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Abstract

We provide a different axiomatization of the core interpreted as a reasonable set (Milnor, 1952) and introduce a new property, called max-intersection, related with the vector lattice structure of cooperative games with transferable utility. In particular, it is shown that the core is the only solution satisfying *projection consistency*, *reasonability*, *max-intersection* and *modularity*.

Keywords: Cooperative TU game, core, reasonable set.

JEL classification: C71, C78

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1 Introduction

The core of a transferable utility game (a *game*) (Gillies, 1959) has been widely studied and axiomatized in game theory. This solution concept can be interpreted as a *reasonable* set in the sense of Milnor (1952), since it assigns to every player at most his maximal marginal contribution. In spite of the fact that this property is desirable, at least from a normative point of view, as far as we know it has not been employed in the well-known axiomatizations of the core. The aim of this work is to provide a different axiomatic characterization of the core using *reasonability*, but also to introduce a new property inspired in the vector lattice structure of the games.

In a previous paper, Llerena and Rafels (2005) show that any game can be expressed as the maximum of a finite collection of convex games, all of which have the same efficiency level. This result suggests to consider the behavior of the core with respect to the maximum operation performed on games. We call this axiom *max-intersection*: the payoff vectors in the solution of the maximum game remain in the solution of the games involved in the decomposition, and vice-versa. This axiom is quite intuitive for solutions based on objections. We complete our axiomatic framework with *projection consistency* and *modularity*. *Consistency* is, perhaps, the most fundamental property used in this field (see Thomson, 1990, 1996 and Driessen, 1991 for surveys on consistency). Roughly speaking, this principle says that there is no inconsistency in what the players of the reduced game will get in both the original game and the reduced game. Projection consistency has been used by Funaki (1995) and recently by Bhattacharya (2004) to characterize the core and the *Equal division core* (Selten, 1972), respectively. Modularity extends the *triviality* axiom (Keiding, 1986) to the class of modular games. To justify this axiom it is important to point out that modularity is satisfied by the main solution concepts. So, in this sense, modularity forces the solution to be the "natural" for modular games.

The paper is organized as follows. Section 2 presents the general notation and some

definitions. Section 3 contains the main result: the core of a game is the only solution satisfying projection consistency, reasonability, max-intersection and modularity.

2 Notation and definitions

Let \mathcal{U} be a set of potential players that may be finite or infinite. A *cooperative game with transferable utility (a game)* is a pair (N, v) where $N = \{1, \dots, n\} \subset \mathcal{U}$ is a finite set of players and $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$, where 2^N denotes the set of all subsets (coalitions) of N . We will use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we will denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by Γ .

Let \mathbb{R}^N stand for the space of real-valued vectors indexed by N , $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T : $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. A game is *convex* (Shapley, 1972) if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. A game (N, v) is called *modular* if there exists a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^N$ such that for every $S \subseteq N$, $v(S) = \sum_{i \in S} x_i$.

The set of *feasible payoff* vectors of the game (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, and the *pre-imputation set* of the game (N, v) by $X(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$. A *solution* is a mapping σ which associates with any game (N, v) a subset $\sigma(N, v)$ of the set $X^*(N, v)$. Notice that the solution set $\sigma(N, v)$ is allowed to be empty or a singleton. We say that a solution σ is *Pareto optimal (P-OPT)* if $\sigma(N, v) \subseteq X(N, v)$ for $(N, v) \in \Gamma$.

The *core* of the game (N, v) is defined by $C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$.

3 An axiomatic characterization of the core

We start this section by defining the properties we use to characterize the core. To introduce consistency first we need to define reduced games.

Definition 1 Let $v \in G^N$, $x \in \mathbb{R}^N$ and $\emptyset \neq T \subset N$. The **projected reduced game** relative to T at x is defined as the game $(T, r_x^T(v))$ such that

$$r_x^T(v)(S) := \begin{cases} 0 & \text{if } S = \emptyset, \\ v(S) & \text{if } \emptyset \neq S \subset T, \\ v(N) - x(N \setminus T) & \text{if } S = T. \end{cases}$$

The intuition here is that if the members of $N \setminus T$ leave, no cooperation with them is possible anymore but the commitment to their payoffs has to be honored by the grand coalition T in the reduced game. Moreover, since no cooperation with the players *out of the game* is possible, the worth of each coalition $S \subset T$ in the reduced game remains what it was in the original game.

Let σ be a solution on Γ . Then, σ satisfies

1. **projection consistency (P-CONS)** whenever the next condition is satisfied: if $(N, v) \in \Gamma$, $\emptyset \neq T \subset N$ and $x \in \sigma(N, v)$, then $(T, r_x^T(v)) \in \Gamma$ and $x_T \in \sigma(T, r_x^T(v))$.
2. **reasonability (REAS)** if, for all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$ and all $i \in N$, $x_i \leq \max_{S \subseteq N \setminus \{i\}} \{v(S \cup \{i\}) - v(S)\}$.
3. **max-intersection (MAX-INT)** if $\sigma(N, \max\{v, w\}) = \sigma(N, v) \cap \sigma(N, w)$, for all games $(N, v), (N, w)$ with $v(N) = w(N)$.
4. **modularity (MOD)** if for any modular game (N, v) generated by the vector $x \in \mathbb{R}^N$, $\sigma(N, v) = \{x\}$.

Theorem 2 *The core is the only solution on Γ satisfying projection consistency, reasonability, max-intersection and modularity.*

PROOF:

Clearly, the core satisfies **P-CONS**, **REAS**, **MAX-INT** and **MOD**.

Let σ be a solution on Γ satisfying the above properties and $(N, v) \in \Gamma$. First we prove that σ satisfies **P-OPT**. Indeed, let $x \in \sigma(N, v)$ and $i \in N$. By **P-CONS**, $x_i \in \sigma(\{i\}, r_x^{\{i\}}(v))$. Since $(\{i\}, r_x^{\{i\}}(v))$ is the modular game generated by $y = r_x^{\{i\}}(v)(\{i\}) \in \mathbb{R}$, by **MOD** $x_i = r_x^{\{i\}}(v)(\{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} x_j$, and thus $x(N) = v(N)$.

To show the inclusion $C(N, v) \subseteq \sigma(N, v)$, let $x \in C(N, v)$ and define the modular game (N, w_x) generated by x : $w_x(S) := x(S)$ for all $S \subset N$. Clearly, $w_x = \max\{w_x, v\}$. By **MAX-INT**, $\sigma(N, w_x) = \sigma(N, w_x) \cap \sigma(N, v)$, which implies $\sigma(N, w_x) \subseteq \sigma(N, v)$. Finally, by **MOD** $\{x\} = \sigma(N, w_x)$, and thus $x \in \sigma(N, v)$.

To show the reverse inclusion, first consider the case (N, v) with $|N| = 1$. Since $C(N, v) = \{v(N)\}$ and $C(N, v) \subseteq \sigma(N, v)$, by **P-OPT** we can conclude that $\sigma(N, v) = \{v(N)\}$. If $|N| \geq 2$, from Llerena and Rafels (2005) we know that there is a finite collection of convex games $(N, v_1), \dots, (N, v_k)$ such that

$$v = \max\{v_1, \dots, v_k\}, \text{ with } v(N) = v_1(N) = \dots = v_k(N). \quad (1)$$

By **MAX-INT**

$$\sigma(N, v) = \bigcap_{j=1}^k \sigma(N, v_j). \quad (2)$$

Let $x \in \sigma(N, v)$ and consider the convex game (N, v_j) , with $j \in \{1, \dots, k\}$. By (2), $x \in \sigma(N, v_j)$, and by **REAS**,

$$x_i \leq \max_{S \subseteq N \setminus \{i\}} \{v_j(S \cup \{i\}) - v_j(S)\} = v_j(N) - v_j(N \setminus \{i\}), \forall i \in N, \quad (3)$$

where the equality follows from the convexity of the game (N, v_j) . Now, by **P-OPT**, $x(N \setminus \{i\}) \geq v_j(N \setminus \{i\})$. Or, equivalently, $x(S) \geq v_j(S)$ for any coalition $S \subset N$ with $|S| = n - 1$. From the convexity of the game (N, v_j) , and taking into account the inequality (3): $x_i \leq v_j(N) - v_j(N \setminus \{i\})$, it is straightforward to check that the projected reduced game $(N \setminus \{i\}, r_x^{N \setminus \{i\}}(v_j))$ is also a convex game. By **P-CONS**, $x_{N \setminus \{i\}} \in \sigma(N \setminus \{i\}, r_x^{N \setminus \{i\}}(v_j))$.

REAS together with the convexity of the projected reduced game implies that, for any player $l \in N \setminus \{i\}$,

$$x_l \leq r_x^{N \setminus \{i\}}(v_j)(N \setminus \{i\}) - r_x^{N \setminus \{i\}}(v_j)(N \setminus \{i, l\}). \quad (4)$$

From the definition of the projected reduced game we have that,

$$x_l \leq v_j(N) - x_i - v_j(N \setminus \{i, l\}). \quad (5)$$

Thus, by **P-OPT** we can conclude that, for any coalition $S \subset N$ with $|S| = n - 2$, $x(S) \geq v_j(S)$.

Following the same argument, and taking into account that the projected reduced game has the transitive property (i.e. for any $(N, v) \in \Gamma$, all $x \in \mathbb{R}^N$ and all $\emptyset \neq S \subset T \subseteq N$, $r_{x_T}^S(r_x^T(v)) = r_x^S(v)$), we can conclude that, for any coalition $S \subseteq N$, $x(S) \geq v_j(S)$. Hence, for any $j \in \{1, \dots, k\}$,

$$\sigma(N, v_j) \subseteq C(N, v_j). \quad (6)$$

Combining expressions (1), (2) and (6), and taking into account that

$$C(N, v) = C(N, \max\{v_1, \dots, v_k\}) = \bigcap_{j=1}^k C(N, v_j),$$

we obtain

$$\sigma(N, v) = \bigcap_{j=1}^k \sigma(N, v_j) \subseteq \bigcap_{j=1}^k C(N, v_j) = C(N, v),$$

which concludes the proof. \square

The following examples show that the above axioms are independent:

1. Let $\sigma^0(N, v) := \emptyset$, for each $(N, v) \in \Gamma$. Then, σ^0 satisfies **P-CONS**, **REAS** and **MAX-INT**, but not **MOD**.
2. Let σ^1 be the equal division core defined by

$$\sigma^1(N, v) := \left\{ x \in X(N, v) \mid \text{for all } S \subseteq N, \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|} \right\}.$$

Then, σ^1 satisfies **P-CONS**, **MAX-INT** and **MOD**, but not **REAS**.

3. Let it be

$$\sigma^2(N, v) := \{x \in X(N, v) \mid v(\{i\}) \leq x_i \leq b_i^v, \text{ for all } i \in N\},$$

where $b_i^v = v(N) - v(N \setminus \{i\})$, for all $i \in N$. Then, σ^2 satisfies **MAX-INT**, **REAS** and **MOD**, but not **P-CONS**.

4. Let it be

$$\sigma^3(N, v) := \{x \in X(N, v) \mid x_i \leq v(\{i\}), \text{ for all } i \in N\}.$$

Then, σ^3 satisfies **P-CONS**, **REAS** and **MOD**, but not **MAX-INT**.

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