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Forming Efficient Networks

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Abstract

In this paper, we suggest a simple sequential mechanism whose subgame perfect equilibria give rise to efficient networks. Moreover, the payoffs received by the agents coincide with their Shapley value in an appropriately defined cooperative game.

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1 Introduction

Recently there has been a surge of interest in economic environments that can be described via graphs, where the activity takes the form of creating links among agents. This type of structure can be applied to different contexts such as the analysis of the internal organization of firms or cost allocation schemes. A major concern for such environments is the attainment of efficient networks, that is, networks for which the output produced exceeds or equals the amount of output achieved by any other network.

In this paper, we suggest a simple sequential mechanism with the property that its equilibrium outcomes generate an efficient network. The mechanism adapts to the network environment the proposal made in Pérez-Castrillo and Wettstein [11].[1]. The mechanism has the agents first choose a proposer among themselves via a bidding game. The proposer then proposes a network structure and a vector of transfers. If the other agents accept his proposal, it is carried out. In case the offer is rejected the proposer is removed and is left on his own. The remaining agents again use the mechanism starting from the bidding stage.

We show that the subgame perfect equilibria of the mechanism generate efficient networks. Moreover, the payoffs received by the agents coincide with their Shapley values in an appropriately defined cooperative game.

Jackson and Wolinsky [6] were the first authors to analyze the problem of generating efficient networks. They focused, to a large degree, on the tension between efficiency and stability. Dutta and Mutuswami [4] and Mutuswami and Winter [9] resolve this tension through the use of mechanisms, assuming that the planner has information regarding the value function. Navarro and Perea [10] suggest a non-cooperative bargaining procedure among pairs of agents whose subgame perfect equilibrium outcomes coincide with the value appearing in Jackson and Wolinsky [6].

An approach similar to ours can be found in Currarini and Morelli [2], who propose two sequential-move mechanisms whose subgame perfect equilibria outcomes generate efficient networks. The payoff configuration in their mechanisms is endogenously generated but is highly asymmetric, being sensitive to the order in which agents move. Also, Matsubayashi and Yamakawa [7] propose a mechanism to share the cost of constructing the network and show that some equilibria involve both efficient networks and equitable allocation rules. Finally, Samejima [12] implements the nucleoulus in subgame perfect equilibria and applies the mechanism to a network environment.

The main advantages of our proposal are the following: The payoff division is equitable, corresponding to the Shapley value of a cooperative game; the efficiency and equity properties of the mechanism hold for a large class of environments; the mechanism is simple and does not require out-of-

[1] See also Mutuswami, Pérez-Castrillo, and Wettstein [8].

equilibrium free disposal.

2 Forming networks

Let $N = \{1, \ldots, n\}$ be the set of agents. For any $S \subseteq N$, let g^S denote the set of all subsets of S of size 2. A *graph* or network, denoted generically by g is some subset of g^N. If $g \subset g^S$ where $S \subsetneq N$, we say that g is a graph restricted to S. A graph, therefore, is a structure of *bilateral relations* among agents. Clearly, agents i and j have a bilateral relation if and only if $\{i, j\} \in g$. We shall refer to the subset $\{i, j\}$ of g as the *link* between i and j and denote it by (ij).[2] We let G_S denote the set of all graphs involving links just between members of S: $g \in G_S$ and $(ij) \in g$ implies that $\{i, j\} \subset S$.

Given a graph g, agents i and j are said to be *connected in* g if there exists a sequence of agents $i = i_0, i_1, \ldots, i_K = j$ such that $(i_k i_{k+1}) \in g$ for all $k = 0, \ldots, K-1$. Let $N(g) \equiv \{i|$ there exists j such that $(ij) \in g\}$ denote the set of agents who have at least one bilateral relation. The graph $h \subset g$ is said to be a *connected component* of g if all agents in $N(h)$ are connected to each other in h, and for all $i \in N(h), j \in N \backslash N(h), (ij) \notin g$. The set of all connected components of g is denoted $C(g)$.

A *value function* is a mapping $v : G_N \to \Re$. We can think of the value of a graph g as representing the total surplus produced by agents when they form a set of bilateral relationships represented by g. We will restrict attention to value functions satisfying *component additivity*, that is, $v(g) = \sum_{h \in C(g)} v(h)$. Component additivity can be interpreted as absence of externalities between different components. We let V denote the set of component additive value functions. Given $v \in V$, a graph g is *strongly efficient* if $v(g) \geq v(g')$ for all $g' \in G_N$. An *allocation rule* is a mapping $Y : V \times G \to \Re^n$ satisfying $\sum_{i \in N} Y_i(v, g) = v(g)$. An allocation rule simply specifies the division of total surplus for each possible graph. An allocation rule Y is *component balanced* if $\sum_{i \in N(h)} Y_i(v, g) = v(h)$ for every $h \in C(g)$.

An example of an allocation rule is the one proposed by Jackson and Wolinsky [6] which associates to each graph, the Shapley value of a transferable utility game associated with the graph. Formally, fix v. For any graph g and $S \subseteq N$, let $g|S \equiv \{(ij)|(ij) \in g$ and $\{i, j\} \subseteq S\}$ denote the restriction of g to S. The Jackson-Wolinsky allocation rule for any graph g, which we denote by ϕ^g, is:[3]

$$\phi_i^g(N, v) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} [v(g|(S \cup \{i\})) - v(g|S)].$$

[2] Our emphasis on bilateral relationships means that the links in our framework are *non-directed*. We say more on the applicability of this mechanism to situations involving directed links later.

[3] This value is also referred to in the literature as the Myerson value.

Jackson and Wolinsky [6] show that ϕ^g is the unique allocation rule satisfying component balance and *equal bargaining power*.[4] They also note that this allocation rule may arise naturally if the allocations result from bargaining between agents. However, this bargaining is not modeled explicitly.

The previous value equates the worth of a coalition S to the surplus generated by looking at the restriction of g to S. This procedure, however, does not take into account the fact that the agents in S can form many other graphs besides $g|S$ and ideally one would like to take this into consideration.[5] However, the way to do this is not clear for an arbitrary graph g. Suppose however that we restrict attention to graphs which are strongly efficient. In this case, a natural possibility is to associate to each coalition S the *maximum surplus that can be derived by the members of S acting on their own.* One can now consider the transferable utility game (N, W) defined by $W(S) = \max\{v(g)|g \in G_S\}$ for all $S \subset N$ and the corresponding Shapley value. The game (N, W) can be easily seen to be super-additive; we denote by ϕ the Shapley value of the game (N, W) :

$$\phi_i(N, W) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}[W(S \cup \{i\}) - W(S)].$$

This Shapley value is the same as the value considered in Jackson [5] for the player-based flexible allocation network case.

It is easy to see that the restricted graph $g|S$ is not necessarily the graph that maximizes the surplus for S even if g itself is strongly efficient. The two approaches outlined above are thus bound to give different results. The following example illustrates this possibility.

Example 1 *Consider the following value function taken from Jackson and Wolinsky [6]. Let* $N = \{1, 2, 3\}$*, and the component-additive value function given by* $v((ij)) = v(g^N) = 1$ *and, for* $i \neq j, i \neq k, j \neq k$*,* $v((ij), (jk)) = 1+\epsilon$ *where* $0 < \epsilon < 1/6$*. The strongly efficient networks are of the form* $g^j = \{(ij), (jk)\}$*. The Jackson-Wolinsky procedure applied to any* g^j *gives the allocation* $(x_i, x_j, x_k) = ((1 + 2\epsilon)/6, (2 + \epsilon)/3, (1 + 2\epsilon)/6)$ *while the Shapley value of* (N, W) *gives the uniform payoff vector* $((1+\epsilon)/3, (1+\epsilon)/3, (1+\epsilon)/3)$*. This example also illustrates that, in contrast to the Jackson-Wolinsky rule, the proposed allocation here need not be component balanced. We view this as a consequence of having to take into account the strategic possibilities open to an agent* outside *the component to which he belongs.*

[4]The allocation rule Y satisfies equal bargaining power if $Y_i(v, g) - Y_i(v, g - (ij)) = Y_j(v, g) - Y_j(v, g - (ij))$ where $g - (ij)$ is the graph obtained by removing the link (ij) from g.

[5]Typically, the literature on social and economic networks assumes that agents have the right to decide which links they want to form. See for instance, the papers of Jackson and Wolinsky [6], Currarini and Morelli [2] or Dutta and Mutuswami [4]. In this context, it is thus necessary that the complete strategic possibilities open to an agent be considered.

The *bidding mechanism* that we propose for the network environment can be considered as a model of network formation in which a bargaining process is modeled explicitly. By playing this mechanism, connected components are formed sequentially. In stage 1 of the mechanism, the agents bid to choose the proposer. Each agent bids one number for each agent (including himself), the bids submitted by an agent must sum up to zero. The agent for whom the aggregate bid is the highest is chosen as the proposer. In stage 2 the proposer offers a vector of payments to all other agents and chooses a connected component he wants to form. The offer is accepted if all the other agents agree. In case of acceptance the connected component is formed and the agents outside it proceed to play the same game again among themselves. In the case of rejection all the agents other than the proposer play the same game again. Formally, the bidding mechanism for network formation operates as follows:

If there is only one agent i (say), he can only form the empty graph, $g = \emptyset$ and therefore, he obtains $v(\emptyset) = 0$.[6]

Given the rules for at most $n-1$ agents, the mechanism for $N = \{1, \ldots, n\}$ works as follows:

$t = 1$: Each agent $i \in N$ makes bids $b_j^i \in \Re$, one for every $j \in N$, with $\sum_{j \in N} b_j^i = 0$. Agents bid simultaneously.

For each $i \in N$, define the *aggregate bid* to agent i by $B_i = \sum_{j \in N} b_i^j$. Let $\alpha = \text{argmax}_i(B_i)$ where an arbitrary tie-breaking rule is used in the case of a non-unique maximizer. Once the winner α has been chosen, every agent $i \in N$ pays b_α^i and receives B_α/n.

$t = 2$: agent α chooses a subset of agents S_α (such that $\alpha \in S_\alpha$), a graph $g_\alpha^* \in G_{S_\alpha}$ (such that g_α^* is connected on S_α) and offers $x_i^\alpha \in \Re$ to every $i \in N \backslash \{\alpha\}$.

$t = 3$: The agents in $N \backslash \{\alpha\}$, sequentially, either accept or reject the offer. If an agent rejects it, then the offer is rejected. Otherwise, the offer is accepted.

If the offer is accepted, then the final payoff to agent $i \in S_\alpha \backslash \{\alpha\}$ is $x_i^\alpha - b_\alpha^i + B_\alpha/n$, agent α receives $v(g_\alpha^*) - \sum_{i \neq \alpha} x_i^\alpha - b_\alpha^\alpha + B_\alpha/n$ and agents in $N \backslash S_\alpha$ receive $x_i^\alpha - b_\alpha^i + B_\alpha/n$ plus what they obtain in the game played by $N \backslash S_\alpha$. If the offer is rejected, the final payoff to α is $-b_\alpha^\alpha + B_\alpha/n$ and final payoff to any $i \neq \alpha$ is the sum of $-b_\alpha^i + B_\alpha/n$ and the payoff obtained in the game played by $N \backslash \{\alpha\}$.

Theorem 1 *At any subgame perfect equilibrium of the bidding mechanism, a strongly efficient graph is always formed. Moreover, the payoffs to the agents are uniquely given by the Shapley value ϕ.*

[6] Component additivity implies that the value of an isolated player (and therefore, the empty graph) is zero.

Proof:. The proof proceeds via induction on the number of agents. The theorem trivially holds for the case of $n = 1$. We assume it holds for all $m \leq n-1$ and show it also holds for $m = n$.

Consider a game (N, v) with n agents. We claim (under the induction hypothesis) that the following strategies are the unique SPE of the mechanism and furthermore generate the Shapley value ϕ:

At $t = 1$, each agent i, $i \in N$, announces $b_j^i = \phi_i(N\backslash\{j\}, W) - \phi_i(N, W)$ for every $j \neq i$ and $b_i^i = W(N) - W(N\backslash\{i\}) - \phi_i(N, W)$.

At $t = 2$, agent i, if he is the proposer, chooses a subset of agents S_i and a graph (a connected component) g_i^* such that $v(g_i^*) + W(N\backslash S_i) = W(N)$. Moreover, he offers $x_j^i = \phi_j(N\backslash\{i\}, W)$ to every $j \in S_i\backslash\{i\}$, and $x_j^i = \phi_j(N\backslash\{i\}, W) - \phi_j(N\backslash S_i, W)$ to every $j \notin S_i$.

At $t = 3$, agent i, if agent $j \neq i$ is the proposer and $i \in S_j$, accepts any offer greater than or equal to $\phi_i(N\backslash\{j\}, W)$ and rejects it otherwise. If agent $j \neq i$ is the proposer and $i \notin S_j$, agent i accepts any offer greater than or equal to $\phi_i(N\backslash\{j\}, W) - \phi_i(N\backslash S_j, W)$ and rejects it otherwise.

By the induction argument all SPE must have all agents different from the proposer behave according to the strategy described in $t = 3$. The proposer at $t = 2$ will not make any offer larger than those prescribed by the strategies at $t = 2$. Moreover given the payoff structure he would be best off if he forms a component that is part of an efficient graph. It can also be shown that in any SPE the aggregate bid to each agent should be zero and as a result an agent's payoff is the same regardless of who is chosen as the proposer. The proposed bids at $t = 1$ are the only feasible bids that satisfy all of these requirements.[7] ■

Remark 1 *In principle, one can apply the bidding mechanism to situations involving directed links. However, we need an additional assumption: namely, that establishing links needs the permission of both concerned parties. This assumption may be valid in some circumstances. For instance, a telephone connection (directed link) may be initiated by one party, but it requires the cooperation of both to carry forth a conversation. A similar consideration, we think, is valid with e-mail. Equally, there may be situations where links can be established unilaterally – in such cases, our mechanism is not valid because an agent whose proposal is rejected may reenter by unilaterally establishing links. Bala and Goyal [1] and Dutta and Jackson [3] both consider network models involving directed links.*

Finally, we would like to emphasize the fact, proved in Mutuswami, Pérez-Castrillo and Wettstein [8], that the equilibrium bidding strategies in the mechanism satisfy very strong properties. In particular, (a) they are

[7]For a more detailed proof see the analysis of the equilibrium strategies in Pérez-Castrillo and Wettstein [11] and Mutuswami, Pérez-Castrillo and Wettstein [8].

unique, (b) they are robust to deviations by coalitions of agents, and (c) they are maxmin strategies at the bidding stage.[8]

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[8] A strategy is a maxmin strategy if it maximizes the minimum payoff that a player can possibly obtain for any choice of strategies by the other players.