is a successful group manipulation by coalition (23).

Suppose that $\varphi(\succeq) = \{(123)\}$. Then, by condition (i), agent 2 can report preferences $\succeq_2''$ where agent 1 is his (unique) friend and agent 3 is his (unique) enemy so that for $\succeq'' \equiv (\succeq_1, \succeq_2'', \succeq_3)$ the unique core partition is $\{(12),(3)\}$. Then, $\varphi(\succeq'') = \{(12),(3)\}$ and $(\succeq_1, \succeq_2'')$ is a successful group manipulation by coalition (12).

We conclude that there is no *core stable* and *group strategy-proof* mechanism on $\widetilde{\mathcal{R}}^{fn}$. $\square$

Next, we discuss (im)possibilities for a domain of friend-oriented problems with neutrals such that neither condition (ii) nor condition (ii)' holds. Let $\widetilde{\mathcal{R}}^{fn}$ be a domain of friend-oriented problems with neutrals such that

(i) $\widetilde{\mathcal{R}}^{fn} \cap \mathcal{R}^f$ is rich.

Assume that $\widetilde{\mathcal{R}}^{fn}$ contains some problem where some agent has a neutral, i.e., $\widetilde{\mathcal{R}}^{fn} \not\subseteq \mathcal{R}^f$.[23] Suppose that neither condition (ii) of Proposition 6 nor condition (ii)' of Proposition 7 holds. Then, for each problem in $\widetilde{\mathcal{R}}^{fn}$ where some agent has a neutral, for each agent with a neutral, the neutral is unique and all other agents are enemies. Formally,

(ii)" for each problem $\succeq \in \widetilde{\mathcal{R}}^{fn}$ and for each agent $i$ such that $N(\succeq_i) \neq \emptyset$, $|N(\succeq_i)| = 1$ and $|E(\succeq_i)| = n-2$ (i.e., $|F(\succeq_i)| = 0$).

Now, a natural question to ask is:

> Under conditions (i) and (ii)", does there exist a *core stable* and *group strategy-proof* mechanism?

As the following two examples show, the answer depends on the specific domain $\widetilde{\mathcal{R}}^{fn}$.

**Example 11 (Adding preferences with one neutral and no friends to the domain of lexicographically friend-oriented preferences).** Let $N = \{1,2,3\}$.[24] Let $\widetilde{\mathcal{R}}^{fn} \equiv \prod_{i \in N} \widetilde{\mathcal{R}}_i^{fn}$ be a domain of problems such that

- for each agent $i = 1,2,3$, $\mathcal{R}_i^{lf} \subseteq \widetilde{\mathcal{R}}_i^{fn}$ and

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[23] Note that if $\widetilde{\mathcal{R}}^{fn} \subseteq \mathcal{R}^f$, then Theorem 3 and Corollary 4 can be applied, i.e., $\varphi^{SCC}$ is the unique *core stable* and *group strategy-proof* mechanism on $\widetilde{\mathcal{R}}^{fn}$.

[24] To extend the discussion to more agents, one can add agents with preferences where all other agents are enemies.

- $\succeq_3^* \in \widetilde{\mathcal{R}}_3^{fn}$,

where $\succeq_3^*$ are preferences such that for agent 3, agent 2 is a neutral and agent 1 is an enemy (i.e., agent 3 has no friends). Formally, we define $\succeq^* \in \widetilde{\mathcal{R}}^{fn}$ such that

| $\succeq_1^*$ | $\succeq_2^*$ | $\succeq_3^*$ |
|---|---|---|
| 12 | 123 | $3 \sim 23$ |
| 123 | 23 | $13 \sim 123$ |
| 1 | 12 | |
| 13 | 2 | |

One easily verifies that $SC(\succeq^*) = \{\{(1),(23)\}\}$ and $C(\succeq^*) = \{\{(12),(3)\},\{(1),(23)\}\}$.

Note that since $\mathcal{R}^{lf} \subseteq \widetilde{\mathcal{R}}^{fn}$ and since $\mathcal{R}^{lf}$ is rich, $\widetilde{\mathcal{R}}^{fn} \cap \mathcal{R}^f$ is rich. Hence, $\widetilde{\mathcal{R}}^{fn}$ satisfies (i) and (ii)". Suppose that there is a *core stable* and *group strategy-proof* mechanism $\varphi$ on $\widetilde{\mathcal{R}}^{fn}$.

Suppose that $\varphi(\succeq^*) = \{(12),(3)\}$. Then, agents 2 and 3 can report preferences $(\succeq_2', \succeq_3')$ such that they are each other's unique friend (and agent 1 is their common enemy). Then, $\succeq' \equiv (\succeq_1^*, \succeq_2', \succeq_3') \in \mathcal{R}^{lf} \subseteq \widetilde{\mathcal{R}}^{fn}$. Since the unique core partition at $\succeq'$ is $\{(1),(23)\}$, we have that $\varphi(\succeq') = \{(1),(23)\}$ and $(\succeq_2', \succeq_3')$ is a successful group manipulation by coalition (23).

Suppose that $\varphi(\succeq^*) = \{(1),(23)\}$. Then, agent 2 can report lexicographical preferences $\succeq_2''$: $123, 12, 23, 2$ (i.e., he switches coalitions (12) and (23) in $\succeq_2^*$). Then, $\succeq'' \equiv (\succeq_1^*, \succeq_2'', \succeq_3^*) \in \widetilde{\mathcal{R}}^{fn}$. Since the unique core partition at $\succeq''$ is $\{(12),(3)\}$, we have that $\varphi(\succeq'') = \{(12),(3)\}$ and $\succeq_2''$ is a successful manipulation by agent 2.

Hence, there is no *core stable* and *group strategy-proof* mechanism $\varphi$ on $\widetilde{\mathcal{R}}^{fn}$. ◇

**Example 12 (Adding preferences with one neutral and no friends to the domain of preferences that satisfy appreciation of friends).** Let $N = \{1,2,3\}$.[25] Let $\widetilde{\mathcal{R}}^{fn} \equiv \prod_{i \in N} \widetilde{\mathcal{R}}_i^{fn}$ be the domain of problems such that

- for each $i = 1, 2$, $\widetilde{\mathcal{R}}_i^{fn} = \mathcal{R}_i^{af}$ and
- $\widetilde{\mathcal{R}}_3^{fn} = \mathcal{R}_3^{af} \cup \{\succeq_3^*\}$,

where, as in Example 11, $\succeq_3^*$ are preferences such that for agent 3, agent 2 is a neutral and agent 1 is an enemy (i.e., agent 3 has no friends). Note that since $\mathcal{R}^{af} \subseteq \widetilde{\mathcal{R}}^{fn}$ and since $\mathcal{R}^{af}$ is rich, $\widetilde{\mathcal{R}}^{fn} \cap \mathcal{R}^f$ is rich. Hence, $\widetilde{\mathcal{R}}^{fn}$ satisfies (i) and (ii)".

Is $\varphi^{SCC}$ a *core stable* and *group strategy-proof* mechanism on $\widetilde{\mathcal{R}}^{fn}$? It follows immediately from Theorem 2 that mechanism $\varphi^{SCC}$ on $\widetilde{\mathcal{R}}^{fn}$ is *core stable*. However, $\varphi^{SCC}$ is not *group strategy-proof* on $\widetilde{\mathcal{R}}^{fn}$. This can be easily seen as follows. Let $\succeq \in \widetilde{\mathcal{R}}^{fn}$ such that agent 1 reports no friends, agent 2 reports that agent 3 is a friend, and agent 3 reports $\succeq_3^*$. Formally, $F(\succeq_1) = \emptyset$, $3 \in F(\succeq_2)$, and $\succeq_3 = \succeq_3^*$. Then, $\varphi^{SCC}(\succeq) = \{(1),(2),(3)\}$. Similarly to Example 11,

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[25] To extend the discussion to more agents, one can add agents with preferences where all other agents are enemies.

agents 2 and 3 can report preferences $(\succeq'_2, \succeq'_3)$ such that they are each other's unique friend (and agent 1 is their common enemy). Then, $\succeq' \equiv (\succeq_1, \succeq'_2, \succeq'_3) \in \widetilde{\mathcal{R}}^{fn}$. Since the unique core partition at $\succeq'$ is $\{(1), (23)\}$, we have that $\varphi^{SCC}(\succeq') = \{(1), (23)\}$ and $(\succeq'_2, \succeq'_3)$ is a successful group manipulation by coalition (23). Hence, $\varphi^{SCC}$ is not *group strategy-proof* on $\widetilde{\mathcal{R}}^{fn}$.

Next, we show the existence of a mechanism that is *core stable* and *group strategy-proof* on $\widetilde{\mathcal{R}}^{fn}$. Specifically, we slightly adjust the SCC mechanism: if agent 1 reports no friends, agent 2 reports that agent 3 is a friend, and agent 3 reports $\succeq^*_3$, then we turn agent 3's neutral (agent 2) into a friend and apply the SCC mechanism; otherwise, the SCC mechanism is applied directly. Formally, let $\varphi^*$ be the mechanism that is defined as follows. For each $\succeq \in \widetilde{\mathcal{R}}^{fn}$,

$$\varphi^*(\succeq) \equiv \begin{cases} \varphi^{SCC}(\succeq_1, \succeq_2, \bar{\succeq}_3) = \{(1), (23)\} & \text{if } F(\succeq_1) = \emptyset,\ 3 \in F(\succeq_2), \text{ and } \succeq_3 = \succeq^*_3; \\ \varphi^{SCC}(\succeq) & \text{otherwise,} \end{cases}$$

where $\bar{\succeq}_3 \in \mathcal{R}^{af}_3$ are preferences of agent 3 such that agent 2 is the unique friend and agent 1 is the unique enemy.

We prove that $\varphi^*$ is *core stable*. Let $\succeq \in \widetilde{\mathcal{R}}^{fn}$. If $F(\succeq_1) = \emptyset$, $3 \in F(\succeq_2)$, and $\succeq_3 = \succeq^*_3$, then one easily verifies that $\varphi^*(\succeq) \in C(\succeq)$. Otherwise, it follows from Theorem 2 that $\varphi^*(\succeq) \in C(\succeq)$. Hence, $\varphi^*$ is *core stable* on $\widetilde{\mathcal{R}}^{fn}$.

Finally, we prove that $\varphi^*$ is *group strategy-proof* on $\widetilde{\mathcal{R}}^{fn}$. Suppose that $\varphi^*$ is not *group strategy-proof*. Then, there exists a problem $\widetilde{\mathcal{R}}^{fn}$ and a coalition $S \subseteq N$ with preferences $\succeq'_S \in \widetilde{\mathcal{R}}^{fn}_S$ such that

(a) for each $i \in S$, $\varphi^*_i(\succeq'_S, \succeq_{-S}) \succeq_i \varphi^*_i(\succeq)$ and

(b) for some $j \in S$, $\varphi^*_j(\succeq'_S, \succeq_{-S}) \succ_j \varphi^*_j(\succeq)$.

**Case 1.** $F(\succeq_1) = \emptyset$, $3 \in F(\succeq_2)$, and $\succeq_3 = \succeq^*_3$.
Then, at $\varphi^*(\succeq)$ each agent is assigned to a most preferred coalition. This contradicts condition (b).

**Case 2.** $F(\succeq_1) \neq \emptyset$, $3 \notin F(\succeq_2)$, or $\succeq_3 \neq \succeq^*_3$.
Suppose $F(\succeq'_1) \neq \emptyset$, $3 \notin F(\succeq'_2)$, or $\succeq'_3 \neq \succeq^*_3$. Then, we have both $\varphi^*(\succeq) = \varphi^{SCC}(\succeq)$ and $\varphi^*(\succeq') = \varphi^{SCC}(\succeq')$. Then, conditions (a) and (b) yield a contradiction with *group strategy-proofness* of the SCC mechanism on $\widetilde{\mathcal{R}}^{fn}$ (Proposition 3).

Now suppose $F(\succeq'_1) = \emptyset$, $3 \in F(\succeq'_2)$, and $\succeq'_3 = \succeq^*_3$. Then, $\varphi^*(\succeq') = \{(1), (23)\}$.

We first prove that agents 2 and 3 are mutual friends at $\succeq$. Since $\succeq'_3 \neq \succeq_3$, agent 3 changes his strategy from $\succeq$ to $\succeq'$ and hence is member of coalition $S$. From condition (a) it follows that $(23) = \varphi^*_3(\succeq') \succeq_3 \varphi^*_3(\succeq) \succeq_3 (3)$. Then, since agent 3 has no neutrals at $\succeq_3$, it follows that agent 2 is a friend of agent 3 at $\succeq_3$.

Finally, we show that agent 3 is a friend of agent 2 at $\succeq_2$. In fact, if agent 2 is member of coalition $S$, then the same arguments as before can be applied. Now suppose agent 2 is not a member of $S$. Then, $\succeq_2'=\succeq_2$ and $\{(1),(23)\} = \varphi^*(\succeq') = \varphi^{SCC}(\succeq_1', \succeq_2, \succeq_3^*)$. Thus, $\{2,3\}$ is a strongly connected component at $(\succeq_1', \succeq_2, \succeq_3^*)$. Hence, agent 3 is a friend of agent 2 at $\succeq_2$.

From condition (b) it follows that $\varphi^*(\succeq') \neq \varphi^*(\succeq)$. Since $\varphi_2^*(\succeq') = (23)$, it follows that $\varphi_2^*(\succeq) = (123)$, $\varphi_2^*(\succeq) = (12)$, or $\varphi_2^*(\succeq) = (2)$. However, since agents 2 and 3 are mutual friends at $\succeq$ and $\varphi^*(\succeq) = \varphi^{SCC}(\succeq)$, we have $3 \in \varphi_2^*(\succeq)$. Thus, $\varphi_2^*(\succeq) \neq (12)$ and $\varphi_2^*(\succeq) \neq (2)$. Hence, $\varphi_2^*(\succeq) = (123)$. Since $\varphi^*(\succeq) = \varphi^{SCC}(\succeq)$, $\varphi_1^{SCC}(\succeq) = (123)$. Hence, agent 1 has at least one friend at $\succeq_1$. Then, $\varphi_1^*(\succeq) = (123) \succ_1 (1) = \varphi_1^*(\succeq')$. From condition (a) it follows that agent 1 is not member of coalition $S$. Then, $\succeq_1'=\succeq_1$. Since $F(\succeq_1) \neq \emptyset$, $F(\succeq_1') \neq \emptyset$, which contradicts the assumption that $F(\succeq_1') = \emptyset$. This completes the proof of *group strategy-proofness* of $\varphi^*$ on $\widetilde{\mathcal{R}}^{fn}$.

We conclude that $\varphi^*$ is a *core stable* and *group strategy-proof* mechanism on $\widetilde{\mathcal{R}}^{fn}$. $\diamond$

## I Appendix: Proof of Proposition 5

We prove that on each subdomain $\widetilde{\mathcal{R}}^{fn}$ of friend-oriented preferences with neutrals, the SCC mechanism is *weakly group strategy-proof* (Proposition 5).

***Proof***. Suppose that $\varphi^{SCC}$ is not *weakly group strategy-proof* on some subdomain $\widetilde{\mathcal{R}}^{fn}$ of friend-oriented preferences with neutrals. Then, there exist a problem $\succeq \in \widetilde{\mathcal{R}}^{fn}$, a coalition $S \subseteq N$, and $\succeq_S' \in \prod_{j\in S} \widetilde{\mathcal{R}}_i$ such that for each $j \in S$,

$$\varphi_j^{SCC}(\succeq_S', \succeq_{-S}) \succ_j \varphi_j^{SCC}(\succeq). \tag{41}$$

For each $i \in N$, let $F_i$ and $E_i$ denote the set of friends and enemies of agent $i$ at $\succeq$. Let $G_1, \ldots, G_K$ be the strongly connected components of graph $\Gamma(\succeq)$. For each $1 \leq k \leq K$, let $G_k = (V_k, A_k)$. Based on the labeling of strongly connected components $G_1, \ldots, G_K$ according to Fact 1, for all $l, l' \in \{1, \ldots, K\}$ with $l < l'$, graph $\Gamma(\succeq)$ contains no edge from any vertex in $V_{l'}$ to any vertex in $V_l$. By definition of $\varphi^{SCC}$, for each $l \in \{1, \ldots, K\}$ and each $i \in V_l$, $\varphi_i^{SCC}(\succeq) = V_l$.

We will complete the proof by showing that for each $l \in \{1, \ldots, K\}$, $S \cap V_l = \emptyset$, which contradicts $\emptyset \neq S = S \cap N = S \cap \bigcup_{l=1}^{K} V_l$. Let $\succeq' \equiv (\succeq_S', \succeq_{-S})$. First we consider $V_K$.

**Case *K*.** We will prove that $S \cap V_K = \emptyset$ and that for each $i \in V_K$, $\varphi_i^{SCC}(\succeq') = V_K$.

Let $j \in S$. Suppose to the contrary that $j \in V_K$. We will obtain a contradiction with assumption (41). It follows from Fact 2 that agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with all his friends, i.e.,

$$F_j \subseteq V_K = \varphi_j^{SCC}(\succeq). \tag{42}$$

Then, from (41) and by Condition (F) of friend-oriented preferences with neutrals, agent $j$ in coalition $\varphi_j^{SCC}(\succeq')$ is still together with all his friends, i.e.,

$$F_j \subseteq \varphi_j^{SCC}(\succeq'). \tag{43}$$

Next, we prove that if agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with an enemy $e$, then that enemy is also in his coalition $\varphi_j^{SCC}(\succeq')$, i.e.,

$$E_j \cap \varphi_j^{SCC}(\succeq) \subseteq E_j \cap \varphi_j^{SCC}(\succeq'). \tag{44}$$

Suppose, by contradiction, that agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with an enemy $e$ who is not in his coalition $\varphi_j^{SCC}(\succeq')$, i.e., $e \in E_j \cap (\varphi_j^{SCC}(\succeq) \setminus \varphi_j^{SCC}(\succeq'))$. Since $\varphi_j^{SCC}(\succeq) = V_K$, $e \in V_K$. By definition of $\varphi^{SCC}(\succeq')$,

$$\text{agents } j \text{ and } e \text{ are in distinct SCC coalitions of } \Gamma(\succeq'). \tag{45}$$

For each $h \in V_K$, let $V'(h)$ denote the SCC coalition of $\Gamma(\succeq')$ that contains agent $h$. By definition of $V'(h)$, $V'(h) \cap V_K \neq \emptyset$. Moreover, from $j, e \in V_K$ and (45) it follows that $|\{V'(h)\}_{h \in V_K}| \geq 2$. Since the condensation graph of $\Gamma(\succeq')$ is acyclic, let $V' \in \{V'(h)\}_{h \in V_K}$ be an SCC coalition without an outgoing edge to any of the other SCC coalitions in $\{V'(h)\}_{h \in V_K} \setminus \{V'\}$.[26] Hence, there is no edge from any vertex in $V'$ to any vertex in $[\bigcup_{h \in V_K} V'(h)] \setminus V'$. In particular, in $\Gamma(\succeq')$, there is no edge from any vertex in $V' \cap V_K$ to any vertex in

$$\left[\bigcup_{h \in V_K} (V'(h) \cap V_K)\right] \setminus (V' \cap V_K) = V_K \setminus (V' \cap V_K).$$

However, since

$$[V' \cap V_K] \cup [V_K \setminus (V' \cap V_K)] = V_K$$

is an SCC coalition of $\Gamma(\succeq)$, there is an edge from some vertex in $V' \cap V_K$ to some vertex in $V_K \setminus (V' \cap V_K)$.[27] Let $(i^*, j^*)$ be an edge from $V' \cap V_K$ to $V_K \setminus (V' \cap V_K)$ in $\Gamma(\succeq)$.

In particular,

$$j^* \notin V'. \tag{46}$$

Since $(i^*, j^*)$ is an edge in $\Gamma(\succeq)$,

$$j^* \in F_{i^*}. \tag{47}$$

Since there is no edge from any vertex in $V' \cap V_K$ to any vertex in $V_K \setminus (V' \cap V_K)$ in $\Gamma(\succeq')$, $(i^*, j^*)$ is not an edge in $\Gamma(\succeq')$. Then, since only agents in $S$ change preferences from $\succeq$ to $\succeq'$,

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[26] Note that since $|\{V'(h)\}_{h \in V_K}| \geq 2$, $|\{V'(h)\}_{h \in V_K} \setminus \{V'\}| \geq 1$.

[27] If $\widetilde{V}$ is an SCC coalition of a graph, then for each $T \subsetneq \widetilde{V}$ with $T \neq \emptyset$, there is an edge from some vertex in $T$ to some vertex in $\widetilde{V} \setminus T$.

we conclude that

$$i^* \in S \cap V_K.$$

Then, we can use arguments similar to those that established (42) and (43) to obtain $F_{i^*} \subseteq \varphi_{i^*}^{SCC}(\succeq')$. From (47), $j^* \in \varphi_{i^*}^{SCC}(\succeq')$. However, since by definition of $V'$, $\varphi_{i^*}^{SCC}(\succeq') = V'$, we obtain $j^* \in V'$; contradicting (46). This proves (44).

Hence, through (42), (43), and (44), we have now shown that in coalition $\varphi_j^{SCC}(\succeq')$ agent $j$ is still together with all his friends and together with the same enemies as before. Thus, Conditions (E) and (N) of friend-oriented preferences with neutrals imply that $\varphi_j^{SCC}(\succeq) \succeq_j \varphi_j^{SCC}(\succeq')$ which contradicts (41). We conclude that $j \notin V_K$. Hence, $S \cap V_K = \emptyset$.

Finally, note that graph $\Gamma(\succeq)$ contains no edge from $V_K$ to $V_1 \cup \cdots \cup V_{K-1}$. Furthermore, since $S \cap V_K = \emptyset$, $\succeq'_{V_K} = \succeq_{V_K}$. Thus, $V_K$ is an SCC coalition of graph $\Gamma(\succeq')$. Then, for each $i \in V_K$, $\varphi_i^{SCC}(\succeq') = V_K$. This completes the proof of Case $K$.

Next, let $l \in \{1, \ldots, K-1\}$. Previous Cases $K, K-1, \ldots, l+1$ imply that

$$S \cap \bigcup_{l' \in \{l+1,\ldots,K\}} V_{l'} = \emptyset \text{ and } V_{l+1}, V_{l+2}, \ldots, V_K \text{ are SCC coalitions of graph } \Gamma(\succeq'). \tag{48}$$

Consider $V_l$.

**Case *l*.** We will prove that $S \cap V_l = \emptyset$ and that for each $i \in V_l$, $\varphi_i^{SCC}(\succeq') = V_l$.

Let $j \in S$. Suppose to the contrary that $j \in V_l$. We will obtain a contradiction with assumption (41). It follows from Fact 2 that agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$, i.e.,

$$\varphi_j^{SCC}(\succeq) \cap \bigcup_{l' \in \{l+1,\ldots,K\}} (F_j \cap V_{l'}) = \emptyset \;\text{ and }\; \bigcup_{l' \in \{1,\ldots,l\}} (F_j \cap V_{l'}) = F_j \cap V_l \subseteq \varphi_j^{SCC}(\succeq). \tag{49}$$

Then, from (48), (41), and by Condition (F) of friend-oriented preferences with neutrals, agent $j$ in coalition $\varphi_j^{SCC}(\succeq')$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$, i.e.,

$$\varphi_j^{SCC}(\succeq') \cap \bigcup_{l' \in \{l+1,\ldots,K\}} (F_j \cap V_{l'}) = \emptyset \;\text{ and }\; \bigcup_{l' \in \{1,\ldots,l\}} (F_j \cap V_{l'}) = F_j \cap V_l \subseteq \varphi_j^{SCC}(\succeq'). \tag{50}$$

Next, we prove that if agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with an enemy $e$, then that enemy is also in his coalition $\varphi_j^{SCC}(\succeq')$, i.e.,

$$E_j \cap \varphi_j^{SCC}(\succeq) \subseteq E_j \cap \varphi_j^{SCC}(\succeq'). \tag{51}$$

Suppose, by contradiction, that agent $j$ in coalition $\varphi_j^{SCC}(\succeq)$ is together with an enemy $e$ who is not in his coalition $\varphi_j^{SCC}(\succeq')$, i.e., $e \in E_j \cap (\varphi_j^{SCC}(\succeq) \setminus \varphi_j^{SCC}(\succeq'))$. Since $\varphi_j^{SCC}(\succeq) = V_l$,

$e \in V_l$. By definition of $\varphi^{SCC}(\succeq')$,

$$\text{agents } j \text{ and } e \text{ are in distinct SCC coalitions of } \Gamma(\succeq'). \tag{52}$$

For each $h \in V_l$, let $V'(h)$ denote the SCC coalition of $\Gamma(\succeq')$ that contains agent $h$. By definition of $V'(h)$, $V'(h) \cap V_l \neq \emptyset$. Moreover, from $j, e \in V_l$ and (52) it follows that $|\{V'(h)\}_{h \in V_l}| \geq 2$. Since the condensation graph of $\Gamma(\succeq')$ is acyclic, let $V' \in \{V'(h)\}_{h \in V_l}$ be an SCC coalition without an outgoing edge to any of the other SCC coalitions in $\{V'(h)\}_{h \in V_l} \setminus \{V'\}$.[28] Hence, there is no edge from any vertex in $V'$ to any vertex in $[\bigcup_{h \in V_l} V'(h)] \setminus V'$. In particular, in $\Gamma(\succeq')$, there is no edge from any vertex in $V' \cap V_l$ to any vertex in

$$\left[\bigcup_{h \in V_l} (V'(h) \cap V_l)\right] \setminus (V' \cap V_l) = V_l \setminus (V' \cap V_l).$$

However, since

$$[V' \cap V_l] \cup [V_l \setminus (V' \cap V_l)] = V_l$$

is an SCC coalition of $\Gamma(\succeq)$, there is an edge from some vertex in $V' \cap V_l$ to some vertex in $V_l \setminus (V' \cap V_l)$. Let $(i^*, j^*)$ be an edge from $V' \cap V_l$ to $V_l \setminus (V' \cap V_l)$ in $\Gamma(\succeq)$. In particular,

$$j^* \notin V'. \tag{53}$$

Since $(i^*, j^*)$ is an edge in $\Gamma(\succeq)$,

$$j^* \in F_{i^*}. \tag{54}$$

Since there is no edge from any vertex in $V' \cap V_l$ to any vertex in $V_l \setminus (V' \cap V_l)$ in $\Gamma(\succeq')$, $(i^*, j^*)$ is not an edge in $\Gamma(\succeq')$. Then, since only agents in $S$ change preferences from $\succeq$ to $\succeq'$, we conclude that

$$i^* \in S \cap V_l. \tag{55}$$

Then, we can use arguments similar to those that established (49) and (50) to obtain $F_{i^*} \cap V_l \subseteq \varphi^{SCC}_{i^*}(\succeq')$. From (54), $j^* \in \varphi^{SCC}_{i^*}(\succeq')$. However, since by definition of $V'$, $\varphi^{SCC}_{i^*}(\succeq') = V'$, we obtain $j^* \in V'$; contradicting (53). This proves (51).

Hence, through (49), (50), and (51), we have now shown that in coalition $\varphi^{SCC}_j(\succeq')$ agent $j$ is together with all his friends that did not join previously considered SCC coalitions $V_{l+1}, \ldots, V_K$ and together with the same enemies as before. Thus, Conditions (E) and (N) of friend-oriented preferences with neutrals implies that $\varphi^{SCC}_j(\succeq) \succeq_j \varphi^{SCC}_j(\succeq')$ which contradicts (41). We conclude that $j \notin V_l$. Hence, $S \cap V_l = \emptyset$.

Finally, note that graph $\Gamma(\succeq)$ contains no edge from $V_l$ to $V_1 \cup \cdots \cup V_{l-1}$. Furthermore, since $S \cap V_l = \emptyset$, $\succeq'_{V_l} = \succeq_{V_l}$. Thus, from (48), $V_l$ is an SCC coalition of graph $\Gamma(\succeq')$. Then, for each

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[28] Note that since $|\{V'(h)\}_{h \in V_l}| \geq 2$, $|\{V'(h)\}_{h \in V_l} \setminus \{V'\}| \geq 1$.

$i \in V_l$, $\varphi_i^{SCC}(\succeq') = V_l$. This completes the proof of Case $l$.

We have recursively shown that for each $l \in \{1, \ldots, K\}$, $S \cap V_l = \emptyset$, which contradicts $\emptyset \neq S = S \cap N = S \cap \bigcup_{l=1}^{K} V_l$. Therefore, $\varphi^{SCC}$ is *weakly group strategy-proof* on each subdomain $\widetilde{\mathcal{R}}^{fn}$ of friend-oriented preferences with neutrals. □

## J Appendix: $\boldsymbol{\psi}$ is *core stable* and *weak group strategy-proof* (Example 6)

Let $\overline{\succeq}_1 \in \mathcal{R}_1^{fn}$ such that $F(\overline{\succeq}_1) = N \backslash \{1\}$. Then, mechanism $\psi$ is defined as follows. For each $\succeq \in \mathcal{R}^{fn}$, if $F(\succeq_1) \cup N(\succeq_1) = N \backslash \{1\}$, then $\psi(\succeq) \equiv \varphi^{SCC}(\overline{\succeq}_1, \succeq_{-1})$; otherwise, $\psi(\succeq) \equiv \varphi^{SCC}(\succeq)$.

***Proof of core stability of $\boldsymbol{\psi}$.*** Let $\succeq \in \mathcal{R}^{fn}$. If $F(\succeq_1) \cup N(\succeq_1) \neq N \backslash \{1\}$, then by *core stability* of $\varphi^{SCC}$ (Theorem 2), $\psi(\succeq) = \varphi^{SCC}(\succeq) \in C(\succeq)$.

Now let $F(\succeq_1) \cup N(\succeq_1) = N \backslash \{1\}$. Then, $\psi(\succeq) = \varphi^{SCC}(\overline{\succeq}_1, \succeq_{-1})$. We show that $\psi(\succeq)$ is not blocked by any coalition $S$ at $\succeq$. Let $S \subseteq N$ with $S \neq \emptyset$. If $S \subseteq N \backslash \{1\}$, then by *core stability* of $\varphi^{SCC}$, $S$ does not block $\psi(\succeq)$ at $\succeq$. Now let $1 \in S$. Suppose to the contrary that $S$ blocks $\psi(\succeq)$ at $\succeq$. Then,

$$\text{for each } j \in S,\ S \succ_j \psi_j(\succeq). \tag{56}$$

In particular, $S \succ_1 \psi_1(\succeq)$. Since $F(\succeq_1) \cup N(\succeq_1) = N \backslash \{1\}$, it follows that $S \not\subseteq \psi_1(\succeq)$. Hence, there exists some $k \in N \backslash \{1\}$ such that $k \in S$ and $k \notin \psi_1(\succeq)$. Then, it is easy to see that there exist lexicographically friend-oriented preferences $\widehat{\succeq}_1 \in \mathcal{R}_1^{lf}$ with $F(\widehat{\succeq}_1) = N \backslash \{1\}$ and

$$S \mathrel{\widehat{\succ}_1} \psi_1(\succeq).^{29} \tag{57}$$

Since $F(\widehat{\succeq}_1) = N \backslash \{1\} = F(\overline{\succeq}_1)$, it follows that $\psi(\succeq) = \varphi^{SCC}(\overline{\succeq}_1, \succeq_{-1}) = \varphi^{SCC}(\widehat{\succeq}_1, \succeq_{-1})$. Thus, from (56) it follows that

$$\text{for each } j \in S \backslash \{1\},\ S \succ_j \varphi_j^{SCC}(\widehat{\succeq}_1, \succeq_{-1});$$

and from (57) it follows that

$$S \mathrel{\widehat{\succ}_1} \varphi_1^{SCC}(\widehat{\succeq}_1, \succeq_{-1}).$$

Hence, $S$ blocks $\varphi^{SCC}(\widehat{\succeq}_1, \succeq_{-1})$ at $(\widehat{\succeq}_1, \succeq_{-1})$, which contradicts *core stability* of $\varphi^{SCC}$. Hence, $\psi$ is *core stable*. □

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[29] Lexicographically friend-oriented preferences with these properties can be obtained by letting $k$ be the highest ranked individual agent.

***Proof of weak group strategy-proofness of $\psi$.*** Consider $\succsim \in \mathcal{R}^{fn}$, a coalition $S \subseteq N$, $\succsim'_S \in \prod_{i\in S} \mathcal{R}^{fn}_i$, and $\succsim' \equiv (\succsim'_S, \succsim_{-S})$. We will show that

$$\text{for some } j \in S,\ \psi_j(\succsim) \succsim_j \psi_j(\succsim'). \tag{58}$$

Suppose $1 \notin S$. Then, there exists $\overline{\succsim}_1 \in \mathcal{R}^{fn}_1$ (possibly $\overline{\succsim}_1 = \succsim_1$) such that $\psi(\succsim) = \varphi^{SCC}(\overline{\succsim}_1, \succsim_{-1})$ and $\psi(\succsim') = \varphi^{SCC}(\overline{\succsim}_1, \succsim'_S, \succsim_{-S\cup\{1\}})$. Then, (58) follows from *weak group strategy-proofness* of $\varphi^{SCC}$ (Proposition 5).

Suppose $1 \in S$. We distinguish among three cases.

**Case 1.** Suppose $F(\succsim_1) \cup N(\succsim_1) \neq N\backslash\{1\}$. Then, $\psi(\succsim) = \varphi^{SCC}(\succsim)$. By definition of $\psi$, there exists $\succsim''_1 \in \mathcal{R}^{fn}_1$ (possibly $\succsim''_1 = \succsim'_1$) such that $\psi(\succsim') = \varphi^{SCC}(\succsim''_1, \succsim'_{S\backslash\{1\}}, \succsim_{-S})$. Then, (58) follows from *weak group strategy-proofness* of $\varphi^{SCC}$.

**Case 2.** Suppose $F(\succsim_1) \cup N(\succsim_1) = F(\succsim'_1) \cup N(\succsim'_1) = N\backslash\{1\}$. Then, $\psi(\succsim) = \varphi^{SCC}(\overline{\succsim}_1, \succsim_{-1})$ and $\psi(\succsim') = \varphi^{SCC}(\overline{\succsim}_1, \succsim'_{S\backslash\{1\}}, \succsim_{-S})$. Thus, if $S = \{1\}$, then $\psi(\succsim) = \psi(\succsim')$ so that (58) holds trivially. If $S \neq \{1\}$, then from *weak group strategy-proofness* of $\varphi^{SCC}$, there exists $k \in S\backslash\{1\}$ with $\psi_k(\succsim) \succsim_k \psi_k(\succsim')$, and (58) follows.

**Case 3.** Suppose $F(\succsim_1) \cup N(\succsim_1) = N\backslash\{1\}$ and $F(\succsim'_1) \cup N(\succsim'_1) \neq N\backslash\{1\}$. Suppose to the contrary that (58) does *not* hold. Then,

$$\text{for each } j \in S,\ \psi_j(\succsim') \succ_j \psi_j(\succsim). \tag{59}$$

Then,

$$\psi(\succsim) = \varphi^{SCC}(\overline{\succsim}_1, \succsim_{-1}) \text{ and } \psi(\succsim') = \varphi^{SCC}(\succsim'_1, \succsim'_{S\backslash\{1\}}, \succsim_{-S}). \tag{60}$$

Since $F(\succsim_1) \cup N(\succsim_1) = N\backslash\{1\}$ and $\psi_1(\succsim') \succ_1 \psi_1(\succsim)$ (from (59)), it follows that $\psi_1(\succsim') \not\subseteq \psi_1(\succsim)$. Hence, there exists some $k \in N\backslash\{1\}$ such that $k \in \psi_1(\succsim')$ and $k \notin \psi_1(\succsim)$. Then, there exist lexicographically friend-oriented preferences $\widehat{\succsim}_1 \in \mathcal{R}^{lf}_1$ with $F(\widehat{\succsim}_1) = N\backslash\{1\}$ and

$$\psi_1(\succsim')\, \widehat{\succ}_1\, \psi_1(\succsim).^{30} \tag{61}$$

Since $F(\widehat{\succsim}_1) = N\backslash\{1\} = F(\overline{\succsim}_1)$, it follows that $\psi(\succsim) = \varphi^{SCC}(\overline{\succsim}_1, \succsim_{-1}) = \varphi^{SCC}(\widehat{\succsim}_1, \succsim_{-1})$. Thus, from (60) and (61),

$$\varphi^{SCC}_1(\succsim'_1, \succsim'_{S\backslash\{1\}}, \succsim_{-S})\, \widehat{\succ}_1\, \varphi^{SCC}_1(\widehat{\succsim}_1, \succsim_{-1}).$$

Moreover, from (59) it follows that

$$\text{for each } j \in S\backslash\{1\},\ \varphi^{SCC}_j(\succsim'_1, \succsim'_{S\backslash\{1\}}, \succsim_{-S}) \succ_j \varphi^{SCC}_j(\widehat{\succsim}_1, \succsim_{-1}).$$

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[30] Lexicographically friend-oriented preferences with these properties can be obtained by letting $k$ be the highest ranked individual agent.

Thus, coalition $S$ strictly improves by misreporting $(\succsim'_1, \succsim'_{S\setminus\{1\}}, \succsim_{-S})$ at $(\widehat{\succsim}_1, \succsim_{-1})$, which contradicts *weak group strategy-proofness* of $\varphi^{SCC}$. Hence, (58) does hold and $\psi$ satisfies *weak group strategy-proofness.* $\square$

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