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Pairwise-Stability and Nash Equilibria in Network Formation

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Pairwise-Stability and Nash Equilibria in Network Formation*

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Abstract

Suppose that individual payoffs depend on the network connecting them. Consider the following simultaneous move game of network formation: players announce independently the links they wish to form, and links are formed only under mutual consent. We provide necessary and sufficient conditions on the network link marginal payoffs such that the set of pairwise stable, pairwise-Nash and proper equilibrium networks coincide, where pairwise stable networks are robust to one-link deviations, while pairwise-Nash networks are robust to one-link creation but multi-link severance. Under these conditions, proper equilibria in pure strategies are fully characterized by one-link deviation checks.

Keywords: network formation, pairwise-stability, proper equilibrium.

JEL Classification: C62, C72, D85, L14

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1 Introduction

When individual payoffs depend on an underlying network of bilateral links, self-interested players may want to manipulate the network structure at their advantage. A model of network formation needs to specify how players set up links with each other, together with a network equilibrium concept compatible with this process. In recent years, different network formation procedures and network stability concepts have been proposed. This paper aims at wedging a bridge between the different stability notions proposed. We relate pairwise-stability, a prominent network equilibrium concept due to Jackson and Wolinsky (1996), to proper equilibrium, a non-cooperative refinement due to Myerson (1978).

Pairwise-stability is a network equilibrium concept that dispenses from any procedure of network formation. By definition, pairwise stable networks are robust to one-link deviations. Such one-link deviations are either promoted by single players in isolation (in the case of link-cutting), or at the coordinated initiative of pairs of players (in the case of link-creation). Pairwise-stability is a weak equilibrium notion, often interpreted as a necessary condition for network stability,[1] introduced initially to document the inherent tension between stability and efficiency in a network context. Pairwise-stability is also extensively used for positive purposes due to its computational (relative) simplicity, and to its ability to generate sharp predictions in many contexts.[2]

Myerson (1991) proposes a simultaneous move game of network formation, where players announce non-cooperatively the links they wish to form and the network that forms results from the collection of these announcements. This game is simple and intuitive. When networks are directed and links need not be reciprocated, standard Nash equilibrium analysis is informative about the geometry of endogenous network architectures.[3] Instead, when networks are undirected and link creation requires the mutual consent of the two involved parties, a coordination problem arises. The game displays a multiplicity of Nash equilibria.[4] This coordination problem results from the multidimensionality of the strategy spaces and the requirement of mutual consent. It is thus a feature of the network formation process itself. Allowing for bilateral moves solves the coordination problem satisfactorily, as no mutually beneficial link is now left aside. We call pairwise-Nash networks the Nash equilibrium outcomes that fulfill this added (coalitional-move) requirement.[5]

Our first result shows that pairwise-stability and pairwise-Nash equilibrium are two sides of the same token when joint-links marginal returns and single-link marginal returns to currently existing links are of the same sign, whenever the sum of the latter is non-negative. We call this condition

[1] Most likely, any equilibrium notion should, at least, check for one-link deviations.

[2] See Jackson (2003) for an exhaustive and recent survey on these issues.

[3] Bala and Goyal (2000) offers an analysis of Nash directed networks.

[4] For instance, the empty network is always a Nash equilibrium outcome (when nobody announces any link).

[5] The set of pairwise-Nash networks is thus at the intersection of the set of Nash equilibrium outcomes and the set of pairwise stable networks. See Goyal and Joshi (2002), Bloch and Jackson (2004) and Calvó-Armengol (2004) for definitions and applications of this equilibrium notion.

$\alpha-$*convexity*. Under this condition, robustness to unilateral or to multilateral link severance are equivalent. Many existing models in the literature fulfill this condition, including the connections model, the co-author model, models of information transmission on the network and oligopoly markets, among others.

Pairwise-Nash equilibrium, though, suffers from a serious conceptual drawback. Although intended to be a non-cooperative equilibrium notion, bilateral and coordinated players moves are explicitly allowed (as part of the equilibration process). Note that the process through which pairs of players might coordinate on their strategies' announcements is left unspecified. To avoid such *ad hoc* coalitional moves, we focus on proper equilibrium, a Nash refinement due to Myerson (1978). In a proper equilibrium, players best respond to perturbations of their opponents' strategies, where perturbations are ordered so that more costly mistakes are made with less probability.[6]

We provide necessary and sufficient conditions on marginal payoffs such that pairwise-Nash equilibria and proper equilibria coincide. This condition is twofold. First, network payoffs must be link-responsive, that is, the returns to any link are never zero. This is a mild requirement: network payoffs with exogenous parameters are (often) generically link-responsive. Second, joint-links marginal returns and single-link marginal returns to currently existing links are of the same sign whenever the sum of the latter is non-positive, and whenever not only players use their discretion to cut existing links, but also to create new ones. We call this condition $\beta-$*strong concavity*. More precisely, $\beta-$strong concavity need only hold for those links not in the network and for which the involved parties have conflicting interests in their creation (that is, one player would gain whereas the other would loose). This condition thus trivially holds in networks where absent links are not formed because neither player consents in its creation.

More precisely, link-responsiveness is enough to guarantee that any proper equilibrium network in pure strategies is also a pairwise-Nash equilibrium. Reciprocally, assuming that payoffs are link-responsive, all pairwise-Nash networks can be supported by proper equilibrium strategies if and only if $\beta-$strong concavity holds on the set of missing links that generate conflicting interests. The proof for the second inclusion is constructive, and is sketched below.

Consider some pairwise-Nash network and an equilibrium strategy that supports it. In this network, some links may not be formed by lack of mutual consent, even though it may be of the interest of one of the two parties to create them. Now, modify the equilibrium strategy so that, the interested parties systematically announce such (extra) links. Note that, at equilibrium, any such players taken in isolation are in fact indifferent between announcing and not announcing any of these links (which are not formed anyway). Yet, the network equilibrium strategy at hand may exploit the network payoff externalities in some intricate way that does require some of these announcements not to be effective (in the equilibrium strategy that supports it). The condition for $\beta-$strong

[6] Ordered mistakes differ by, at least, one order of magnitude. This hierarchy of mistakes distinguishes properness from trembling-hand perfection.

concavity is equivalent to stating that the strategy where all these extra announcements are made yields the same equilibrium network. This condition is met for a variety of models, including the connections model and information transmission. In particular, this shows that $\alpha-$concavity and $\beta-$strong concavity are perfectly compatible with one another.

With this new equilibrium pure strategy profile, we construct a completely mixed strategy profile by allowing players to make mistakes on link announcements. When mistakes vanish, the mixed profile converges to the equilibrium one. Mistakes are chosen so that links absent from the network for which neither of the two parties agree on their creation are announced with a (tremble) probability one order of magnitude higher than any other tremble. We show that this mixed strategy profile induces a quasi-perfect equilibrium in any extensive form equivalent[7] of the simultaneous move game of link formation. We then rely on an extensive form characterization of properness by Van Damme (1984) and Mailath, Samuelson and Swinkels (1997) to establish the relationship between proper and pairwise-Nash equilibrium. Incidentally, our analysis also sheds light on the extensive form approach to network formation (and corresponding extensive form equilibrium refinements), pioneered by Aumann and Myerson (1988).

Section 2 describes networks, payoffs and defines $\alpha-$convexity and $\beta-$strong concavity. Section 3 introduces network formation games and stability concepts. Section 4 contains the main results and some examples. The proofs of the main results are gathered in Section 5.

2 Network payoffs

Networks $N = \{1, \dots, n\}$ is a set of players.

Players on N are connected by a network g, a collection of direct links. Let $ij \in g$ be a direct link between players i and j in g . We focus on undirected networks, where $ij \in g$ is equivalent to $ji \in g$.

Let g^N denotes the complete network where each player $i \in N$ is directly linked to any other player $j \in N \backslash \{i\}$. Then, $\mathcal{G} = \left\{g \subseteq g^N\right\}$ is the set of undirected networks on N.

Let $g \in \mathcal{G}$ and $ij \in g$ a *link*. For all $ij \notin g$, $g + ij$ is the network obtained by adding the new link ij to g. Similarly, for all $ij \in g$, $g - ij$ is the network obtained by eliminating the current link ij from g.

Network payoffs A network payoff function is a mapping $u : \mathcal{G} \to \mathbb{R}^N$ that assigns to each network g a payoff $u_i(g)$ for each player $i \in N$. Some examples are provided below.

Link marginal payoffs Let $g \in \mathcal{G}$. For all player $i \in N$ and link $ij \in g$, we denote by:

$$m_{ij}u_i(g) = u_i(g) - u_i(g - ij)$$

[7] Both with complete and incomplete information.

the marginal payoff accruing to i from the link ij in the network g. Given a collection of current links $ij_1, ..., ij_\ell \in g$ of player i in g, the marginal joint value of these links accruing to i in g is:

$$m_{ij_1,...,ij_\ell} u_i(g) = u_i(g) - u_i(g - ij_1 - ... - ij_\ell).$$

Consider now some new direct link $ij \notin g$ of player i, absent from g. Then, $m_{ij} u_i(g + ij)$ is the marginal payoff accruing to i from the new link ij added to g, and the joint marginal value of a collection of new links $ij_1, ..., ij_\ell \notin g$ is $m_{ij_1,...,ij_\ell} u_i(g + ij_1 + ... + ij_\ell)$.

$\alpha-$convexity and $\beta-$strong concavity in new links The following definitions documents two possible features displayed by link marginal payoffs.

Let u be a network payoff function, $\mathcal{A} \subseteq \mathcal{G}$ and $\alpha \geq 0$.

Definition 1 *Let $i \in N$. Then, u_i is $\alpha-$convex on $\mathcal{A}$ if and only if, for all $g \in \mathcal{A}$ and $ij_1, ..., ij_\ell \in g$, we have:*

$$m_{ij_1,...,ij_\ell} u_i(g) \geq \alpha \sum_{p=1}^{\ell} m_{ij_p} u_i(g). \tag{1}$$

u is $\alpha-$convex on $\mathcal{A}$ if and only if u_i is $\alpha-$convex on $\mathcal{A}$ for each $i \in N$.

The condition for *$\alpha-$convexity* states that the joint marginal value of a group of links already in the network is higher than the sum of the marginal values of each single link, scaled by α. The case $\alpha = 1$ corresponds to convexity. $\alpha-$convexity is a property that applies to marginal payoff returns from existing links only.[8]

Let $g \in \mathcal{G}$. For all $i \in N$, let $g_i = \{ij \in g\}$ be the set of direct links stemming from i in g, and $\overline{g}_i = \{ij \notin g\}$ the set of i's missing direct links in g. Let $\gamma_i \subseteq \overline{g}_i$ be a subset of i's missing links in g, and $\beta > 0$.

Definition 2 *Let $i \in N$. Then, u_i is $\beta-$strongly concave in own new links on γ_i if and only if, for all $ij_{\ell+1}, ..., ij_{\ell+\ell'} \in g$ and $ij_1, ..., ij_\ell \in \gamma_i$, we have:*

$$m_{ij_1,...,ij_\ell} u_i(g + ij_1 + ... + ij_\ell - ij_{\ell+1} - ... - ij_{\ell+\ell'}) \leq \beta \sum_{p=1}^{\ell} m_{ij_p} u_i(g + ij_p). \tag{2}$$

Suppose first that $\gamma_i = \emptyset$. Then, *$\beta-$strong concavity* states that the joint marginal value of a group of links already in the network is higher than the sum of the marginal values of each single link, scaled by β. The case $\beta = 1$ corresponds to concavity. When $\gamma_i = \emptyset$, $\beta-$strong concavity is simply the dual of $\alpha-$convexity.

Suppose now that $\gamma_i \neq \emptyset$. Then, $\beta-$strong concavity compares the joint-links marginal value of current links to the sum of their single-link marginal values, both in the original network g and

[8] Given that marginal link values are only computed for links already in the network, a more precise (but introverted) wording would be $\alpha-$convexity in own current links. We keep $\alpha-$convexity for short.

in every supernetwork of g where subsets of links in γ_i are added. $\beta-$strong concavity is thus a stronger version of $\beta-$concavity that allows for link addition for the comparison of marginal payoffs.[9]

Note that $\alpha-$convexity on $\mathcal{A}$ implies $\alpha'-$convexity on $\mathcal{A}$, for all $\alpha > \alpha' \geq 0$, and similarly for $\beta-$strong concavity, for all $\beta' > \beta > 0$. In fact, both conditions are equivalent to a sign preserving condition for joint-link marginal payoffs versus the sum of single-link marginal payoffs. Formally, for all $x \in \mathbb{R}$, let:

$$\omega(x) = \begin{cases} +1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases} .$$

Proposition 1 *Let $\mathcal{A} \subseteq \mathcal{G}$, $\alpha \geq 0$ and $i \in N$. Then, u_i is $\alpha-$convex on $\mathcal{A}$ if and only if, for all $ij_1, ..., ij_\ell \in g$ and $g \in \mathcal{A}$:*

$$\omega(\sum_{p=1}^{\ell} m_{ij_p} u_i(g)) \geq 0 \text{ implies } \omega(m_{ij_1,...,ij_\ell} u_i(g)) \geq 0.$$

Let $\gamma_i \subseteq \overline{g}_i$ and $\beta > 0$. Then, u_i is $\beta-$strongly concave in own new links on γ_i if and only if, for all $ij_1, ..., ij_\ell \in \gamma_i$ and $ij_{\ell+1}, ..., ij_{\ell+\ell'} \in g_i$:

$$\begin{aligned} \omega(\sum_{p=1}^{\ell} m_{ij_p} u_i(g + ij_p)) &\leq 0 \text{ implies} \\ \omega(m_{ij_1,...,ij_\ell} u_i(g + ij_1 ... + ij_\ell - ij_{\ell+1} ... - ij_{\ell+\ell'})) &\leq \omega(\sum_{p=1}^{\ell} m_{ij_p} u_i(g + ij_p)). \end{aligned}$$

3 Network equilibrium

In what follows, we first define *pairwise-stability*, due to Jackson and Wolinsky (1996), and often interpreted as a necessary condition for network equilibrium.[10]

We then formulate a simultaneous move game of network formation due to Myerson (1991). This game is simple and intuitive, but generally displays a multiplicity of Nash equilibria. To accommodate this fact, we define *pairwise-Nash equilibrium*, a variation of Nash equilibrium where players are allowed to deviate by pairs.[11]

Finally, we recall the definition of *proper equilibrium*, a Nash equilibrium refinement for simultaneous move games introduced by Myerson (1978).

[9] Again, a more precise wording for this condition on marginal payoffs would be $\beta-$strong concavity in own current links and own new links in γ_i.

[10] See also Jackson and Watts (2002) for a dynamic foundation of pairwise-stability.

[11] See Bloch and Jackson (2004), Calvó-Armengol (2004) and Goyal and Joshi (2002) for applications of this concept. See also Dutta and Mutuswami (1997) and Jackson and van den Nouweland (2000) for alternatives to pairwise stability and Nash equilibrium that allow for coalitional moves.

Pairwise-stability Pairwise stable networks are robust to one-link deviations, where link severance is unilateral, while link creation is bilateral and under mutual consent of the two involved players.

Definition 3 *A network $g \in \mathcal{G}$ is pairwise stable with respect to the network payoff function u if and only if, for all $ij \in g$, both $m_{ij}u_i(g) \geq 0$ and $m_{ij}u_j(g) \geq 0$, while for all $ij \notin g$, if $m_{ij}u_i(g+ij) > 0$, then $m_{ij}u_j(g+ij) < 0$, for all $i \in N$.*

We denote by $PS(u)$ the set of pairwise stable networks with respect to u.

A simultaneous move game of network formation This game is due to Myerson (1991).[12] The set of players is N. All players $i \in N$ simultaneously announce the direct links they wish to form. Formally, $S_i = \{0,1\}^{n-1}$ is i's set of pure strategies. Let $s_i = (s_{i1}, \ldots s_{i,i-1}, s_{i,i+1}, \ldots s_{in}) \in S_i$. Then, $s_{ij} = 1$ if and only if i wants to set up a direct link with $j \neq i$ (and thus $s_{ij} = 0$, otherwise). We assume that mutual consent is needed to create a direct link, that is, ij is created if and only if $s_{ij}s_{ji} = 1$.

Let $S = S_1 \times ... \times S_n$. A pure strategy profile $s = (s_1, \ldots, s_n) \in S$ induces an undirected network $g\left(s\right)$.

Let $\Sigma_i = \Delta(\{0,1\}^{n-1})$ be i's set of mixed strategies, where $\Delta(\{0,1\}^{n-1})$ denotes the set of probability distributions over $\{0,1\}^{n-1}$. A mixed strategy $\sigma_i \in \Sigma_i$ is thus a joint (multivariate Bernoulli) distribution that allows for rich correlation patterns in individual link announcements. In particular, given a mixed strategy σ_i, i announces a link with j with marginal probability:[13]

$$\sigma_i(s_{ij} = 1, \cdot) = \sum_{s \in \{0,1\}^{n-1}; s_{ij}=1} \sigma_i(s)$$

Let $\Sigma = \Sigma_1 \times ... \times \Sigma_n$. A mixed strategy $\sigma = (\sigma_1, ..., \sigma_n) \in \Sigma$ generates a probability distribution over $\mathcal{G}$, a random graph.[14]

Pairwise-Nash equilibrium A pure strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a Nash equilibrium of the game of network formation if and only if $u_i\left(g\left(s^*\right)\right) \geq u_i\left(g\left(s_i, s_{-i}^*\right)\right)$, for all $s_i \in S_i$ and $i \in N$. Nash equilibrium, though, is too weak an equilibrium concept to single out equilibrium networks. For instance, the empty network is always a Nash equilibrium.[15] Building upon pairwise-stability,

[12] To quote Myerson: "Now consider a link-formation process in which each player independently writes down a list of players with whom he wants to form a link (...) and the payoff allocation is (...) for the graph that contains a link for every pair of players who have named each other." (p. 448)

[13] Note that, for $n \geq 3$, the collection of such marginal probabilities does not define univocally a mixed strategy. Fréchet (1951) gives an early and thorough account of the relationship between joint distributions (here, mixed strategies on the collection of all link announcements) and marginals (here, single link probabilities) when $n = 3$.

[14] See Jackson and Rogers (2004) for a relation between random graphs and individual incentives.

[15] When nobody announces any link.

we further require that any mutually beneficial link be formed at equilibrium. Pairwise-Nash equilibrium networks are robust to bilateral commonly agreed one-link creation, and to unilateral multi-link severance.

Definition 4 *A network $g \in \mathcal{G}$ is a pairwise-Nash equilibrium network with respect to the network payoff function u if and only if there exists a Nash equilibrium strategy profile s^* that supports g, that is, $g = g(s^*)$, and, for all $ij \notin g$, if $m_{ij}u_i(g+ij) > 0$, then $m_{ij}u_j(g+ij) < 0$, for all $i \in N$.*

We denote by $PNE(u)$ the set of pairwise-Nash equilibrium networks with respect to u.

Proper equilibrium

Definition 5 *A strategy profile $\sigma \in \Sigma$ is proper if there exists a sequence of completely mixed equilibrium strategy profiles $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ with limit σ such that, for all $i \in N$, $s_i', s_i'' \in S_i$, and $t \in \mathbb{N}$ we have:*

$$Eu_i(s_i', \sigma_{-i}^{\varepsilon_t}) > Eu_i(s_i'', \sigma_{-i}^{\varepsilon_t}) \text{ implies that } \sigma_i^{\varepsilon_t}(s_i'') \leq \varepsilon_t \cdot \sigma_i^{\varepsilon_t}(s_i').$$

This refinement is due to Myerson (1978). In a proper equilibrium, players best respond to perturbations of their opponents' strategies, where costly mistakes are made with less probability. This hierarchy of mistakes differentiates properness from trembling-hand perfection. Myerson (1978) showed that every game has a proper equilibrium. We denote by $PRE(u)$ the set of proper Nash equilibrium networks *in pure strategies* with respect to u.[16]

4 The results

We first provide necessary and sufficient conditions on the payoff function u for the set of pairwise stable networks and the set of pairwise-Nash equilibrium networks to coincide. We latter characterize the equivalence between pairwise-Nash equilibrium networks and proper equilibrium networks.

Pairwise stable and pairwise-Nash equilibrium networks

Theorem 1 *$PS(u) = PNE(u)$ if and only if u is $\alpha-$convex on $PS(u)$, for some $\alpha \geq 0$.*

[16] We want to establish necessary and sufficient conditions such that the set of pairwise stable, pairwise-Nash and proper equilibrium networks coincide. Given that both pairwise-stability and pairwise-Nash networks are only defined for pure strategies (that is, deterministic networks), we restrict attention to proper equilibria in pure strategies. Myerson (1978) shows that any finite normal form game has a proper equilibria. Given our restricttion to pure strategies, existence is not warranted here. See Jackson and Watts (2001) for a perspective on the existence of pairwise stable networks.

The proof derives from the simple observation that, when payoffs are $\alpha-$convex on $PS(u)$ for some $\alpha \geq 0$, if a player does not benefit from severing any single link, then he does not gain from cutting any group of links simultaneously. Robustness to unilateral or to multilateral link severance are equivalent when marginal payoffs from existing links satisfy (1) on some particular set.

Pairwise-Nash and proper equilibrium networks We now define link-responsive network payoffs functions.

Definition 6 *The network payoff function u is link-responsive if and only if $m_{ij_1,...,ij_\ell} u_i(g) \neq 0$, for all $i \in N$, $ij_1, ..., ij_\ell \in g$, and $g \in \mathcal{G}$.*

Link-responsiveness states that no link removal or addition is innocuous for the players directly involved. Network payoff functions that depend on some exogenous set of parameters (e.g., a constant marginal-link cost) are generically link-responsive..

For all $g \in \mathcal{G}$ and u, let $\nu_i(g,u) = \{ij \notin g : m_{ij}u_i(g+ij) \leq 0\}$ be the set of i's direct links not in g with non-positive marginal returns.

Theorem 2 *Suppose that u is link-responsive. Then, $PNE(u) = PRE(u)$ if and only if u_i is $\beta-$strongly concave in own new links on $\nu_i(g,u)$, for all $i \in N$ and $g \in PNE(u)$, for some $\beta > 0$.*

Link-responsiveness is enough for any proper equilibrium network to be pairwise-Nash, while $\beta-$strong concavity on $\nu_i(g,u)$ is needed to sustain pairwise-Nash equilibria as proper. Consider some pairwise-Nash equilibrium g, and s an equilibrium strategy profile that supports it. Now, modify the profile s the following way. For each single link not in g, say $ij \notin g$, if one of the two parties involved agrees to create such a link, say i,[17] then modify s by letting i announce this link, that is, set $s_{ij} = 1$. The new strategy profile obtained from this procedure is still an equilibrium strategy that supports g if and only if the condition in Theorem 2 holds.

With this new equilibrium pure strategy profile, we construct a completely mixed strategy profile by allowing players to make mistakes on link announcements. When mistakes vanish, the mixed profile converges to the equilibrium one. Mistakes are chosen so that links (not in g) for which neither of the two parties agree on their creation are announced with a (tremble) probability one order of magnitude higher than any other tremble probability on any other link announcement. This mixed strategy profile induces a quasi-perfect equilibrium in any extensive form equivalent of the simultaneous move game of link formation.[18] We finally invoke a result by Mailath, Samuelson and Swinkels (1997) who provide an extensive form characterization of properness by means of quasi-perfect equilibrium strategies.

[17] That is, $m_{ij}u_i(g+ij) > 0$.

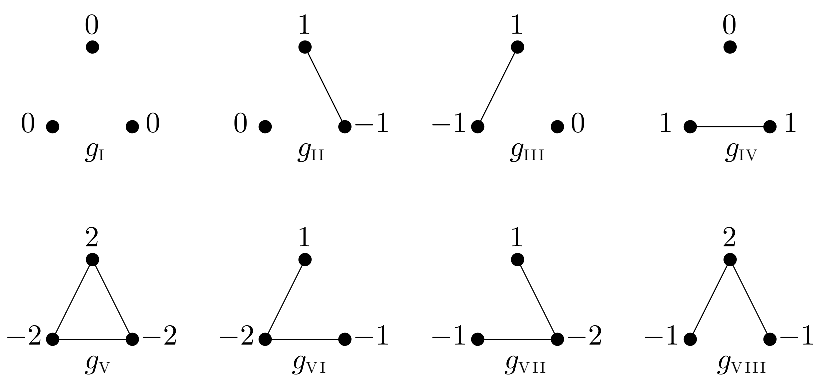
[18] Quasi-perfection, a notion due to van Damme (1984), requires that, with the limit strategy, players are best-responding at all information sets to each element of the sequence of perturbed strategies. Quasi-perfection also forces players to ignore their perturbations in their own strategies.

When links absent from the network are such that no player consents in their creation, then $\nu_i(g, u) = \emptyset$, for all $i \in N$, and any pairwise-Nash equilibrium outcome is proper. In particular, when the complete network is a pairwise-Nash equilibrium outcome, it is also proper.

Corollary 1 *Let $g \in PNE(u)$. Suppose that, for all $ij \notin g$, we have $m_{ij}u_i(g + ij) < 0$ and $m_{ij}u_j(g + ij) < 0$. Then, $g \in PRE(u)$.*

In what follows, we first show with an example that trembling-hand perfect equilibrium outcomes need not be pairwise-Nash, even when the conditions in Theorem 2 hold. This justifies the need to resort to proper equilibrium, a refinement that differs from trembling-hand by imposing a hierarchy on trembles related to the relative cost values associated with these mistakes.

Example (trembling-hand networks need not be pairwise-Nash) Let $N = \{1, 2, 3\}$ and consider the network payoffs defined below:



Denote by 1 the player at the top, and by 2 and 3 the players at the bottom, and let $S_1 = \{(s_{12}, s_{13}) \in \{0,1\}^2\}$, $S_2 = \{(s_{21}, s_{23}) \in \{0,1\}^2\}$ and $S_3 = \{(s_{31}, s_{32}) \in \{0,1\}^2\}$. The only pairwise-Nash equilibrium network is g_{IV}. The network payoff function u is link-responsive and satisfies (2). The empty network g_{I} is a trembling-hand network for the pure strategy $s^* = ((1,1),(0,0),(0,0))$ in which player 1 announces all the links, and players 2 and 3 do not announce any link. This strategy s^* is the limit of the sequence of best-replying trembles of the perturbed game, where:

$$\begin{aligned}
\sigma_1^{\varepsilon}((0,0)) &= \sigma_1^{\varepsilon}((1,0)) = \sigma_1^{\varepsilon}((0,1)) = \varepsilon, \sigma_1^{\varepsilon}((1,1)) = 1 - 3\varepsilon \\
\sigma_2^{\varepsilon}((1,0)) &= \sigma_2^{\varepsilon}((0,1)) = \sigma_2^{\varepsilon}((1,1)) = \varepsilon, \sigma_2^{\varepsilon}((0,0)) = 1 - 3\varepsilon \\
\sigma_3^{\varepsilon}((1,0)) &= \sigma_3^{\varepsilon}((0,1)) = \sigma_3^{\varepsilon}((1,1)) = \varepsilon, \sigma_3^{\varepsilon}((0,0)) = 1 - 3\varepsilon
\end{aligned}$$

Next, we apply our results to the connections model and the co-author model, both due to Jackson and Wolinsky (1996), and to a model of information transmission due to Calvó-Armengol (2004).

Example (connections model) This model is due to Jackson and Wolinsky (1996). Suppose that network payoffs are given by:

$$u_i(g) = \sum_{j \in N} \delta^{d_{ij}(g)} - cn_i(g),$$

for some $\delta \in (0,1)$, where $n_i(g) = \#g_i$ and $d_{ij}(g)$ is the geodesic distance between i and j. As marginal link costs are constant, we focus on link marginal gross returns. A lower bound for marginal gross returns is $\delta\ell$ (when i is the hub of a star), and an upper bound is $[\delta + (n-2)\delta^2]\ell$ (when i is a spoke of a star encompassing all players). Therefore, u is $1/(n-1)-$convex in own current links on $\mathcal{G}$, implying that the set of pairwise stable and pairwise-Nash equilibrium networks coincide. Next, note that u satisfies (2) for the complete network and for stars on the cost ranges where these are pairwise stable (resp. $c < \delta - \delta^2$ and $\delta - \delta^2 < c < \delta$).[19] This is a consequence of Corollary 1, as peripheral players in the star network get a negative return from any direct link with one another. Note also that u is generically link-responsive. Therefore, stars and the complete network are (generically) proper equilibria when $c < \delta - \delta^2$ and $\delta - \delta^2 < c < \delta$, respectively.

Example (co-author model) This model is due to Jackson and Wolinsky (1996). Suppose that network payoffs are given by:

$$u_i(g) = (1 + \frac{1}{n_i(g)}) \sum_{ij \in g} \frac{1}{n_j(g)},$$

if $\#g_i \geq 1$, and $u_i(g) = 0$, otherwise. Then, straight calculations show that u is $2/n^3-$convex in own current links on $\mathcal{G}$. The set of pairwise stable and pairwise-Nash equilibrium networks thus coincide. Pairwise stable networks consist on a sequence of fully connected networks such that, if we order this sequence by components' sizes, the square of the size of any component is smaller than the next size in this sequence.[20] It is plain that any player in a component gains by cutting current links with its component mates and rewiring with players in components of smaller size. Such links, though, are not formed because it is not in the interest of the latter (the players in the components of smaller size) to consent to this rewiring process. In fact, (2) does not hold, and pairwise stable networks are not proper (except for the complete network, which is pairwise stable, pairwise-Nash and proper).

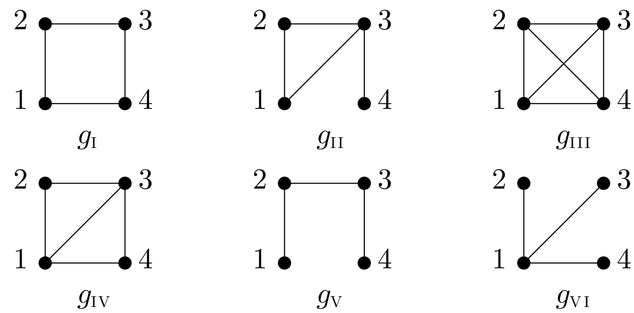
Example (information transmission) This model is due to Calvó-Armengol (2004). Suppose that network payoffs are given by:

$$u_i(g) = 1 - \prod_{j \in N} p_{ji}(g) - cn_i(g),$$

[19] See Proposition 2 in Jackson and Wolinsky (1996).

[20] See Proposition 4 in Jackson and Wolinsky (1996).

where $p_{ji}(g) \in (0,1)$, for all $i, j \in N$. In this case, u is $1-$convex in own current links on $\mathcal{G}$, and the set of pairwise stable and pairwise-Nash equilibrium networks coincide.[21] Note that u is generically link-responsive. Let $N = \{1, 2, 3, 4\}$: There are six different connected networks on N:



When the p_{ij}'s take a particular expression,[22] g_{I}, g_{II} and g_{III} are the only pairwise stable networks, each for some cost range. Corollary 1 implies that only g_{I} and g_{III} are (generically) proper on the cost ranges where these are pairwise stable (and only there).

5 Main proofs

Proof of Proposition 1 Derives from the fact that $\mathcal{G}$ is a finite set.

Proof of Theorem 1 First, it is clear that $PNE(u) \subseteq PS(u)$. Hence, if $PS(u) = \emptyset$, the result follows. Suppose now that $PS(u) \neq \emptyset$, and let $g^* \in PS(u)$. Define:

$$\phi(g^*, u) \in \min \{m_{ij} u_i(g^*) : ij \in g, i \in N\}.$$

Pairwise-stability implies that $\phi(g^*, u) \geq 0$. Suppose that u is $\alpha-$convex in own current links on $PS(u)$ for some $\alpha \geq 0$. Then, (1) implies that $m_{ij_1,\ldots,ij_\ell} u_i(g^*) \geq \alpha \ell \phi(g^*, u) \geq 0$, for all $ij_1, \cdots, ij_\ell \in g^*$ and $i \in N$. Moreover, by definition of pairwise-stability, if $m_{ij} u_i(g^* + ij) > 0$, then $m_{ij} u_j(g^* + ij) < 0$, for all $ij \notin g^*$ and $i \in N$. Therefore, $g^* \in PNE(u)$.

Now, suppose that there exists $g^* \in PS(u)$ such that, for all $\alpha \geq 0$, (1) does not hold for g^*. Then, for some $i \in N$, there exists $ij_1, \cdots, ij_\ell \in g^*$ such that $m_{ij_1,\ldots,ij_\ell} u_i(g^*) < 0$, implying that $g^* \notin PNE(u)$. ∎

[21] This results from Claim 2 in Calvó-Armengol (2004).

[22] Let $u_i(g) = 1 - b \prod_{ij \in g} q(n_j(g)) - c n_i(g)$, where

$$q(n_j(g)) = \frac{1 - (1-c)^{n_j(g)}}{c n_j(g)}.$$

This is i's expected payoff for a stochastic process of information transmission in the network. Then, g_{I}, g_{II} and g_{III} are the unique pairwise-stable networks for some cost ranges (see Calvó-Armengol 2004).

Proof of Theorem 2 We decompose the proof in two parts. Proposition 2 establishes that $PRE(u) \subseteq PNE(u)$ when u is link-responsive. Suppose that u is link-responsive. Proposition 3 then shows that $PNE(u) \subseteq PRE(u)$ if and only if u is u_i is $\beta-$strongly concave in own new links on $\nu_i(g,u)$, for all $i \in N$ and $g \in PNE(u)$, for some $\beta > 0$.

Proposition 2 *If the network payoff u is link-responsive, then $PRE(u) \subseteq PNE(u)$.*

Proof. Let u be link-responsive. We show that $g \notin PNE(u)$ implies that $g \notin PRE(u)$.

Let g^* be a Nash equilibrium outcome such that $m_{ij}u_i(g^* + ij) > 0$ and $m_{ij}u_j(g^* + ij) \geq 0$, for some $ij \notin g^*$. Then, $g^* \notin PNE(u)$. Suppose that $g^* \in PRE(u)$, and let s^* be a pure strategy proper equilibrium that supports it. Then, $g^* = g(s^*)$. Let $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ be a sequence of completely mixed strategy profiles such that $\lim_{t\to+\infty} \sigma^{\varepsilon_t}(s^*) = 1$, and σ^{ε_t} is an ε_t-proper equilibrium of the link formation game with network payoffs u, for all $t \in \mathbb{N}$.

Given that s^* is also a Nash equilibrium strategy and that $ij \notin g^*$, necessarily, $s^*_{ij} = s^*_{ji} = 0$.

For all $j \neq i$, define $e(ij) = (0, ..., s_{ij} = 1, 0, ..., 0)$. In $e(ij)$, player i only announces the link with j. Let $s'_i = s^*_i \vee e(ij)$. In s'_i, player i announces exactly the same links than in s^*_i plus an extra link with player j. This extra link is not reciprocated by player j in s^*.

For all $t \in \mathbb{N}$, define:

$$\Delta_i(s'_i, s^*_i; \sigma^{\varepsilon_t}_{-i}) = Eu_i(g(s'_i, \sigma^{\varepsilon_t}_{-i})) - Eu_i(g(s^*_i, \sigma^{\varepsilon_t}_{-i})) = \sum_{\widetilde{s}_{-i}\in S_{-i}} \sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})\, \Delta_i(s'_i, s^*_i; \widetilde{s}_{-i}), \tag{3}$$

where $\Delta_i(s'_i, s^*_i; \widetilde{s}_{-i}) = u_i(g(s'_i, \widetilde{s}_{-i})) - u_i(g(s^*_i, \widetilde{s}_{-i}))$. For all $\widetilde{s}_{-i}$ such that $\widetilde{s}_{ji} = 0$, we have $g(s'_i, \widetilde{s}_{-i}) = g(s^*_i, \widetilde{s}_{-i})$, and $\Delta_i(s'_i, s^*_i; \widetilde{s}_{-i}) = 0$. Therefore,

$$\Delta_i(s'_i, s^*_i; \sigma^{\varepsilon_t}_{-i}) = \sum_{\widetilde{s}_{-i}\in S_{-i}:\widetilde{s}_{ji}=1} \sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})\, \Delta_i(s'_i, s^*_i; \widetilde{s}_{-i}).$$

Let $\widetilde{s}_{-i} \in S_{-i}$ such that $\widetilde{s}_{ji} = 1$. Let $\widetilde{g} = g(s^*_i, \widetilde{s}_{-i})$. Note that $\widetilde{g}_{ij} = 0$, and that $g(s'_i, \widetilde{s}_{-i}) = \widetilde{g} + ij$. Also, $s_{ik} = 0$ implies that $\widetilde{g}_{ik} = 0$. Define $\mathcal{G}(s^*_i) = \{g \in \mathcal{G} : s^*_{ik} = 0 \Rightarrow g_{ik} = 0\}$. It is readily checked that

$$\mathcal{G}(s^*_i) = \{g(s^*_i, \widetilde{s}_{-i}) : \widetilde{s}_{-i} \in S_{-i}, \widetilde{s}_{ji} = 1\}.$$

Therefore, we can write:

$$\Delta_i(s'_i, s^*_i; \sigma^{\varepsilon_t}_{-i}) = \sum_{\widetilde{g}\in\mathcal{G}(s^*_i)} \mu_{\varepsilon_t}(\widetilde{g})\, m_{ij}u_i(\widetilde{g} + ij),$$

where

$$\mu_{\varepsilon_t}(\widetilde{g}) = \sum_{\substack{\widetilde{s}_{-i}\in S_{-i}:\widetilde{s}_{ji}=1 \\ g(s^*_i,\widetilde{s}_{-i})=\widetilde{g}}} \sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i}).$$

Given that the mixed strategy profiles σ^{ε_t}, $t \in \mathbb{N}$ have full support, $\mu_{\varepsilon_t}(\widetilde{g}) \neq 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*)$ and $t \in \mathbb{N}$. Therefore, $\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t}) > 0$ is equivalent to

$$m_{ij} u_i(g^* + ij) + \sum_{\widetilde{g} \in \mathcal{G}(s_i^*), \widetilde{g} \neq g^*} \frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g^*)} m_{ij} u_i(\widetilde{g} + ij) > 0.$$

Since $\Delta_i(s_i', s_i^*; \sigma_{-i}^{\varepsilon_t})$ is continuous in $\sigma_{-i}^{\varepsilon_t}$, and given that $m_{ij} u_i(g^* + ij) > 0$, it suffices to show that $\lim_{t \to +\infty} \mu_{\varepsilon_t}(\widetilde{g}) / \mu_{\varepsilon_t}(g^*) = 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*), \widetilde{g} \neq g^*$. Note that $\lim_{t \to +\infty} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}) = 0$, for all $\widetilde{s}_{-i} \in S_{-i}$ such that $\widetilde{s}_{ji} = 1$. Therefore, $\lim_{t \to +\infty} \mu_{\varepsilon_t}(\widetilde{g}) = 0$, for all $\widetilde{g} \in \mathcal{G}(s_i^*)$, including $\widetilde{g} = g^*$. Establishing that

$$\lim_{t \to +\infty} \frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g^*)} = 0, \text{ for all } \widetilde{g} \in \mathcal{G}(s_i^*), \widetilde{g} \neq g^*,$$

is thus equivalent to showing that the rate of convergence of $\mu_{\varepsilon_t}(\widetilde{g}), \widetilde{g} \neq g^*$ is at least one order of magnitude higher than that of $\mu_{\varepsilon_t}(g^*)$. This is implied by properness, as detailed below.

For each $k \in N$, we partition the strategy set S_k into two disjoint sets S_k^+ and S_k^- where:

$$\begin{cases} S_k^+ = \{s_k \in S_k : u_k(g(s_k, s_{-k}^*)) \geq u_k(g^*)\} \\ S_k^- = \{s_k \in S_k : u_k(g(s_k, s_{-k}^*)) < u_k(g^*)\} \end{cases}.$$

It is plain that $S_k = S_k^+ \cup S_k^-$ and that $S_k^+ \cap S_k^- = \emptyset$. Given that u is link-responsive and that s^* is a Nash equilibrium strategy supporting g^*, we have $g(s_k', s_{-k}^*) = g^*$, for all $s_k' \in S_k^+$. Moreover, as $\lim_{t \to +\infty} \sigma^{\varepsilon_t} = s^*$, and given that each player's expected payoff is continuous in the vector of other players' (completely) mixed strategies, there exists some t_k such that, for all $t \geq t_k$, we have $u_k(g(s_k^+, \sigma_{-k}^{\varepsilon_t})) < u_k(g(s_k^-, \sigma_{-k}^{\varepsilon_t}))$, for all $s_k^+ \in S_k^+$ and $s_k^- \in S_k^-$. Given that $\{\sigma^{\varepsilon_t}\}_{t \in \mathbb{N}}$ is a sequence of ε_t-proper equilibria, this implies that, for all $t \geq t_k$, $s_k^+ \in S_k^+$ and $s_k^- \in S_k^-$ we have $\sigma_k^{\varepsilon_t}(s_k^-) \leq \varepsilon_t . \sigma_k^{\varepsilon_t}(s_k^+)$. Note, also, that $s_j' \in S_j^+$. Define:

$$\begin{cases} \mathcal{G}' = \{g \in \mathcal{G}(s_i^*) : g \neq g^*, g = g(\widetilde{s}_j, s_{-j}^*), \widetilde{s}_j \in S_j, \widetilde{s}_{ji} = 1\} \\ \mathcal{G}'' = \{g \in \mathcal{G}(s_i^*) : g \neq g^*, g = g(s_i^*, \widetilde{s}_{-i}), \widetilde{s}_{-i} \in S_{-i}, \widetilde{s}_{ji} = 1, \widetilde{s}_k \neq s_k^* \text{ for some } k \neq j\} \end{cases}.$$

In words, $\mathcal{G}'$ is the set of networks derived from g^* under s^* where only player j makes a mistake (including always the announcement of the link ij), whereas $\mathcal{G}''$ uncovers all the networks where at least some other player (besides j) makes a mistake. Clearly, $\mathcal{G}(s_i^*) \backslash g^* = \mathcal{G}' \cup \mathcal{G}''$. Let $\widetilde{g} \in \mathcal{G}'$. Then,

$$\mu_{\varepsilon_t}(\widetilde{g}) = \sigma_{-i-j}^{\varepsilon_t}(s_{-i-j}^*) \sum_{\substack{\widetilde{s}_j \in S_j : \widetilde{s}_{ji} = 1 \\ g(\widetilde{s}_j, s_{-j}^*) = \widetilde{g}}} \sigma_j^{\varepsilon_t}(\widetilde{s}_j).$$

But, for all $\widetilde{s}_j \in S_j$ such that $g(\widetilde{s}_j, s_{-j}^*) \in \mathcal{G}'$, necessarily, $\widetilde{s}_j \in S_j^-$, implying in turn that $\sigma_j^{\varepsilon_t}(\widetilde{s}_j) \leq \varepsilon_t \sigma_j^{\varepsilon_t}(s_j')$, for all $t \geq t_j$. Therefore, for all $t \geq t_j$, we have $\mu_{\varepsilon_t}(\widetilde{g}) \leq K(\widetilde{g}) \varepsilon_t \sigma_{-i}^{\varepsilon_t}(s_{-i}')$, where $K(\widetilde{g}) = \# \left\{ \widetilde{s}_j \in S_j : \widetilde{s}_{ji} = 1, g(\widetilde{s}_j, s_{-j}^*) = \widetilde{g} \right\} \geq 1$. Then, given that

$$\mu_{\varepsilon_t}(g) = \sigma_{-i}^{\varepsilon_t}(s_{-i}') + \sum_{\substack{\widetilde{s}_{-i} \in S_{-i} \backslash \{s_{-i}'\}: \\ \widetilde{s}_{ji} = 1, g(s_i^*, \widetilde{s}_{-i}) = \widetilde{g}}} \sigma_{-i}^{\varepsilon_t}(\widetilde{s}_{-i}),$$

we have

$$\frac{\mu_{\varepsilon_t}(\widetilde{g})}{\mu_{\varepsilon_t}(g)} \leq K(\widetilde{g})\varepsilon_t \frac{1}{1+\sum_{\substack{\widetilde{s}_{-i}\in S_{-i}\setminus\{s'_{-i}\}:\\ \widetilde{s}_{ji}=1, g(s^*_i,\widetilde{s}_{-i})=\widetilde{g}}} \frac{\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})}{\sigma^{\varepsilon_t}_{-i}(s'_{-i})}}.$$

But, any $\widetilde{s}_{-i} \in S_{-i}$ such that $\widetilde{s}_{-i} \neq s'_{-i}$ and $\widetilde{s}_{ji} = 1$ contains at least one additional mistake with respect to s'_{-i}, implying that $\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})/\sigma^{\varepsilon_t}_{-i}(s'_{-i}) \to 0$ as $t \to +\infty$, for any such strategy profile, which implies in turn that $\lim_{t\to+\infty} \mu_{\varepsilon_t}(\widetilde{g})/\mu_{\varepsilon_t}(g^*) = 0$.

Let now $\widetilde{g} \in \mathcal{G}''$. Let $\widetilde{s}_{-i} \in S_{-i}$ such that $g = g(s^*_i, \widetilde{s}_{-i})$. Define $L = \{k \neq j : \widetilde{s}_k \neq s^*_k\}$. By definition, $L \neq \emptyset$. Now, $\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i}) = \sigma^{\varepsilon_t}_j(\widetilde{s}_j)\sigma^{\varepsilon_t}_L(\widetilde{s}_L)\sigma^{\varepsilon_t}_{-i-j-L}(s^*_{-i-j-L})$ and, thus,

$$\begin{aligned}
\frac{\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})}{\mu_{\varepsilon_t}(g)} &\leq \frac{\sigma^{\varepsilon_t}_j(\widetilde{s}_j)\sigma^{\varepsilon_t}_L(\widetilde{s}_L)\sigma^{\varepsilon_t}_{-i-j-L}(s^*_{-i-j-L})}{\sigma^{\varepsilon_t}_j(s'_j)\sigma^{\varepsilon_t}_{-i-j}(s^*_{-i-j}) + \sum_{\substack{\widetilde{s}_{-i}\in S_{-i}\setminus\{s'_{-i}\}:\\ \widetilde{s}_{ji}=1, g(s^*_i,\widetilde{s}_{-i})=\widetilde{g}}} \sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})} \\
&\leq \frac{\sigma^{\varepsilon_t}_L(\widetilde{s}_L)}{\sigma^{\varepsilon_t}_L(s^*_L)} \frac{\sigma^{\varepsilon_t}_j(\widetilde{s}_j)}{\sigma^{\varepsilon_t}_j(s'_j)} \frac{1}{1+\sum_{\substack{\widetilde{s}_{-i}\in S_{-i}\setminus\{s'_{-i}\}:\\ \widetilde{s}_{ji}=1, g(s^*_i,\widetilde{s}_{-i})=\widetilde{g}}} \frac{\sigma^{\varepsilon_t}_{-i}(\widetilde{s}_{-i})}{\sigma^{\varepsilon_t}_{-i}(s'_{-i})}}.
\end{aligned}$$

Now, $\lim_{t\to+\infty} \sigma^{\varepsilon_t}_L(\widetilde{s}_L) = 0$, $\lim_{t\to+\infty} \sigma^{\varepsilon_t}_L(s^*_L) = 1$, and

$$\lim_{t\to+\infty} \frac{\sigma^{\varepsilon_t}_j(\widetilde{s}_j)}{\sigma^{\varepsilon_t}_j(s'_j)} = \begin{cases} 1, & \text{if } \widetilde{s}_j = s'_j \\ 0, & \text{otherwise} \end{cases},$$

implying that $\lim_{t\to+\infty} \mu_{\varepsilon_t}(\widetilde{g})/\mu_{\varepsilon_t}(g^*) = 0$.

But then, given that σ^{ε_t} is an ε_t-proper equilibrium, there exists some T such that $\sigma^{\varepsilon_t}_i(s^*_i) \leq \varepsilon_t\, \sigma^{\varepsilon_t}_i(s'_i)$, for all $t \geq T$, implying that $\lim_{t\to+\infty} \sigma^{\varepsilon_t}_i(s^*_i) = 0$, which is a contradiction. ■

Proposition 3 *Suppose that u is link-responsive. Then, $PNE(u) \subseteq PRE(u)$ if and only if u_i is $\beta-$strongly concave in own new links on $\nu_i(g,u)$, for all $i \in N$ and $g \in PNE(u)$, for some $\beta > 0$.*

Proof. Let u be link-responsive. Suppose first that u_i is $\beta-$strongly concave in own new links on $\nu_i(g,u)$, for all $i \in N$ and $g \in PNE(u)$, for some $\beta > 0$. Let $g^* \in PNE(u)$, and denote by $S^{-1}(g^*) \subseteq S$ the set of all pure strategy Nash equilibrium profiles that support g^*. Let $s' \in S^{-1}(g^*)$. Recall that $\nu_i(g^*,u) = \{ij \notin g^* : m_{ij}u_i(g^*+ij) < 0\}$ and $\overline{g}^*_i\setminus\nu_i(g^*,u) = \{ij \notin g^* : m_{ij}u_i(g^*+ij) > 0\}$.[23] Given that g^* is a pairwise-Nash equilibrium outcome, necessarily $ij \in \overline{g}^*_i\setminus\nu_i(g^*,u)$ implies that $m_{ij}u_j(g^*+ij) < 0$ (as, otherwise, creating the currently missing link ij would be profitable for both i and j). For all $i \in N$, let $s^*_i = s'_i \vee \cup_{ij\in\overline{g}^*_i\setminus\nu_i(g^*,u)} e(ij)$. By definition, $s^*_{ij} = 0$ if and only if $ij \in \nu_i(g^*,u)$. Indeed, suppose that $ij \notin \nu_i(g^*,u)$. Then, either $ij \in g_i$ and $s'_{ij} = 1$, implying that $s^*_{ij} = 1$. Or $ij \in \overline{g}^*_i\setminus\nu_i(g^*,u)$, in which case $s'_{ij} = 0$ but, by construction, $s^*_{ij} = 1$. By $\beta-$strong

[23] Note that the strict inequality derives from link-responsiveness.

concavity, s^* is still a pure strategy Nash equilibrium profile that supports g^*, that is, $s^* \in S^{-1}(g^*)$ and $g^* = g(s^*)$.

For all $t \in \mathbb{N}$, $t \neq 0$, define the following marginal probabilities:

$$\sigma^{*t}(s_{ij} = 1, \cdot) = \begin{cases} \frac{1}{t}, \text{ if } ij \in \nu_i(g^*, u) \\ 1 - \frac{1}{t^2}, \text{ otherwise} \end{cases}.$$

Given that $s^*_{ij} = 0$ if and only if $ij \in \nu_i(g^*, u)$ (and, thus, $s^*_{ij} = 1$, otherwise) these marginal probabilities for each single link announcement are just a tremble away from the pure strategy link announcement in s^*. Tremble probabilities for links $ij \notin \nu_i(g^*, u)$ are one order of magnitude higher than trembles for links $ij \in \nu_i(g^*, u)$.[24] Under the marginal distribution, each single link ij follows a Bernoulli process with probability $\sigma^{*t}(s_{ij} = 1, \cdot)$. Denote by σ^{*t} the (independent) product of such marginal probabilities. By construction, this is a multivariate Bernoulli process that constitutes a completely mixed strategy such that $\lim_{t \to +\infty} \sigma^{*t}(s^*) = 1$.

For each player $k \in N$, we partition the strategy set $S_k \backslash \{s^*_k\}$ into two disjoint sets $S^{=}_k$ and $S^{\neq}_k$ defined as follows:

$$\begin{cases} S^{\neq}_k = \{s_k \neq s^*_k : g(s_k, s^*_{-k}) \neq g^*\} \\ S^{=}_k = \{s_k \neq s^*_k : g(s_k, s^*_{-k}) = g^*\} \end{cases}.$$

It is plain that $S_k = S^{\neq}_k \cup S^{=}_k$ and that $S^{\neq}_k \cap S^{=}_k = \emptyset$. We now distinguish two cases.

First, let $s'_i \in S^{\neq}_k$. Then, by link-responsiveness of the network payoff function u, and given that s^*_i is a best-response to s^*_{-i}, we have $u_i(g^*) > u_i(g(s'_i, s^*_{-i}))$. Using the notations in (3) we get:

$$\Delta_i(s^*_i, s'_i; \sigma^t_{-i}) = \sigma^t_{-i}(s^*_{-i})[u_i(g^*) - u_i(g(s'_i, s^*_{-i}))] + \sum_{\widetilde{s}_{-i} \neq s^*_{-i}} \sigma^t_{-i}(\widetilde{s}_{-i}) \left[u_i(g(s^*_i, \widetilde{s}_{-i})) - u_i(g(s'_i, \widetilde{s}_{-i}))\right].$$

But, $\lim_{t \to +\infty} \sigma^{*t}(s^*_{-i}) = 1$ whereas $\lim_{t \to +\infty} \sigma^{*t}(\widetilde{s}_{-i}) = 0$, for all $\widetilde{s}_{-i} \neq s^*_{-i}$. Therefore, there exists some $T'_i > 0$ such that, for all $t \geq T_1$, we have $Eu_i(g(s^*_i, \sigma^{*t}_{-i})) > Eu_i(g(s'_i, \sigma^{*t}_{-i}))$, for all $s'_i \in S^{\neq}_i$.

Second, let $s'_i \in S^{=}_k$. Now, by definition, $g(s'_i, s^*_{-i}) = g^*$, and we get:

$$\Delta_i(s^*_i, s'_i; \sigma^t_{-i}) = \sum_{\widetilde{s}_{-i} \neq s^*_{-i}} \sigma^t_{-i}(\widetilde{s}_{-i}) \left[u_i(g(s^*_i, \widetilde{s}_{-i})) - u_i(g(s'_i, \widetilde{s}_{-i}))\right],$$

and all the probability weights $\sigma^t_{-i}(\widetilde{s}_{-i})$ now converge to zero as $t \to +\infty$. We control for the sign of $\Delta_i(s^*_i, s'_i; \sigma^t_{-i})$ by looking at the rates of convergence of the different probabilities.

[24] If $\nu_i(g^*, u) = \varnothing$, for all $i \in N$ then, necessarily, $\overline{g}^*_i = \varnothing$, for all $i \in N$ (as, otherwise, link creation would be strictly profitable for any pair of players not yet linked in g^*), implying that g^* is the complete network, which is a proper equilibrium under link-responsiveness for the following reasons. First, given that $\{\sigma^{*t}\}$ have full support, the expected payoffs accruing to each player under σ^{*t} are a continuous function of the trembles. Second, any link cutting is strictly harmful for every player (from link-responsiveness and the fact that the network is a pairwise-Nash equilibrium). Therefore, by continuity on the size of the trembles, it is strictly optimal not to cut any link for small enough trembles. Therefore, we assume throughout that $\nu_i(g^*, u) \neq \varnothing$ for at least some $i \in N$. In words, there is some tremble over (at least) one link one order of magnitude higher than some other trembles on other links (that is, trembles are not all of the same order of magnitude).

Suppose first that $ij \in g^*$ for all $j \in N$, that is, player i is fully directly connected in g^*. Then, $S_k^{=} = \emptyset$, and we are done.

Suppose, on the contrary, that $ij \notin g^*$ for some $j \in N$. Take one such link ij. Then, we distinguish two cases. Either $m_{ij}u_i(g^* + ij) < 0$ and $s_{ij}^* = 0$, or $m_{ij}u_i(g^* + ij) > 0$. In this latter case, by definition of s^*, we have $s_{ij}^* = 1$.

Recall also that $m_{ij}u_i(g^*) > 0$, for all $ij \in g^*$. For all such links, by definition of s^*, we have $s_{ij}^* = 1$.

Take $\widetilde{s}_{-i} \neq s_{-i}^*$. Now, $u_i(g(s_i^*, \widetilde{s}_{-i})) \neq u_i(g(s_i', \widetilde{s}_{-i})$ if and only if (by link-responsiveness) $g(s_i^*, \widetilde{s}_{-i}) \neq g(s_i', \widetilde{s}_{-i})$, which is equivalent to the fact that $\widetilde{s}_{-i}$ is reciprocating at least one link that is announced in s_i' but that is not announced in s_i^*. Formally, there exists at least one $ij \notin g^*$ such that $s_{ij}^*\widetilde{s}_{ji} = 0$ but $s_{ij}'\widetilde{s}_{ji} = 1$, equivalent to $s_{ij}^* = 0$ and $s_{ij}' = \widetilde{s}_{ji} = 1$. Necessarily, for any such link, $m_{ij}u_i(g^* + ij) < 0$. Also, given that, $g(s_i', s_{-i}^*) = g^*$, this link ij is not announced in s_{-i}^* as, otherwise, $g(s_i', s_{-i}^*)$ and g^* would differ for at least this link. Therefore, it is also true that $m_{ij}u_j(g^* + ij) < 0$. Then, by definition, $\sigma_{-i}^{*t}(\widetilde{s}_{ji}) = 1/t$.

More generally, let $\widehat{S}_{-i}$ be the set of strategy profiles $\widetilde{s}_{-i} \neq s_{-i}^*$ such that $\widetilde{s}_{ji} = 1$ for some $ij \notin g^*$, $m_{ij}u_i(g^* + ij) < 0$ and $m_{ij}u_j(g^* + ij) < 0$, and $\widetilde{s}_{kl} = s_{kl}^*$, otherwise. In words, $\widehat{S}_{-i}$ is the set of strategy profile for i's opponents that differ from s_{-i}^* only for one link which corresponds to a mistake of probability $1/t$. Given that any other strategy profile $\widetilde{s}_{-i} \in \widehat{S}_{-i}$, $\widetilde{s}_{-i} \neq s_{-i}^*$, differs from s_{-i}^* from either more than one mistake (with probability at most $1/t$ each) or from only one mistake of probability $1/t^2$, we have:

$$\Delta_i(s_i^*, s_i'; \sigma_{-i}^t) = \sum_{\widetilde{s}_{-i} \in \widehat{S}_{-i}} \frac{1}{t}\left[u_i(g(s_i^*, \widetilde{s}_{-i})) - u_i(g(s_i', \widetilde{s}_{-i}))\right] + o(\frac{1}{t}).$$

Let $\widetilde{s}_{-i} \in \widehat{S}_{-i}$. Then, $u_i(g(s_i^*, \widetilde{s}_{-i})) - u_i(g(s_i', \widetilde{s}_{-i})) = -m_{ij}u_i(g^* + ij)$ for some $ij \notin g^*$ such that $m_{ij}u_i(g^* + ij) < 0$. Therefore, there exists some $T_i'' > 0$ such that, for all $t \geq T_i^{''}$, we have:

$$\sum_{\widetilde{s}_{-i} \in \widehat{S}_{-i}} \frac{1}{t}\left[u_i(g(s_i^*, \widetilde{s}_{-i})) - u_i(g(s_i', \widetilde{s}_{-i}))\right] + o(1) > 0, \text{ for all } s_i' \in S_i^{=},$$

equivalent to $Eu_i(g(s_i^*, \sigma_{-i}^{*t})) > Eu_i(g(s_i', \sigma_{-i}^{*t}))$, for all $s_i' \in S_i^{=}$. Let now $T = \max\{T_i', T_i'' : i \in N\}$. Then, for all $i \in N$ and for all $t \geq T$, s_i^* is a strict best response to σ_{-i}^{*t}. Formally,

$$Eu_i(g(s_i^*, \sigma_{-i}^{*t})) > Eu_i(g(s_i', \sigma_{-i}^{*t})), \text{ for all } s_i' \in S_i,\ t \geq T \text{ and } i \in N. \tag{4}$$

We first introduce a definition. Consider a normal form game $\gamma = (N, S, u)$ with an extensive form equivalent Γ. The extensive form is characterized by a set of histories H, a function P that assigns to each non-terminal history a player in N, a partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$. The set of actions available after the non-terminal history h to $P(h)$ is denoted by $A(h)$, and we assume that $h, h\prime \in \mathcal{I}_i$ implies that $A(h) = A(h')$.

Definition 7 (Van Damme, 1984) *A sequence* $\{\sigma^{\varepsilon_t}\}_{t\in\mathbb{N}}$ *of completely mixed strategies in* γ *with limit* σ *induces a quasi-perfect equilibrium in* Γ *if, for the corresponding sequence of completely mixed behavior strategies* ξ^{ε_t} *with limit* ξ*, for all* $i \in N$ *and for all information set* $h \in H$ *such that* $P(h) = i$*, contingent on having reached* h*, there exists some* T *such that:*

$$Eu_i(\xi_i, \xi_{-i}^{\varepsilon_t}) \geq Eu_i(\xi_i', \xi_{-i}^{\varepsilon_t}),\ \textit{for all } \xi_i' \textit{ and } t \geq T.$$

Let $\Gamma(u)$ be any extensive form game equivalent to the simultaneous move game of link formation with network payoffs u. We prove the following result.

Lemma 1 *The sequence* $\{\sigma^{*t}\}_{t\in\mathbb{N}}$ *with limit* s^* *induces a quasi-perfect equilibrium in* $\Gamma(u)$.

Proof. Denote by ξ^{*t} the behavioral strategies induced by σ^{*t}, for all $t \in \mathbb{N}$. This behavioral strategies generate a probability distribution over the set of terminal nodes to which we associate a probability distribution over networks in $\mathcal{G}$ denoted by $\pi^{*t} \in \Delta(\mathcal{G})$. If $\mu^{*t} \in \Delta(\mathcal{G})$ denote the probability distribution on $\mathcal{G}$ induced by σ^{*t}, it is plain that $\pi^{*t} \in O(\mu^{*t})$, when $t \to +\infty$. In words, the probabilities assigned to any single network on $\mathcal{G}$ by π^{*t} and by σ^{*t} are of the same order of magnitude. Then, by (4) there exists some $T > 0$ such that, for all $t \geq T$, we have $Eu_i(\xi_i^*, \xi_{-i}^{*t}) \geq Eu_i(\xi_i', \xi_{-i}^{*t})$, for all ξ_i', and for all i.$\square$

Therefore, when u_i is $\beta-$strongly concave in own new links on $\nu_i(g,u)$, for all $i \in N$ and $g \in PNE(u)$, for some $\beta > 0$, we can construct a sequence $\{\sigma^{*t}\}$ that induces a quasi-perfect equilibrium in any extensive form equivalent to Myerson's simultaneous move game in network formation. By the extensive form characterization of properness by Mailath, Samuelson and Swinkels (1997), we conclude that this sequence sustains a proper equilibrium in the original normal form game.[25]

Suppose now that $\beta-$strong concavity does not hold for some i. Then, using the arguments in the proof of Theorem 1 we prove, mutatis mutandis, that, for any sequence of completely mixed strategy profiles $\{\sigma^{\varepsilon_t}\}$ converging to s^*, there always exists a strategy $s_i' \neq s_i^*$ yielding strictly higher payoffs against $\sigma_{-i}^{\varepsilon_t}$ than s_i^*. ■

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