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# Stochastic Dominance and Absolute Risk Aversion

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# Stochastic Dominance and Absolute Risk Aversion*

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**Abstract**

In this paper we propose the infimum of the Arrow-Pratt index of absolute risk aversion as a measure of global risk aversion of a utility function. We show that, for any given arbitrary pair of distributions, there exists a threshold level of global risk aversion such that all increasing concave utility functions with at least as much global risk aversion would rank the two distributions in the same way. Furthermore, this threshold level is sharp in the sense that, for any lower level of global risk aversion, we can find two utility functions in this class yielding opposite preference relations for the two distributions.

*JEL classification codes:* D81, D30.

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## 1. Introduction

The famous papers of Hadar and Russell (1969) and Rothschild and Stiglitz (1970) developed a notion of riskiness. One distribution is riskier than another when the former dominates the latter according to the *monotonic second order stochastic dominance* (MSOSD, henceforth) criterion, that is, when it is unanimously preferred by all expected utility maximizers who prefer more to less and who are risk averters. Unanimity requires thus that all decision makers agree with the most extreme risk averse preferences, that is, those giving all the weight to the worst possible outcome. Clearly, for these extreme preferences, which of the two distributions will be preferred depends on the lower tail of the distributions only. However, when we want to verify that individuals with lower degrees of risk aversion will agree on that ordering as well, we need to compare the two distributions over the entire support. Indeed, the main result in Hadar and Russell (1969) and Rothschild and Stiglitz (1970) is that a pair of distributions can be ranked according to the MOSOSD criterion if and only if a strong integral condition relating the two distributions is satisfied. This condition is quite stringent so that the ordering on the set of distributions induced by the MSOSD criterion is indeed very partial.

A question that naturally arises in the theory of decision under risk is whether the comparison between risky prospects would be facilitated by requiring unanimity only on a subset of the class of increasing and concave utility functions with appealing properties. This task has proven quite unproductive since many additional natural properties imposed on utility functions, like *decreasing absolute risk aversion* (DARA, henceforth),[1] do not yield a non-dense basis through which an operative condition relating two distribution functions can be obtained (see Gollier and Kimball, 1996). One exception is the class of mixed utility functions, that are those having non-negative odd derivatives and non-positive even derivatives. Caballé and Pomansky (1996) show that the set of negative exponential functions constitutes a basis for that family of utilities. Therefore, a distribution is preferred to another by all individuals with increasing utilities exhibiting sign-alternating derivatives if and only if the Laplace transform of the former is smaller than that of the latter.[2]

In the present paper we take back the approach of Hadar and Russell (1969) and Rothschild and Stiglitz (1970). However, instead of classifying the pairs of

[1] A Bernoulli utility belonging to the DARA class exhibits a demand for a risky asset that increases with wealth (Arrow, 1970; Pratt, 1964).

[2] The class of mixed utility functions constitutes a subset of the DARA class and includes all the DARA utilities typically found in some economic applications, like the hyperbolic absolute risk aversion, the isoelastic, or the exponential functions. In fact, mixed utilities satisfy other appealing properties found in the literature, like risk vulnerability (Gollier and Pratt, 1996), properness (Pratt and Zeckhauser, 1987) or standardness (Kimball, 1993).

distributions into “uncontroversial” –those that can be ranked according to MSOSD– and “controversial” –those that cannot–, we wish to associate to every pair of distributions a parameter reflecting how controversial their ranking is. Specifically, for any arbitrary pair of distributions we wish to characterize the “lowest” degree of risk aversion such that all decision makers with at least this degree of risk aversion would unanimously prefer one distribution over the other. The lower the degree of required risk aversion the less controversial the ranking will be. Notice that the main point of our analysis is not the ordering, since this is given by the preferences of the most risk averse individuals. The critical question is how many more individuals will agree with such an ordering.

Our analysis is based on the use of the key observation, made by Diamond and Stiglitz (1974), Meyer (1977) and Lambert and Hey (1979), that two random variables can be ranked according to MSOSD if and only if any common concave transformation of these random variables can be ranked according to MSOSD. Using this fact, the strategy we follow is to identify the “least” concave utility function $u$ for which the distributions of the transformed random variables can be ranked according to MSOSD. We know then that all the increasing and concave transformations of this utility function $u$ will rank the two original distributions as the function $u$ does and, hence, all the individuals having utilities displaying more absolute risk aversion at each point than that of the threshold utility $u$ will choose unanimously the same random variable.

The concept of “lowest” risk aversion (or “least” concavity) requires to define a previous notion of global risk aversion (or global concavity) permitting a complete order over the set of increasing and concave utility functions. Such a notion of global concavity can be made precise in a number of ways. In this paper we consider the infimum of the Arrow-Pratt index of *absolute risk aversion* (ARA, henceforth) of a utility function over its domain as a measure of global concavity. The main result of our analysis appears in Theorem 3.1, where we demonstrate that the partition of utilities generated by that global measure of concavity has the property that, for any given arbitrary pair of distributions, we can find a threshold level of global risk aversion $s^*$ such that all increasing and concave utility functions with a level $s \in [s^*, \infty)$ unanimously rank one distribution over the other and that for any level $s \in (0, s^*)$ we can find two utility functions with this level of global concavity that will reverse the previous ranking. We also explore alternative notions of global risk aversion like the supremum or the average of the Arrow-Pratt index of ARA and show that we cannot obtain such sharp a characterization.

The paper is organized as follows. Section 2 reviews some concepts appearing in the literature of decision under risk. Section 3 contains our main results concerning alternative measures of global risk aversion and their relationship with MSOSD. Our concluding remarks in Section 4 are followed by a fairly lengthy and technical section with the proofs.

## 2. Orderings on distributions

Consider the set of random variables taking values on the interval $[a,b]$. If $F_{\tilde{x}}$ is the distribution function of the random variable $\tilde{x}$, then the expectation (or mean) of the distribution of $\tilde{x}$ is $E_{F_{\tilde{x}}} = \int_{[a,b]} z dF_{\tilde{x}}(z)$.[3] Suppose that an agent has a state-independent preference relation defined on the space of random variables and that this preference relation has an expected utility representation (or Bernoulli utility) $u$. This means that the agent prefers the random variable $\tilde{x}$ with distribution function $F_{\tilde{x}}$ to the random variable $\tilde{y}$ with distribution function $F_{\tilde{y}}$ whenever

$$\int_{[a,b]} u(z) dF_{\tilde{x}}(z) \geq \int_{[a,b]} u(z) dF_{\tilde{y}}(z). \tag{2.1}$$

It is well known that the Bernoulli utility $u$ is unique up to a strictly increasing affine transformation. Note that a state-independent preference relation defined on the space of random variables induces a preference relation on the set of distribution functions. Therefore, we will say that $F_{\tilde{x}}$ is preferred to $F_{\tilde{y}}$ by an individual having the Bernoulli utility $u$, $F_{\tilde{x}} \underset{u}{\succsim} F_{\tilde{y}}$ , if (2.1) holds. Moreover, $F_{\tilde{x}}$ is strictly preferred to $F_{\tilde{y}}$ by an individual with Bernoulli utility $u$, $F_{\tilde{x}} \underset{u}{\succ} F_{\tilde{y}}$, whenever (2.1) holds with strict inequality.

**Definition 1.**

*(a) The distribution function $F_{\tilde{x}}$ dominates the distribution function $F_{\tilde{y}}$ according to the monotonic second order stochastic dominance (MSOSD) criterion, $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ , if $F_{\tilde{x}} \underset{u}{\succsim} F_{\tilde{y}}$ for all the Bernoulli utility functions $u$ that are increasing and concave.*

*(b) The distribution function $F_{\tilde{x}}$ strictly dominates the distribution function $F_{\tilde{y}}$ according to the MSOSD criterion, $F_{\tilde{x}} \underset{D}{\succ} F_{\tilde{y}}$ , if $F_{\tilde{x}} \underset{u}{\succ} F_{\tilde{y}}$ for all the Bernoulli utility functions $u$ that are increasing and strictly concave.*

Therefore, if $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ , then all the individuals who prefer more to less and are risk averse will prefer the random variable $\tilde{x}$ to the random variable $\tilde{y}$. According to the well known analysis of Hadar and Russell (1969) and Rothschild and Stiglitz (1970), we can state the following famous result:

[3] The integral appearing in the expression is the Lebesgue integral with respect to the Lebesgue-Stieltjes measure (or distribution) associated with the distribution function $F$ (see section 1.4 of Ash, 1972).

**Proposition 1.** *$F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ if and only if*

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \le 0 \quad \text{for all } x \in [a,b]\,. \tag{2.2}$$

*Moreover, $F_{\tilde{x}} \underset{D}{\succ} F_{\tilde{y}}$ if the previous inequality is strict for all $x \in (a,b)$.*

Consider now an increasing and concave function $u$ and two random variables $\tilde{x}$ and $\tilde{y}$. Let $F_{u(\tilde{x})}$ and $F_{u(\tilde{y})}$ be the distribution functions associated with the composite random variables $u(\tilde{x})$ and $u(\tilde{y})$, respectively. The following corollary, arising from the papers of Diamond and Stiglitz (1974), Meyer (1977), and Lambert and Hey (1979), will play a crucial role in our analysis:

**Corollary 1.** *$F_{u(\tilde{x})} \underset{D}{\succsim} (\underset{D}{\succ}) F_{u(\tilde{y})}$ if and only if $F_{v(\tilde{x})} \underset{D}{\succsim} (\underset{D}{\succ}) F_{v(\tilde{y})}$ for all the Bernoulli utility functions $v$ that are increasing and (strictly) concave transformations of $u$.*

**Proof.** Obvious from Definition 1, since $F_{u(\tilde{x})} \underset{D}{\succsim} (\underset{D}{\succ}) F_{u(\tilde{y})}$ if and only if $F_{u(\tilde{x})} \underset{v}{\succsim} (\underset{v}{\succ}) F_{u(\tilde{y})}$ for all the Bernoulli utility functions $v$ that are increasing and (strictly) concave transformations of $u$. ■

The order induced on the set of distribution functions by the MSOSD criterion is very partial as the distributions that can be ranked according to that criterion constitute indeed a very small subset of distribution functions. This can be easily deduced from just looking at the stringent integral condition (2.2). In contrast, the max-min criterion discussed in Rawls (1974), which makes preferable the distribution with the better worst possible outcome, induces a quite complete ordering on the set of distributions. Before defining more precisely this lexicographic criterion we need the following definition that will be used extensively in the rest of the paper:

**Definition 2.** *The right-continuous function $g$ defined on $[a,b]$ changes sign at $x$ if there exist two real numbers $\varepsilon > 0$ and $\eta \ge 0$ such that the following two conditions hold:*

*(i) $g(z) \cdot g(y) \le 0$ for all $(z,y) \in (x-\varepsilon, x) \times [x, x+\eta]$ and*

*(ii) $g(z) \cdot g(y) < 0$ for some $(z,y) \in (x-\varepsilon, x) \times [x, x+\eta]\,$.*

**Definition 3.**

*(a) The distribution function $F_{\tilde{x}}$ strictly dominates the distribution function $F_{\tilde{y}}$ according to the max-min criterion, $F_{\tilde{x}} \underset{M}{\succ} F_{\tilde{y}}$, if there exists a $\hat{z} \in (a,b)$ such that $F_{\tilde{x}}(z) \le F_{\tilde{y}}(z)$ for all $z \in [a, \hat{z})$, and $F_{\tilde{x}}(z) < F_{\tilde{y}}(z)$ for some $z \in [a, \hat{z})$.*

*(b) The distribution function $F_{\tilde{x}}$ dominates the distribution function $F_{\tilde{y}}$ according to the max-min criterion, $F_{\tilde{x}} \underset{M}{\gtrsim} F_{\tilde{y}}$, if either $F_{\tilde{x}} \underset{M}{\succ} F_{\tilde{y}}$ or $F_{\tilde{x}}(z) = F_{\tilde{y}}(z)$ for all $z \in [a,b]$.*

Clearly, the ordering induced by the max-min criterion is much more complete than that induced by the MSOSD criterion. For instance, all pairs $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ of distribution functions for which the function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign a finite number (including zero) of times can be ranked according to the former criterion.

We will restrict our attention throughout the paper to pairs of distributions functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ of random variables taking values on the interval $[a, b]$ that satisfy the following assumption:

**Assumption M.** The function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign a finite number of times and $F_{\tilde{x}} \underset{M}{\succ} F_{\tilde{y}}$, whereas neither $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ nor $F_{\tilde{y}} \underset{D}{\succsim} F_{\tilde{x}}$.

On the one hand, the assumption of a finite number of changes of sign is sufficient to ensure that there is a point on $[a, b]$ of first sign change according to Definition 2.[4] On the other hand, the assumption of $F_{\tilde{x}} \underset{M}{\succ} F_{\tilde{y}}$ and neither $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ nor $F_{\tilde{y}} \underset{D}{\succsim} F_{\tilde{x}}$ is made without loss of generality whenever the two random variables under consideration cannot be ranked according to the MSOSD criterion and the function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign a finite number of times.

Consider the case where the distribution functions $F_{\tilde{x}}$ and $F_{\tilde{y}}$ cannot be ranked by MSOSD. Suppose that we could find a utility function $u$ such that the composite random variables satisfy $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$. Then, by Corollary 1, the random variable $\tilde{x}$ will be preferred to $\tilde{y}$ by all agents having Bernoulli utility functions that are increasing and concave transformations of $u$. The rest of the paper is devoted to find the "least concave" utility function permitting the MSOSD ranking of the distributions associated with the composite random variables when the original pair of distributions satisfies Assumption M.

## 3. Results

### 3.1. The Main Result

Let $u$ be a twice continuously differentiable function on $(a, b)$. The Arrow-Pratt index of *absolute risk aversion* (ARA) of the function $u$ at $z \in (a, b)$ is $A_u(z) = -u''(z)/u'(z)$ (see Arrow, 1971; and Pratt, 1964). We will first propose a global measure of risk aversion (the infimum of the ARA index) inducing a partition over the set of increasing and concave functions, such that all the utility functions displaying more global

[4] Consider in this respect the case where $F_{\tilde{x}}$ and $F_{\tilde{y}}$ satisfy $F_{\tilde{x}} < F_{\tilde{y}}$ on $[a, c)$, $F_{\tilde{x}} = F_{\tilde{y}}$ at $c$ and they intersect at all points of the form $c + (1/n)$ for every positive integer $n$. Obviously, the function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign infinitely many times and there is no point of first change of sign according to Definition 2.

risk aversion than a threshold level rank one distribution over the other. Consider thus the following partition of the set of increasing and concave utility functions on $[a, b]$ that are twice continuously differentiable on $(a, b)$. A function $u$ belongs to the class $I(s)$ if the infimum of the ARA index over its domain is $s$,

$$u \in I(s), \text{ whenever } \inf_{z \in (a,b)} A_u(z) = s.$$

Consider the class of increasing, concave and twice continuously differentiable utility functions $r(\cdot; s)$ exhibiting an ARA index, $A_{r(\cdot;s)}(z)$, equal to the constant $s > 0$ for all $z \in (a, b)$. These functions exhibiting *constant absolute risk aversion* (CARA) have a functional form that is an increasing affine transformation of the function $-e^{-sz}$. Notice that all $u \in I(s)$ with $s \geq s^*$ are increasing and concave transformations of the CARA utility with an ARA index equal to $s^*$. Therefore, an implication of Corollary 1 is that, if $F_{r(\tilde{x};s^*)} \underset{D}{\succ} F_{r(\tilde{y};s^*)}$ then $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ for all $u \in I(s)$ with $s \geq s^*$. From this observation we derive the main result of our paper, which is stated in the following theorem:

**Theorem 3.1.** *Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M. Then, there exists a real number $s^* > 0$ such that*

*(a) $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ for all $u \in I(s)$ with $s > s^*$.*

*(b) There exists a $u \in I(s)$ such that $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ for all $s \in (0, s^*)$.*

*(c) There exists a $u \in I(s)$ such that neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $s \in (0, s^*)$.*

Parts (a) and (b) of the previous theorem imply that we can always find a function $u$, with an arbitrarily given value of the infimum of its ARA index, for which the random variables $u(\tilde{x})$ and $u(\tilde{y})$ can be compared according to the MSOSD criterion. In fact, part (a) says that, for sufficiently large values of the infimum of the ARA index, MSOSD between two random variables *always* holds. On the contrary, part (c) tells us that, if a concave transformation of two random variables does not generate MSOSD, then that transformation must exhibit a small value of the infimum of its ARA index.

Consider the class of increasing, concave and twice continuously differentiable utility functions $r(\cdot; s)$ exhibiting an ARA index, $A_{r(\cdot;s)}(z)$, equal to the constant $s > 0$ for all $z \in (a, b)$. These functions exhibiting *constant absolute risk aversion* (CARA) have a functional form that is an increasing affine transformation of the function $-e^{-sz}$. Notice that all $u \in I(s)$ with $s \geq s^*$ are increasing and concave transformations of the CARA utility with an ARA index equal to $s^*$. Therefore, an implication of the previous theorem is that, if $F_{r(\tilde{x};s^*)} \underset{D}{\succ} F_{r(\tilde{y};s^*)}$ then $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ for all $u \in I(s)$ with $s \geq s^*$.

The following corollary characterizes explicitly the critical value $s^*$ of the infimum of the ARA index above which unanimity is reached:

**Corollary 2.** *The critical value $s^*$ defined in Theorem 3.1 is the smallest positive real number $s$ satisfying*

$$\int_a^x \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] e^{-sz} dz \leq 0 \quad \text{for all } x \in [a,b]\,. \tag{3.1}$$

### 3.2. Alternative Global Notions of Risk Aversion

Obviously, there are many alternative ways by which one can define a global measure based on the ARA index. We will next discuss two of them, namely, the supremum of the ARA index and the average of the ARA index over the utility domain. We will see that the kind of results that can be obtained with these two measures are much less appealing than those obtained with the measure based on the infimum of the ARA index.

Consider now the following partition of the set of increasing and concave utility functions on $[a,b]$ that are twice continuously differentiable on $(a,b)$. A function $u$ belongs to the class $P(s)$ if the supremum of the ARA index over its domain is $s$,

$$u \in P(s), \text{ whenever } \sup_{z \in (a.b)} A_u(z) = s.$$

The following theorem parallels Theorem 3.1 for the partition formed by the sets $P(s)$ with $s \in (0, \infty)$ :

**Theorem 3.2.** *Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M. Then, there exists a real number $s^* > 0$ such that,*

*(a) Neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $u \in P(s)$ with $s \in (0, s^*)$.*

*(b) There exists a $u \in P(s)$ such that $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ for all $s > s^*$.*

*(c) There exists a $u \in P(s)$ such that neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $s > s^*$.*

Note that parts (a) and (c) of the previous theorem imply that we can always find a function $u$ with an arbitrarily given value of the supremum of its ARA index, for which the random variables $u(\tilde{x})$ and $u(\tilde{y})$ cannot be ranked according to the MSOSD criterion. Part (b) tells us that, for a sufficiently high value $s$ of the supremum of the ARA index, it is possible to order two given random variables for some utility function belonging to $P(s)$. However, as follows from part (a), MSOSD turns out to be unfeasible for sufficiently small values of the supremum of the ARA index.

Obviously, the ARA index cannot be properly applied to non-differentiable utility functions. Consider then the index of *thriftiness* that has been proposed as a global measure of concavity for general strictly increasing functions (see Chateauneuf et al., 2000). This index captures the maximal relative drop of the slope of the function $u$ along its domain and is given by

$$T(u) = \sup_{z_1 < z_2 \leq z_3 < z_4} \left[ \frac{u(z_2) - u(z_1)}{z_2 - z_1} \Big/ \frac{u(z_4) - u(z_3)}{z_4 - z_3} \right].$$

For functions defined on $[a, b]$ that are differentiable, strictly increasing and concave, the index of thriftiness becomes simply $T(u) = u'(a)/\, u'(b)$ . In this case this index measures how significative is the reduction in the slope of the utility function along its domain. It is plain that the same value of the thriftiness index is compatible with a plethora of local behaviors. For instance, the reduction in the slope can be uniformly distributed over the domain, as occurs with the CARA functions, or it can be concentrated on a very small interval. In the latter case the utility function could exhibit a local ARA index that is zero at all points of its domain except on an arbitrarily small interval where the ARA index could become arbitrarily large. In fact, if we allow for non-differentiable functions, the drop of the slope can occur at a single point and, of course, all concave transformations of such a function will not be differentiable at that point. Note that any increasing and strictly concave transformation of a given function $u$ will exhibit an index of thriftiness larger than that of $u$. It should also be noticed that the index of thriftiness is a measure equivalent to the average value of the ARA index displayed by a twice continuously differentiable utility function $u$ over its domain. Clearly, the average ARA of the function $u$ is

$$\frac{1}{b-a} \int_a^b \frac{-u''(z)}{u'(z)} dz = \frac{1}{b-a} \left[ -\ln \left( u'(b) \right) + \ln \left( u'(a) \right) \right] =$$

$$\frac{1}{b-a} \ln \left( \frac{u'(a)}{u'(b)} \right) = \frac{1}{b-a} \ln \left( T(u) \right).$$

The following theorem establishes the connection between the index of thriftiness and the MSOSD ordering:

**Theorem 3.3.** *Assume that the pair of distribution functions* $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ *satisfies Assumption M. Then, there exists a real number* $t^* > 0$ *such that,*

*(a) For all* $t \in [t^*, \infty)$ *there exists an increasing and concave Bernoulli utility function* $v$ *with* $T(v) = t$ *satisfying* $F_{v(\tilde{x})} \underset{D}{\succsim} F_{v(\tilde{y})}$.

*(b) For all* $t \in (t^*, \infty)$ *there exists a smooth, increasing and concave utility function* $u$ *with* $T(u) = t$ *satisfying* $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$.

The previous theorem provides a very partial result, since it just allows us to establish the existence of a lower bound $t^*$ on the index of thriftiness so that stochastic dominance between two distributions holds for *some* utility function displaying an index of thriftiness larger than that lower bound. The following corollary provides the explicit characterization of that critical value $t^*$ of the thriftiness index:

**Corollary 3.** *The critical value $t^*$ defined in Theorem 3.3 is given by*

$$t^* = -\frac{\displaystyle\int_{z_1}^{z_M} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz}{\displaystyle\int_{a}^{z_1} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz},$$

*where*

$$z_1 = \max\left\{\operatorname*{arg\,min}_{x\in[a,c]} \int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz\right\},$$

*$c$ is the smallest real number at which the integral*

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz$$

*changes sign, and*

$$z_M = \max\left\{\operatorname*{arg\,max}_{x\in[a,b]} \int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz\right\}.$$

## 4. Final remarks

The main result of our paper (Theorem 3.1) provides a sharp characterization of how controversial is the ordering of two distributions on the basis of the extreme most risk averse preferences when the global degree of concavity is measured by the infimum of the ARA index over the support. It establishes that $s^*$ is indeed the lowest degree of global concavity for which we can obtain unanimity in the ranking of the two distributions. Below this threshold we will always find preferences with the same degree of global concavity yielding transforms of the variables that cannot be ranked according to MSOSD.

In contrast, the supremum of the ARA index does not provide such a sharp characterization. We only can find a lower bound on the degree of concavity below which there are no preferences such that all their concave transformation agree in ranking one distribution over the other. Finally, when we use the average ARA index as a global measure of concavity, we cannot find a threshold level of concavity above which we obtain unanimity.

To conclude we simply wish to establish the bridge between our results and the classical analysis of Hadar and Russell (1969) and Rothschild and Stiglitz (1970). In our paper we have introduced a partition on the set of twice continuously differentiable utility functions according to the infimum of their ARA index over their common domain. The class corresponding to the value $s$ of the infimum of the ARA index is $I(s)$. Notice that the class of functions with $s \geq 0$ exactly corresponds to the set of increasing twice continuously differentiable concave functions. The aforementioned classical analysis relating concavity of the Bernoulli utility functions and MSOSD provides a limited answer to the question of whether it is possible to rank two risks by simply knowing that the utility function belongs to a particular set. The most celebrated result of that analysis says that, if we restrict to pairs of distributions satisfying the integral condition (2.2), then $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ for all $u \in I(s)$ with $s \geq 0$. This corresponds to part (a) of our Theorem 3.1. In line with part (b) of the same theorem, when (2.2) is satisfied, it can be shown that there exist increasing functions $u \in I(s)$ with $s < 0$ such that $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ and (again in line with (c)) there exists a $u \in I(s)$ such that neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $s < 0$. Therefore, we can find non-concave utility functions whose increasing and concave transformations would rank one distribution over the other. Furthermore, paralleling our Theorem 3.2, when the two distributions satisfy (2.2), it is also a known result that neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $u \in P(s)$ with $s < 0$, where $P(s)$ is the class of utility functions with a supremum of their ARA index equal to $s$.

In our analysis we show that, when two distributions cannot be ranked by MSOSD, one can nevertheless obtain stochastic dominance, but restricted to a class of increasing and concave functions displaying sufficiently high global risk aversion, namely, the class $I(s)$ with $s$ being greater than an appropriate value $s^*$. We also show that neither $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ nor $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$ for all $u \in P(s)$ with $s \in (0, s^*)$. Thus, our results generalize the aforementioned classical results to any given arbitrary pair of distributions satisfying Assumption M.

## 5. Proofs

### 5.1. A road map

As shown by Meyer (1977), when two random variables cannot be ranked by MSOSD there is always some utility function $u$ (not necessarily concave) such that the resulting distribution of the utility satisfies the integral condition for MSOSD. Notice, however, that by not imposing that the critical utility function be concave, its corresponding concave transformations need not be concave either. This is indeed an undesirable feature when individuals face typical optimization problems under risk. Using this fact, the strategy we follow is to identify the "least" concave utility function $u$ for which the distributions of the transformed random variables can be ranked according to MSOSD. We know then that all the increasing and concave transformations of this utility function $u$ will rank the two original distributions as the function $u$ does and, hence, all the individuals having utilities displaying more absolute risk aversion at each point than that of the threshold utility $u$ will choose unanimously the same random variable.

It is obvious that finding the "least" concave utility function cannot have an unambiguous answer, even when one chooses to measure the local concavity of a utility function by its ARA index. To set the ground we start by analyzing the two extreme types of increasing and concave transformations of the original random variables. First, we consider transformations that are linear (or risk neutral) everywhere except at a single point around which they concentrate all the concavity. Secondly, we will consider transformations that display an ARA index uniformly distributed over its domain, that is, these transformations exhibit *constant absolute risk aversion* (CARA, henceforth). Clearly, the two types of functions under consideration exhibit a very different behavior of their local ARA indexes. If the function is essentially linear, the infimum (supremum) of the local ARA index over its domain becomes zero (infinite) and, thus, no operative lower bound is obtained in terms of the ARA index. In contrast, the global concavity of a utility belonging to the CARA family is perfectly summarized by the ARA index evaluated at any arbitrary point of its domain. For the first type of functions we obtain the smallest drop of the slope at the kink permitting the ordering of the transformed random variables by MSOSD. As for the second type we prove the existence of a critical minimum value of the ARA index allowing for the MSOSD ranking of the two transformed risks.

Our main result follows immediately from the analysis made for the previous two families of functions. If there exists a minimal value of the ARA index for which MSOSD holds for the corresponding CARA transformation of the original random variables, then MSOSD will hold for all utility functions whose infimum of the ARA index is larger than that threshold value. This is so because the latter functions turn out to be concave transformations of the critical CARA function. However, for all lower values of the infimum of the ARA index, it is possible to find functions for which the MSOSD ranking does not apply. Furthermore, if there is a piecewise linear utility function allowing for the MSOSD ranking of the given pair of distributions, then we can find functions with an infimum of their ARA index arbitrarily close to zero permitting this ranking.

We then go into examining whether similar results can be obtained with other reasonable measures of global concavity. To this end, we consider two natural alternative measures of global concavity: the supremum of the ARA index and the average ARA index of the utility function over its domain. These alternative measures turn out to yield much weaker results concerning our original problem. For the supremum our results say that there is a threshold level such that there is no utility function with lower global concavity giving a common transformation of the original random variables permitting their ranking by MSOSD. Further, we demonstrate that for higher degrees of global concavity unanimity is possible but not pervasive. For the case of the average ARA as a global measure of concavity, we show that above some threshold value of this measure we can always find utility functions allowing for the ranking of the two commonly transformed risks by MSOSD. Clearly, none of the two measures yields such a sharp characterization of preferences as the one we obtained with the infimum of the ARA index as the global measure of concavity.

We start by examining transformations that are linear (or risk neutral) everywhere except at a single point and then move on to the case of transformations with the concavity uniformly distributed over the support (CARA). These two sets of results permit a simple proof of the main results.

### 5.2. Local risk neutrality almost everywhere

#### 5.2.1. The non-differentiable case

We start by considering essentially linear transformations of two given random variables. This means that, if we view these transformations as utility functions, they display risk neutrality everywhere except at a point where they exhibit a kink.

The next proposition shows explicitly how we can construct a common increasing and concave transformation of two random variables having distributions that cannot be ranked according to the MSOSD criterion, in order to obtain MSOSD for the corresponding transformed random variables. If one of the two random variables is strictly preferred to the other according to the max-min criterion, then the integral condition (2.2) will be satisfied for an interval of low realizations of these variables. Our strategy consists on scaling down the larger values of both random variables so that the previous integral condition will hold for the whole range of values of the transformed random variables.

**Proposition 2.** *Consider the class of continuous functions with the following functional form:*

$$k(z;\alpha,z_1)=\begin{cases} z & \text{for } z\in[a,z_1) \\ \\ \alpha z+(1-\alpha)z_1 & \text{for } z\in[z_1,b]\,. \end{cases} \tag{5.1}$$

*Assume that the pair of distribution functions $\{F_{\tilde{x}},F_{\tilde{y}}\}$ satisfies Assumption M. Then, there exist two real numbers $\alpha^*\in(0,1)$ and $z_1\in(a,b)$ such that*

$F_{k(\tilde{x};\alpha^*,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha^*,z_1)}$ *and* $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succ} F_{k(\tilde{y};\alpha,z_1)}$ *for all* $\alpha < \alpha^*$ , *whereas neither* $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,z_1)}$ *nor* $F_{k(\tilde{y};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{x};\alpha,z_1)}$ *for all* $\alpha > \alpha^*$ .

**Proof.** Note first that the function $k(\cdot;\alpha,z_1)$ is continuous. From Assunption M and Definition 3, we know that the integral

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \tag{5.2}$$

must change its sign at least once on $(a,b)$ and be non-positive on an interval $[a,c]$, with $c < b$, before becoming positive for the first time. We can thus define the interval $[a,c]$ with $c < b$ satisfying

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \le 0 \qquad \text{for all } x \in [a,c]\,, \tag{5.3}$$

and

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz > 0 \qquad \text{for all } x \in (c, c+h) \text{ and for some } h > 0. \tag{5.4}$$

We can also define the real number $z_1 \in (a,b)$ as

$$z_1 = \max\left\{ \underset{x\in[a,c]}{\arg\min} \int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \right\}, \tag{5.5}$$

and the number $z_M$ as

$$z_M = \max\left\{ \underset{x\in[a,b]}{\arg\max} \int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \right\}. \tag{5.6}$$

Therefore, $z_1$ is the largest value that minimizes the integral (5.2) on the interval $[a,c]$. Note also that the function $F_{\tilde{x}} - F_{\tilde{y}}$ must change sign at $z_1$. Moreover, $z_M$ is the largest value that maximizes the integral (5.2) on the interval $[a,b]$.

Since $F_{k(\tilde{x};\alpha,z_1)}(k) = F_{\tilde{x}}\left(k^{-1}(k;\alpha,z_1)\right)$ and $F_{k(\tilde{y};\alpha,z_1)}(k) = F_{\tilde{y}}\left(k^{-1}(k;\alpha,z_1)\right)$, we have that the MSOSD condition given in (2.2) for the transformed random variables,

$$\int_a^y \left[F_{k(\tilde{x};\alpha,z_1)}(k) - F_{k(\tilde{y};\alpha,z_1)}(k)\right] dk \le 0 \;\; \text{for all } y \in [k(a;\alpha,z_1), k(b;\alpha,z_1)]\,,$$

will be satisfied if and only if the following two inequalities hold:

$$\int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \le 0 \;\; \text{for all } x \in [a, z_1]\,, \tag{5.7}$$

and

$$\int_a^{z_1} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \int_{z_1}^{y} \left[F_{\tilde{x}}\left(\frac{k-(1-\alpha)z_1}{\alpha}\right) - F_{\tilde{y}}\left(\frac{k-(1-\alpha)z_1}{\alpha}\right)\right] dk \leq 0,$$

$$\text{for all } y \in \left[z_1, \alpha b + (1-\alpha)z_1\right]. \qquad (5.8)$$

Making the change of variable, $z = \dfrac{k-(1-\alpha)z_1}{\alpha}$, the integral condition (5.8) becomes

$$\left.\int_a^{z_1} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \alpha \int_{z_1}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \right\} \leq 0 \quad \text{for all } x \in [a, b]. \qquad (5.9)$$

Note that condition (5.7) always holds, as dictated by the definition of $z_1$. Moreover, condition (5.9) holds if and only if

$$V(\alpha, z_1) \equiv \left.\int_a^{z_1} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \alpha \int_{z_1}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \right\} \leq 0. \qquad (5.10)$$

This is so because, according to the definition of $z_M$,

$$\int_a^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \geq \int_a^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz =$$

$$\int_a^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \int_{z_M}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz, \qquad \text{for all } x \in [a, b].$$

Therefore, as $\alpha \in (0, 1)$,

$$\alpha \int_{z_M}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \leq 0 \quad \text{for all } x \in [z_M, b]. \qquad (5.11)$$

The function $V(\alpha, z_1, )$ defined in (5.10) is strictly increasing in $\alpha$, since

$$\frac{\partial V(\alpha, z_1, z_M)}{\partial \alpha} = \int_{z_1}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz > 0,$$

where the strict inequality comes from the definitions of $z_1$ and $z_M$. Moreover,

$$V(0, z_1) = \int_a^{z_1} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz < 0.$$

and

$$V(1, z_1) = \int_a^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz > 0.$$

Therefore, we can choose the unique value $\alpha^* \in (0, 1)$ for which

$$V(\alpha^*, z_1) = 0. \qquad (5.12)$$

The real number $\alpha^*$ is the largest value of $\alpha$ satisfying $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,z_1)}$, that is,

$$\alpha^* = \max\Big\{\alpha \in \mathbb{R} \text{ such that } \int_a^y \Big[F_{k(\tilde{x};\alpha,z_1)}(k) - F_{k(\tilde{y};\alpha,z_1)}(k)\Big]\,dk \le 0$$
$$\text{for all } y \in [k(a;\alpha,z_1), k(b;\alpha,z_1)]\Big\}.$$

Therefore, according to (5.10) and (5.12)), $\alpha^*$ would be given by

$$\alpha^* = -\frac{\displaystyle\int_a^{z_1} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\,dz}{\displaystyle\int_{z_1}^{z_M} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\,dz}, \tag{5.13}$$

where $z_1$ and $z_M$ are given in (5.5) and (5.6).

As follows from (5.10), the inequality $V(\alpha, z_1) < 0$ holds for all $\alpha < \alpha^*$, and this implies that

$$\int_a^y [F_{\tilde{x}}(k) - F_{\tilde{y}}(k)]\,dk < 0 \quad \text{for all } y \in [a, \alpha b - (1-\alpha)z_1],$$

which in turn means that $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succ} F_{k(\tilde{y};\alpha,z_1)}$ for all $\alpha < \alpha^*$.

Finally, for all $\alpha > \alpha^*$ there exists a number $y \in (z_1, \alpha b - (1-\alpha)z_1)$ such that

$$\int_a^y [F_{\tilde{x}}(k) - F_{\tilde{y}}(k)]\,dk > 0,$$

while

$$\int_a^{z_1} [F_{\tilde{x}}(k) - F_{\tilde{y}}(k)]\,dk < 0.$$

According to Proposition 1, the previous two inequalities mean that neither $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,z_1)}$ nor $F_{k(\tilde{y};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{x};\alpha,z_1)}$ for all $\alpha > \alpha^*$. ■

The next corollary shows that the pair $(\alpha^*, z_1)$ is in some sense unique. In particular, if we had chosen a point different from $z_1$, as defined in (5.5), in the functional form of the function $k(\cdot;\alpha,z_1)$, the value of the maximal slope $\alpha^*$ should be smaller in order to preserve stochastic dominance. It follows then that our characterization of $\alpha^*$ is sharp.

**Corollary 4.** *Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M. Consider the set of pairs of numbers $\{\hat{\alpha}, \hat{z}\} \in (0,1) \times (a,b)$ for which $F_{k(\tilde{x};\alpha,\hat{z})} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,\hat{z})}$ and $F_{k(\tilde{x};\alpha,\hat{z})} \underset{D}{\succ} F_{k(\tilde{y};\alpha,\hat{z})}$ for all $\alpha < \hat{\alpha}$, whereas neither $F_{k(\tilde{x};\alpha,\hat{z})} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,\hat{z})}$ nor $F_{k(\tilde{y};\alpha,\hat{z})} \underset{D}{\succsim} F_{k(\tilde{x};\alpha,\hat{z})}$ for all $\alpha > \hat{\alpha}$. Then, $\hat{\alpha} \le \alpha^*$.*

**Proof.** Note from (5.10) and (5.12) that the pairs $\{\hat{\alpha}, \hat{z}\}$ must satisfy

$$V(\hat{\alpha}, \hat{z}) \equiv \int_a^{\hat{z}} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz + \hat{\alpha} \int_{\hat{z}}^{z_M} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz = 0. \tag{5.14}$$

In order to preserve MSOSD for the transformed random variables, $\hat{z}$ must belong to the interval $[a, c]$ satisfying conditions (5.3) and (5.4). Furthermore, both the definition of $z_1$ in (5.5) and the fact that $\alpha^* \in (0,1)$ imply that $V(\alpha^*, \hat{z}) \geq V(\alpha^*, z_1) = 0$. Therefore, since the function $V(\alpha, z)$ is strictly decreasing in $\alpha$, it follows from (5.14) that $\hat{\alpha} \leq \alpha^*$. ■

We conclude this section with a technical remark concerning the location of the value $z_M$ defined in (5.6) when the two original distributions have the same mean.[5] Note that the value $z_M$ could be located at the upper limit of the interval $[a, b]$. However, if we assume that the distributions of the random variables $\tilde{x}$ and $\tilde{y}$ satisfy $E_{F_{\tilde{x}}} = E_{F_{\tilde{y}}}$, then $z_M < b$. We first state the following lemma:

**Lemma 1.** *Assume that the distribution functions $F_{\tilde{x}}$ and $F_{\tilde{y}}$ satisfy $E_{F_{\tilde{x}}} = E_{F_{\tilde{y}}}$ and neither $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ nor $F_{\tilde{y}} \underset{D}{\succsim} F_{\tilde{x}}$. Then, the function $F_{\tilde{x}} - F_{\tilde{y}}$ must change sign on $(a, b)$ at least twice.*

**Proof.** Since neither $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ nor $F_{\tilde{y}} \underset{D}{\succsim} F_{\tilde{x}}$, it is well known that $F_{\tilde{x}} - F_{\tilde{y}}$ must change sign at least once on $(a, b)$.[6] Let us proceed by contradiction and assume that the right-continuous function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign only once so that, without loss of generality, assume that $F_{\tilde{x}}(x) \leq F_{\tilde{y}}(x)$ for all $x \in [a, x^*)$, and $F_{\tilde{x}}(x) > F_{\tilde{y}}(x)$ for all $x \in (x^*, b)$. Therefore, letting $H(x) = \int_a^x [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz$ , we have that $H(x^*) \leq 0$. Clearly, $H(x)$ is increasing for $x \in [x^*, b]$. Moreover, $H(b) = 0$ since, by integrating by parts,

$$\int_a^b [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz = -\int_{[a,b)} z dF_{\tilde{x}}(z) + \int_{[a,b]} z dF_{\tilde{y}}(z) = -E_{F_{\tilde{x}}} + E_{F_{\tilde{y}}} = 0. \tag{5.15}$$

Therefore $H(x) \leq 0$ for all $x \in [a, b]$, which means that $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$, and this is the desired contradiction. ■

**Corollary 5.** *Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}\}$ satisfies Assumption M and $E_{F_{\tilde{x}}} = E_{F_{\tilde{y}}}$. Then, $z_M \in (a, b)$ and the function $F_{\tilde{x}} - F_{\tilde{y}}$ changes sign at $z_M$.*

[5]This is the scenario considered by Atkinson (1970).

[6]See Hadar and Rusell (1969).

**Proof.** From (5.6) and the fact that $F_{\tilde{x}} \underset{D}{\succsim} F_{\tilde{y}}$ does not hold, we get

$$\int_a^{z_M} [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz \neq 0.$$

Moreover, from (5.15), we have

$$\int_a^b [F_{\tilde{x}}(z) - F_{\tilde{y}}(z)]\, dz = 0.$$

Therefore, $z_M < b$. Finally as $z_M$ is interior, it is clear from (5.6) that $F_{\tilde{x}} - F_{\tilde{y}}$ must change sign at $z_M$. ■

Note then that when the pair $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M with $E_{F_{\tilde{x}}} = E_{F_{\tilde{y}}}$, the function $F_{\tilde{x}} - F_{\tilde{y}}$ must change sign both at $z_1$ and at $z_M$, which agrees with the statement of Lemma 1.

### 5.2.2. The differentiable case

The transformation $k(\cdot; \alpha^*, z_1)$ of the original random variables proposed in Proposition 2 in order to obtain MSOSD has the undesirable property of being non-differentiable. Obviously, all the increasing and concave transformations of the function $k(\cdot; \alpha^*, z_1)$ are also non-differentiable at $z_1$. However, these functions can be arbitrarily approximated by a differentiable function, as the next proposition shows:

**Proposition 3.** *Consider the class of functions defined on $[a, b]$ with the following functional form:*

$$q(z; \varepsilon, \beta, z_1) = \begin{cases} (1+\varepsilon)z - \varepsilon z_1 + \varepsilon^2 & \text{for} \quad z \in [a, z_1 - \varepsilon] \\ \\ g(z) & \text{for} \quad z \in (z_1 - \varepsilon, z_1 + \varepsilon) \\ \\ (\beta - \varepsilon)z + (1 - \beta + \varepsilon)z_1 + \varepsilon^2 & \text{for} \quad z \in [z_1 + \varepsilon, b]\,. \end{cases} \tag{5.16}$$

*Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M. Then, for all $\beta \in (0, \alpha^*)$, there exists a real number $\varepsilon > 0$ and a function $g(\cdot)$ such that the function $q(\cdot; \varepsilon, \beta, z_1)$ is smooth, increasing, concave, and satisfies $F_{q(\tilde{x};\eta,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\eta,\beta,z_1)}$ for all $\eta \in (0, \varepsilon]$ .*

**Proof.** According to Proposition 2, we can choose a real number $\beta \in (0, \alpha^*)$ for which $F_{k(\tilde{x};\beta,z_1)} \underset{D}{\succ} F_{k(\tilde{y};\beta,z_1)}$, where $z_1$ and $\alpha^*$ are defined in (5.5) and (5.13), respectively. Since we have strict MSOSD, we can slightly perturb the continuous function $k(\cdot, \beta, z_1)$, whose functional form is given in (5.1), while preserving strict

MSOSD. Hence, for a given smooth, increasing and concave function $g$, there exist a sufficiently small real number $\varepsilon > 0$ such that the function $q(z;\varepsilon,\beta,z_1)$ with the functional form given in (5.16) satisfies $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$. Note that in order to make the function $q(\cdot;\varepsilon,\beta,z_1)$ smooth we need to pick $g$ so that $g'(z_1-\varepsilon) = 1+\varepsilon$, $g'(z_1+\varepsilon) = \beta-\varepsilon$, and all the higher order derivatives of $g$ evaluated both at $z_1-\varepsilon$ and at $z_1+\varepsilon$ must be equal to zero. Such a function $g$ exists according to Borel's theorem. In particular, consider the smooth function

$$f_1(z) = \begin{cases} e^{-1/z} & \text{for} \quad z > 0 \\ \\ 0 & \text{for} \quad z \leq 0. \end{cases} \tag{5.17}$$

Let

$$f_2(z) = f_1\left(m \cdot [z - z_1 + \varepsilon]\right) \cdot f_1(z_1 + \varepsilon - z), \tag{5.18}$$

where $m$ is a strictly positive real number, and define

$$f_3(z) = \frac{\int_{-\infty}^{z} f_2(x)dx}{\int_{-\infty}^{\infty} f_2(x)dx}. \tag{5.19}$$

Clearly $f_3$ is a smooth function with $f_3(z) = 0$ for $z \leq z_1 - \varepsilon$, $f_3(z) = 1$ for $z \geq z_1 + \varepsilon$, and $f_3$ is strictly increasing on $(z_1 - \varepsilon, z_1 + \varepsilon)$. Consider now the smooth decreasing function $f_4(z) = 1 + \varepsilon - (1 + 2\varepsilon - \beta)f_3(z)$. Therefore, $g(z) = \int_a^z f_4(x)dx - \varepsilon z_1 + \varepsilon^2$ is the desired smooth function provided we choose the positive scalar $m$ so that the condition $g(z_1 + \varepsilon) = z_1 + \beta\varepsilon$ is met. This condition ensures the continuity of the function $q(z;\varepsilon,\beta,z_1)$ at $z_1 + \varepsilon$. Note also that

$$g'(z) = f_4(z) = 1 + \varepsilon - (1 + 2\varepsilon - \beta)f_3(z) \geq -\varepsilon + \beta > 0, \tag{5.20}$$

where the week inequality comes from the fact that $f_3(z) \leq 1$, and the strict inequality holds for a sufficiently small value of $\varepsilon$. Moreover,

$$g''(z) = f_4'(z) = -(1 + 2\varepsilon - \beta)f_3'(z) = -(1 + 2\varepsilon - \beta)\frac{f_2(z)}{\int_{-\infty}^{\infty} f_2(x)dx} < 0, \tag{5.21}$$

where the inequality holds since $\beta < 1$ and $f_2(z) > 0$ for $z \in (z_1 - \varepsilon, z_1 + \varepsilon)$. It is then obvious that $F_{q(\tilde{x};\eta,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\eta,\beta,z_1)}$ for all $\eta \in (0,\varepsilon)$. ■

The next proposition shows that, if the non-differentiable function $k(z;\alpha,z_1)$ is picked so that neither $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,z_1)}$ nor $F_{k(\tilde{y};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{x};\alpha,z_1)}$, then this function can also be approximated by a smooth function:

**Proposition 4.** *Consider the class of functions with the functional form given in* (5.16). *Assume that the pair of distribution functions* $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ *satisfies Assumption*

*M. Then, for all $\beta \in (\alpha^*, 1)$, there exists a real number $\varepsilon > 0$ and a function $g(\cdot)$ such that the function $q(\cdot;\varepsilon,\beta,z_1)$ is smooth, increasing, concave, and neither $F_{q(\tilde{x};\eta,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{y};\eta,\beta,z_1)}$ nor $F_{q(\tilde{y};\eta,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{x};\eta,\beta,z_1)}$ for all $\eta \in (0,\varepsilon]$.*

**Proof.** Construct a function $q(z;\varepsilon,\beta,z_1)$ having the functional form given in (5.16) with $\beta \in (\alpha^*, 1)$, where $z_1$ and $\alpha^*$ are defined in (5.5) and (5.13), respectively. The function $q(z;\varepsilon,\beta,z_1)$ can be obviously constructed so that neither $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$ nor $F_{q(\tilde{y};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{x};\varepsilon,\beta,z_1)}$ for a sufficiently small real number $\varepsilon > 0$, by following the same steps of the proof of Proposition 3. ■

It is important to point out that in our previous two propositions we have chosen the functional form (5.16) and not another because that form allow us to analyze explicitly the global behavior of its ARA index. The following corollary characterizes the limiting behavior of the infimum and the supremum of the ARA index of the function $q(\cdot;\varepsilon,\beta,z_1)$ defined in (5.16) as $\varepsilon$ becomes arbitrarily small:

**Corollary 6.**

*(a)* $\inf\limits_{z\in(a.b)} \left( \lim\limits_{\varepsilon\to 0} A_{q(\cdot;\varepsilon,\beta,z_1)}(z) \right) = 0;$

*(b)* $\sup\limits_{z\in(a.b)} \left( \lim\limits_{\varepsilon\to 0} A_{q(\cdot;\varepsilon,\beta,z_1)}(z) \right) = \infty.$

**Proof.** (a) Obvious, since the function $q(z;\varepsilon,\beta,z_1)$ is linear for $z \notin (z_1-\varepsilon, z_1+\varepsilon)$.

(b) From the construction of $q(\cdot;\varepsilon,\beta,z_1)$, we can compute the ARA index on the interval $(z_1-\varepsilon, z_1+\varepsilon)$ as

$$A_{q(\cdot;\varepsilon,\beta,z_1)}(z) = \frac{-q''(z;\varepsilon,\beta,z_1)}{q'(z;\varepsilon,\beta,z_1)} = \frac{-g''(z)}{g'(z)} = \frac{(1+2\varepsilon-\beta)f_2(z)}{1+\varepsilon-(1+2\varepsilon-\beta)\int_{-\infty}^{z} f_2(x)dx},$$

where the third equality comes from the definition of $f_3$ in (5.19) and from (5.20) and (5.21). Therefore,

$$\lim_{\varepsilon\to 0}\left(\lim_{z\to z_1}\frac{-q''(z;\varepsilon,\beta,z_1)}{q'(z;\varepsilon,\beta,z_1)}\right) = \left(\frac{1-\beta}{\beta}\right)\lim_{\varepsilon\to 0}\left(\lim_{z\to z_1}\left(\frac{f_2(z)}{\int_{-\infty}^{z} f_2(x)dx}\right)\right) =$$

$$\left(\frac{1-\beta}{\beta}\right)\lim_{\varepsilon\to 0}\left(\lim_{z\to z_1}\left(\frac{\exp\left\{-\frac{1-m}{m\cdot(z-z_1)}\right\}}{\int_{-\infty}^{z} f_2(x)dx}\right)\right), \tag{5.22}$$

where the last equality comes from the expressions for $f_1$ and $f_2$ given in (5.17) and (5.18), respectively. In order to compute the limit (5.22) observe that, on the one hand, $\lim_{\varepsilon\to 0}\left(\lim_{z\to z_1}\left(\int_{-\infty}^{z} f_2(x)dx\right)\right)=0$, since $f_2$ is bounded and continuous. On the other hand, we get

$$\lim_{z\to z_1}\left(\exp\left\{-\frac{1-m}{m\cdot(z-z_1)}\right\}\right)=\begin{cases}0 & \text{for } m\in[0,1)\\ 1 & \text{for } m=1\\ \infty & \text{for } m>1.\end{cases}$$

As $\beta\in(0,1)$, we thus have

$$\lim_{\varepsilon\to 0}\left(\lim_{z\to z_1}\frac{-q''(z;\varepsilon,\beta,z_1)}{q'(z;\varepsilon,\beta,z_1)}\right)=\infty \quad \text{for } m\geq 1.$$

For $m\in[0,1)$, we apply L'Hôpital's rule so that

$$\lim_{z\to z_1}\left(\frac{\exp\left\{-\frac{1-m}{m\cdot(z-z_1)}\right\}}{\int_{-\infty}^{z} f_2(x)dx}\right)=\lim_{z\to z_1}\left(\frac{1-m}{m\cdot(z-z_1)^2}\right)=\infty.$$

Therefore, we obtain the desired conclusion for all $m>0$. ■

We have considered so far utility functions that exhibit local risk neutrality almost everywhere except in a small neighborhood of a point where all the risk aversion is concentrated. In the next section we will use a completely different approach, since instead of concentrating all the concavity in a small interval, we are going to consider transformations of the original random variables through functions that have all the risk aversion uniformly distributed over its domain.

### 5.3. Constant absolute risk aversion

In this section we will show that, given two random variables $\tilde{x}$ and $\tilde{y}$ such that $F_{\tilde{x}} \underset{M}{\succ} F_{\tilde{y}}$ , then there exists a CARA utility function $r(\cdot;s)$ having an absolute risk aversion index $s$ for which $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$. Recall that, if one of the two random variables is strictly preferred to the other according to the max-min criterion, then the integral condition (2.2) will hold for an interval of low realizations of these variables. A CARA transformation of these variables attaches a relative lower weight to high realizations, and this relative weight decreases with the ARA index $s$. Therefore, for a sufficiently large value of $s$ the integral condition (2.2) will be satisfied over the whole range of values of the transformed random variables. The following proposition establishes the basic existence result:

**Proposition 5.** *Assume that the pair of distribution functions $\{F_{\tilde{x}},F_{\tilde{y}}\}$ satisfies Assumption M. Then, there exists a real number $s^*$ such that $F_{r(\tilde{x};s^*)} \underset{D}{\succsim} F_{r(\tilde{y};s^*)}$ and*

$F_{r(\tilde{x};s)} \underset{D}{\succ} F_{r(\tilde{y};s)}$ *for all* $s > s^*$*, whereas neither* $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$ *nor* $F_{r(\tilde{y};s)} \underset{D}{\succsim} F_{r(\tilde{x};s)}$ *for all* $s < s^*$*.*

**Proof.** Consider first the class of continuously differentiable, increasing and concave utility functions with the following functional form:

$$v(z; s, z_0) = \begin{cases} -\dfrac{1}{s}e^{-s(z-z_0)} + z_0 + \dfrac{1}{s} & \text{for} \quad x \in [a, z_0) \\ \\ z & \text{for} \quad x \in [z_0, b]\,, \end{cases} \tag{5.23}$$

where $s > 0$ and $z_0$ is the smallest value on $[a, b]$ at which the function $F_{\tilde{x}}(z) - F_{\tilde{y}}(z)$ changes sign (see Definition 2). Let us find a value of $s$ for which $F_{v(\tilde{x};s,z_0)} \underset{D}{\succsim} F_{v(\tilde{y};s,z_0)}$, that is,

$$\int_{v(a;s)}^{y} \left[F_{v(\tilde{x};s,z_0)}(v) - F_{v(\tilde{y};s,z_0)}(v)\right] dv \leq 0 \quad \text{for all } y \in [v(a; s, z_0), b]\,.$$

By performing the corresponding change of variable, the previous inequality becomes:

$$\int_{a}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] v'(z; s, z_0) dz \leq 0 \quad \text{for all } x \in [a, b]\,,$$

which in turn can be decomposed into the following two inequalities:

$$\int_{a}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] v'(z; s, z_0) dz \leq 0 \quad \text{for all } x \in [a, z_0]\,, \tag{5.24}$$

and

$$\int_{a}^{z_0} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] v'(z; s, z_0) dz + \int_{z_0}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \leq 0$$

$$\text{for all } x \in [z_0, b]\,. \tag{5.25}$$

Note that inequality (5.24) always holds, since the integrand is non-positive by the definition of $z_0$. Taking into account the definition of $z_M$ in (5.6), we know that

$$\int_{z_M}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \leq 0, \;\; \text{for all } x \in [z_M, b]$$

Define the real number $z_N$ as

$$z_N = \max\left\{\underset{x\in[a,b]}{\arg\min} \int_{a}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz\right\}. \tag{5.26}$$

Hence, we get

$$\int_{z_N}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \geq \int_{z_0}^{x} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \quad \text{for all } x \in [z_0, b]\,.$$

Therefore, (5.25) holds whenever

$$\int_a^{z_0} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] v'(z;s)dz + \int_{z_N}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz = 0. \tag{5.27}$$

Let $K = \int_{z_N}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz$ and

$$H(x) = \int_a^x \left[F_{\tilde{y}}(z) - F_{\tilde{x}}(z)\right] dz \quad \text{for } x \in [a, z_0].$$

The mapping $H(x)$ is an increasing and right-continuous function which induces a Lebesgue-Stieltjes measure on $[a, z_0]$. Therefore, (5.27) can be written as

$$\int_a^{z_0} v'(z;s,z_0)dH(z) = K. \tag{5.28}$$

Moreover, by the definition of $z_M$,

$$-\int_a^{z_0} dH(z) + K = \int_a^{z_0} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \int_{z_N}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz \geq$$

$$\int_a^{z_0} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz + \int_{z_0}^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz = \int_a^{z_M} \left[F_{\tilde{x}}(z) - F_{\tilde{y}}(z)\right] dz > 0.$$

Therefore, letting $C = \int_a^{z_0} dH(z)$, we can conclude that $K > C > 0$. Moreover, by noticing that $v'(z;s) = e^{-s(z-z_0)}$, equation (5.28) becomes

$$\int_a^{z_0} e^{-s(z-z_0)} dH^*(z) = \frac{K}{C}, \tag{5.29}$$

where $H^*(z) = \dfrac{H(z)}{C}$ is a distribution function on $[a, z_0]$ because $H^*(z_0) = 1$. Equation (5.29) has a unique solution for $s$, since $\frac{K}{C} > 1$, the LHS of (5.29) is strictly increasing in $s$, $\lim_{s \to 0} \int_a^{z_0} e^{-s(z-z_0)} dH^*(z) = 1$ and $\lim_{s \to \infty} \int_a^{z_0} e^{-s(z-z_0)} dH^*(z) = \infty$. Let $\hat{s}$ be the unique solution of equation (5.29). Clearly, the inequality in (5.25) becomes strict whenever $s > \hat{s}$.

Consider now the increasing and concave function

$$w(z;\hat{s},z_0) = \begin{cases} z & \text{for} \quad z \in [a, z_0) \\ -\dfrac{1}{\hat{s}} e^{-\hat{s}(z-z_0)} + z_0 + \dfrac{1}{\hat{s}} & \text{for} \quad z \in [z_0, b]. \end{cases}$$

The increasing and concave transformation of $v(z;\hat{s},z_0)$ given by $w(v(z;\hat{s},z_0);\hat{s},z_0)$ exhibits a constant ARA index, since

$$w(v(z;\hat{s},z_0);\hat{s},z_0) = -\frac{1}{\hat{s}} e^{-\hat{s}(z-z_0)} + z_0 + \frac{1}{\hat{s}} \quad \text{for } z \in [a, b].$$

Obviously, $F_{w(v(\tilde{x};s,z_0);s,z_0)} \underset{D}{\succ} F_{w(v(\tilde{y};s,z_0);s,z_0)}$ for all $s > \hat{s}$. Hence, $F_{r(\tilde{x};s)} \underset{D}{\succ} F_{r(\tilde{y};s)}$ for all $s > \hat{s}$, where $r(\cdot;s)$ is a CARA utility function with an ARA index equal to $s$. Since, for $s$ sufficiently close to zero, neither $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$ nor $F_{r(\tilde{y};s)} \underset{D}{\succsim} F_{r(\tilde{x};s)}$, we can find by continuity the value $s^* \in (0, \hat{s})$ for which $F_{r(\tilde{x};s^*)} \underset{D}{\succsim} F_{r(\tilde{y};s^*)}$ and $F_{r(\tilde{x};s)} \underset{D}{\succ} F_{r(\tilde{y};s)}$ for all $s > s^*$, whereas neither $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$ nor $F_{r(\tilde{y};s)} \underset{D}{\succsim} F_{r(\tilde{x};s)}$ for all $s < s^*$. ■

The following corollary extends the previous proposition to functions that are not necessarily CARA. In order to obtain MSOSD between two random variables we only require a sufficiently large value of the ARA index on some interval $(a, z_0)$ with $z_0 < b$.

**Corollary 7.** *Assume that the pair of distribution functions $\{F_{\tilde{x}}, F_{\tilde{y}}\}$ satisfies Assumption M. Then, there exists a pair of real numbers $\{\hat{s}, z_0\} \in (0, \infty) \times (a, b)$ such that $F_{\tilde{x}} \underset{u}{\succsim} (\underset{u}{\succ}) F_{\tilde{y}}$ for every twice differentiable, increasing and concave Bernoulli utility function $u$ satisfying $A_u(z) \geq (>) \hat{s}$ for $z \in (a, z_0)$.*

**Proof.** Obvious from the proof of Corollary 1 and from Pratt (1964), since $u$ is an increasing and (strictly) concave transformation of the utility function $v(\cdot; \hat{s}, z_0)$, whose functional form is given in the expression (5.23) and that satisfies $F_{v(\tilde{x};s^*,z_0)} \underset{D}{\succsim} F_{v(\tilde{y};s^*,z_0)}$. ■

As follows from the proofs of the previous corollary and of Proposition 5, the upper limit $z_0$ of the interval where strict concavity is required turns out to be the smallest value at which the function $F_{\tilde{x}}(z) - F_{\tilde{y}}(z)$ changes sign. Moreover the critical value $\hat{s}$ of the ARA index on the interval $(0, z_0)$ is given by the value of $s$ solving equation (5.29).

### 5.4. The proofs of the main results

We have considered in Sections 5.2.2 and 5.3 two basic families of functions for which the transformations of the random variables $\tilde{x}$ and $\tilde{y}$ through these functions can be ranked according to the MSOSD criterion. One family was that of the CARA functions, which was analyzed in Section 5.3. Recall that $r(\cdot; s)$ denotes a CARA utility displaying an ARA index equal to $s$. The other family is formed by functions that have all the risk aversion concentrated on a small interval of its domain. The functional form of a function belonging to the latter class is given in (5.16). Note that the function $q(\cdot; \varepsilon, \beta, z_1)$ is an increasing, concave and smooth function that is linear for all values that do not belong to the interval $(z_1 - \varepsilon, z_1 + \varepsilon)$. Moreover, the derivative of $q(\cdot;\varepsilon,\beta,z_1)$ is equal to $1+\varepsilon$ for all values of the interval $(a, z_1-\varepsilon)$, while its derivative is equal to $\beta-\varepsilon$ for all values belonging to $(z_1+\varepsilon, b)$.

The proof our main result (Theorem 3.1) relates Propositions 3, 4 and 5 with the partition formed by the sets $I(s)$:

**Proof of Theorem 3.1.** (a) Let $s^*$ be the real number defined in Proposition 5. Note that, if $u \in I(s)$ with $s > s^*$, then $u$ is an increasing and concave transformation of the CARA utility $r(\cdot;s)$, since $A_u(z) \geq s$ for all $z \in (a,b)$ (see Pratt, 1964). Then, as $F_{r(\tilde{x};s)} \underset{D}{\succ} F_{r(\tilde{y};s)}$, we must have $F_{u(\tilde{x})} \underset{D}{\succ} F_{u(\tilde{y})}$ as follows from Corollary 1 .

(b) Consider the utility function $q(z;\varepsilon,\beta,z_1)$ characterized in Proposition 3 so that $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$. Clearly, the infimum of the ARA index of $q(z;\varepsilon,\beta,z_1)$ is zero. Obviously, any concave function $w$ will satisfy $F_{w(q(\tilde{x};\varepsilon,\beta,z_1))} \underset{D}{\succ} F_{w(q(\tilde{y};\varepsilon,\beta,z_1))}$ as follows from Corollary 1. Therefore, the infimum of the ARA index of $w(q(\cdot;\varepsilon,\beta,z_1))$ can take any positive value.

(c) Obvious from Proposition 5 since the CARA utility $r(\cdot;s)$ belongs to $I(s)$. ∎

**Proof of Corollary 2.** Obviously, the critical value $s^*$ of the ARA index yielding stochastic dominance for CARA utilities is the smallest positive real value of $s$ satisfying

$$\int_{r(a;s)}^{y} \left[F_{r(\tilde{x};s)}(r) - F_{r(\tilde{y};s)}(r)\right] dr \leq 0 \quad \text{for all } y \in [r(a;s), r(b;s)]\,. \tag{5.30}$$

From Proposition 5 we know that such a critical value $s^*$ exists. Therefore, by performing the corresponding change of variable in (5.30), $s^*$ turns out to be the smallest positive real value of $s$ satisfying (3.1). ∎

The proof of Theorem 3.2 uses arguments similar to those of Theorem 3.1, as we next show:

**Proof of Theorem 3.2.** (a) Let $s^*$ be the real number defined in Proposition 5. Note that, if $u \in P(s)$ with $s < s^*$, then the CARA utility $r(\cdot;s)$ is an increasing and concave transformation of $u$ since $A_u(z) \leq s$ for all $z \in (a,b)$ (see Pratt, 1964). However, if either $F_{u(\tilde{x})} \underset{D}{\succsim} F_{u(\tilde{y})}$ or $F_{u(\tilde{y})} \underset{D}{\succsim} F_{u(\tilde{x})}$, then either $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$ or $F_{r(\tilde{y};s)} \underset{D}{\succsim} F_{r(\tilde{x};s)}$, respectively, as follows from Corollary 1. But this cannot occur since, by construction, neither $F_{r(\tilde{x};s)} \underset{D}{\succsim} F_{r(\tilde{y};s)}$ nor $F_{r(\tilde{y};s)} \underset{D}{\succsim} F_{r(\tilde{x};s)}$ for all $s < s^*$.

(b) Obvious from Proposition 5, since the CARA utility $r(\cdot;s)$ belongs to $P(s)$.

(c) Consider the utility function $q(z;\varepsilon,\beta,z_1)$ characterized in Proposition 4 so that neither $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$ nor $F_{q(\tilde{y};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{x};\varepsilon,\beta,z_1)}$. By making $\varepsilon$ arbitrarily small, the supremum of the ARA index of $q(\cdot;\varepsilon,\beta,z_1)$ can be made arbitrarily large (see part (b) of Corollary 6). Therefore, for every set $P(s)$ with $s$ finite, there exists a function $u \in P(s)$ that is an increasing and convex transformation $w$ of the function $q(\cdot;\varepsilon,\beta,z_1)$. Note that in order to make $u$ concave, the function $w$ must be linear for all values that do not belong to the interval $(z_1-\varepsilon, z_1+\varepsilon)$. It immediately follows that neither $F_{w(q(\tilde{x};\varepsilon,\beta,z_1))} \underset{D}{\succsim} F_{w(q(\tilde{y};\varepsilon,\beta,z_1))}$ nor $F_{w(q(\tilde{y};\varepsilon,\beta,z_1))} \underset{D}{\succsim} F_{w(q(\tilde{x};\varepsilon,\beta,z_1))}$ for some function $w$, since otherwise all concave transformations of $w\left(q(z;\varepsilon,\beta,z_1)\right)$ will make one random variable preferred to the other according to the MSOSD criterion, and this contradicts the fact that neither $F_{q(\tilde{x};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{y};\varepsilon,\beta,z_1)}$ nor $F_{q(\tilde{y};\varepsilon,\beta,z_1)} \underset{D}{\succsim} F_{q(\tilde{x};\varepsilon,\beta,z_1)}$ . ■

Finally, the proof of Theorem 3.3 applies the concept of thriftiness to the class of essentially linear utility functions discussed in Sections 4 and 5. It is obvious that the thriftiness of the function $k(\cdot;\alpha,z_1)$ defined in (5.1) is $T(k(\cdot;\alpha,z_1)) = 1/\alpha$. Hence, the following proof is just a straightforward application of Propositions 2 and 3:

**Proof of Theorem 3.3.** (a) Just note that the function $k(\cdot;\alpha,z_1)$ defined in Proposition 2 satisfies $F_{k(\tilde{x};\alpha,z_1)} \underset{D}{\succsim} F_{k(\tilde{y};\alpha,z_1)}$ for all $\alpha \leq \alpha^*$, and $T\left(k(\cdot;\alpha,z_1)\right) = 1/\alpha$ .

(b) The smooth, increasing and concave function $q(\cdot;\eta,\beta,z_1)$ defined in Proposition 3 satisfies $F_{q(\tilde{x};\eta,\beta,z_1)} \underset{D}{\succ} F_{q(\tilde{y};\eta,\beta,z_1)}$, and

$$T\left(q(\cdot;\eta,\beta,z_1)\right) = \frac{1+\varepsilon}{\beta-\varepsilon} > \frac{1}{\beta} > \frac{1}{\alpha^*},$$

where the inequalities follow from the construction of the function $q(\cdot;\eta,\beta,z_1)$. To finish the proof of this part we just have to define $t = 1/\alpha$ and $t^* = 1/\alpha^*$. ■

**Proof of Corollary 3.** It follows directly from expressions (5.3), (5.4), (5.5), (5.6) and (5.13), and because $t^* = 1/\alpha^*$. ■

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