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A Study of the Interaction of Insurance and Financial Markets: Efficieny and Full Insurance Coverage

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Abstract

The first contribution of this paper is to provide a framework, a model together with a corresponding equilibrium notion, suitable for the study of the interaction between insurance and dynamic financial markets. This framework is used to prove the central result in the paper: in equilibrium agents purchase full insurance coverage, despite insurance prices that are not actuarially fair. The paper identifies three conditions which together explain why buying full insurance is optimal for any risk-averse individual even in the presence of loaded insurance prices: (i) insurance contracts are priced competitively, (ii) financial prices include a risk premium only for undiversifiable risk, and (iii) financial markets are effectively complete. An implication is that in this model disasters can be insured by fully reserved stock insurance companies.

"Consumers trading in both markets at once use the financial market to diversify their investment portfolio and use the insurance market to insure their personal risk. In ignoring trade in financial assets, the formal model in this paper bypasses an important aspect of consumer behavior under risk." Marshall (1974b, p675)

This paper provides a novel framework with which to study insurance without 'bypassing trading in financial markets'. Including financial markets as an integral part of the analysis provides rich insights on the workings of modern insurance markets. The first such insight, and the main contribution of this paper, is that when financial markets are sufficiently flexible, efficiency and equilibrium go hand in hand with full insurance coverage even in the presence of loaded insurance prices. This paper shows that it is optimal to purchase full coverage, even when insurance prices are not actuarially fair. This is demonstrated using a general equilibrium model, where agents not only buy insurance, but can also invest in shares of the companies that sell insurance. Moreover, it is shown that equilibrium is efficient in the sense that it decentralizes a Pareto optimal risk sharing rule.

To illustrate the main ideas in this paper, consider the following simple example:[1] there are 50 villagers living on an island which is the only source of coconuts for the nearby archipelago. Each villager owns a coconut tree which will produce 1.000 coconuts in one year's time. All villagers are exposed to the risk of losing 80% of the coconuts if their tree is hit by lightning, which occurs with 10% probability per year (assume the risks are independent from one tree to another). Suppose that insurance is issued by three insurance companies that compete fiercely for customers in prices. Insurance is sold at the beginning of the year, it is not traded during the year, and the corresponding indemnities are paid at the end of the year. Insurance companies also issue shares in the island's booming stock market, which is open every day, and where a riskless bond is also traded. Anyone holding shares of an insurance company at the end of the year will receive a dividend equal to the

[1] All numbers for the example are made up (coconut trees only produce around 50 coconuts per year) but they are representative of the kind of numbers that can be obtained from a numerical example. The interest rate is assumed to be zero for convenience.

insurance premia collected by the insurance company at the beginning of the year minus the indemnity payments that are due—if indemnity payments are greater than the premia collected, the shareholders will have to make up the difference.

The results in this paper say that in an equilibrium of this economy all villagers will buy full coverage (i.e., receive 800 coconuts in the event of a loss) and they will pay a premium which will be more than 80 coconuts, say 90 coconuts, to the insurance companies. The 10 coconuts the insurance companies receive from each villager over and above the actuarially fair price of 80 will be redistributed to the insurance companies' shareholders to compensate them for the risk of owning shares—the risk that indemnity payments are not equal to their expected value and that the indemnities tend to be high (and dividends low) when coconuts are scarce. This paper shows that all the villagers will own shares and their asset trading behavior can be described using a single fully diversified portfolio, for example, a portfolio that puts the same weight on each of the three companies. But, if villagers have different degrees of risk aversion, they will invest different amounts of wealth in the portfolio relative to the riskless bond: those with greater risk aversion will have more wealth invested in risky insurance company stocks. As time passes and storms come and go, sometimes lightning will hit some trees and sometimes not. These news affect the probability and amount of indemnities each insurance company will pay and hence the dividends to shareholders and current share prices. Villagers will react to these changes in share prices and future dividends by readjusting the amount of money invested in the risky portfolio. The current paper shows that not only is the villager's behavior just described optimal, but also, that the economy's institutions: private insurance combined with trading on insurance company shares, leads to an optimal distribution of the risk in the island amongst all villagers.

The first step is to set up a general model of an economy in which private insurance is available and insurance company shares are traded. Then, an appropriate notion of equilibrium is defined. Having developed the necessary tools, the paper presents the main results: first, it is shown that there exist efficient insurance market equilibria and that the number of actively traded financial assets needed to attain efficient allocations can be as few as two. Then, the central result of the paper is given: in efficient equilibria agents buy full insurance. This is shown to hold despite strictly positive loadings on insurance. The paper identifies three conditions which together explain why buying full insurance is optimal for any risk-averse individual even in the presence of loaded insurance prices. These three conditions are formalized in the paper and can described as requiring that (i) insurance contracts be priced competitively, (ii) financial prices include a risk premium only for undiversifiable risk, and (iii) financial markets are effectively complete. Conditions (i) and (ii) ensure that the price of insurance is "economically fair", i.e. it exactly compensates for the actuarial and the economic risk (the probability and magnitude of indemnity payments plus the market price for the undiversifiable component of insurance risk, minus the time value of premium payments), and Condition (iii) says that given a sufficient amount of initial capital, any Pareto efficient consumption allocation can be constructed by trading in financial markets.

These results imply that in this model disasters can be insured by stock insurance companies when those companies are required to reserve in full (so that there will never be a shortfall in insurance company assets) or when they are fully assessable (owners are liable for any liabilities not covered by firm assets, as are private Names for the liabilities of their Syndicates at Lloyd's of London).

Another result obtained here is to show that optimal trading strategies, which are usually obtained as the solution of a stochastic differential equation, can be described explicitly in terms of (deterministic) equations.

Methodologically, the paper makes two contributions: the model and the notion of equilibrium. The model developed in this paper generalizes those in the insurance literature by allowing the joint analysis of the market for private insurance and continuously open financial markets. The model also permits great heterogeneity in preferences and endowments. The insurance market equilibrium is an equilibrium concept specially suited for problems of insurance, where contracts that are not traded dynamically (personal insurance) coexist with actively traded financial ones (bonds and stock company shares).

A final contribution of this paper is to provide a unifying framework for a number of existing results. For example, as markets are frictionless and there are no agency costs reinsurance is redundant as put forward in Doherty and Tini˘ (1981). In fact, as investorsc will always hold a fully diversified portfolio of shares, the number of insurance companies is irrelevant (as long as they act competitively). A second example is the characterization of the loading using the market price for risk and its actuarial properties. This is the approach pursued in Ellickson and Penalva (1997), Aase (1999) and Schweitzer (2001), and it contrasts with the approach that the loading is determined from the insured's willingness to pay or the interaction of the risk aversions of insurance companies and reinsurers (as in many papers from Borch (1962) to Aase (2002)).[2]

The paper proceeds as follows: this introduction concludes with a brief overview of some of the more relevant articles in the literature. The next section introduces the model. As the model is new, the section provides a detailed description of its different aspects: preferences and risks, the insurance market and the stock market (contracts and pricing), and agents' budget constraints. The section concludes with a short review of the model plus the definition of an insurance market equilibrium. Then, the first results on efficiency are presented in the second section. These set the stage for the central result in the third section, where the characteristics of efficient equilibria are determined, and it is shown that it is optimal to buy full insurance coverage despite unfair prices. The fourth section analyzes the causes behind the central result and identifies the three conditions that help explain the insurance decision, as well as including other interesting results. The final section puts the efficiency of equilibrium result in context, considers several extensions of the model and their effect on the main results of the paper, discusses the special case where agents have HARA preferences, and concludes with some comments on future research.

Only the short and simpler proofs are in the text, the rest are in the Appendix.

Related Literature

The central result in this paper formalizes for the first time the following description of equilibrium in Kihlstrom and Pauly (1971): "Persons who bear part of the total loss might be thought of as having a kind of split personality in which they make a certain payment in return for coverage which does not depend on total loss but in which they hold "stock" in an

[2] MacMinn and Witt (1987) analyze the determination of the loading with monopolistically competitive and risk-averse insurance firms that invest in a competitive financial market. These firms set premium prices that will depend on the aggregate price of risk and their corporate risk aversion and the loading they obtain equals the one obtained here for the special case of risk-neutral insurers.

insurance "firm" which makes their final wealth positions vary with the total loss" (quotes in the original). In the current paper insurance firms are explicitly included and the results in Kihlstrom and Pauly (1971) are greatly generalized. Also, the analysis goes further by identifying the conditions under which equilibrium implies full insurance demand and show that these conditions could also hold out of equilibrium.

While the main contribution of the current paper is entirely original, it also contains additional results that extend and complement a large number of results in the existing literature. We will restrict attention to the closest references. For example, there is an extensive literature on the efficiency of insurance markets.[3] Borch (1962) and Wilson (1968) established the mutuality principle for characterizing efficiency under risk. Marshall (1974a) discusses how this (mutuality) approach to insurance provides a solution to the provision of catastrophic insurance that cannot be obtained from a reserves-based approach (which relies on the Law of Large Numbers)—an argument repeated in different forms in the literature and one that motivates our analysis of the role of financial markets as an institution to implement efficient risk-sharing. Based on these early results, many researchers have studied mechanisms other than financial markets to implement efficient risk sharing arrangements, e.g. via mutual insurance companies (Doherty and Dionne (1993)), via the design of private insurance contracts (Cummins and Mahul (2003), Cass Chichilninsky and Wu (1996)), or via reinsurance (Borch (1984), Doherty and Tiniˇ (1981), Froot (2001), Jaffee Russell (1997),c Zanjani (2002)).

The financial market based approach to insurance is studied in Ellickson and Penalva (1997), Harrington and Niehaus (1999), Aase (2001), Christensen et al (2001), Penalva (2001). The current paper extends the literature by formally including insurance as a nontradeable asset and studying its price and demand. The interaction of individual insurance and investment decisions is treated in Smith and Mayers (1983), Eeckhoudt et al (1997) and Somerville (2004), as a partial equilibrium problem while Penalva (2001) takes a general equilibrium perspective. The present analysis is a general equilibrium one and differs from Penalva (2001) in a number of ways. Most importantly, in this paper insurance contracts

[3] The relationship between the efficiency results in this paper and existing ones is analyzed in detail in Section 5.1.

are not tradable. Also, Penalva (2001) looks for conditions that will ensure efficiency while the current paper focuses on the details of insurance demand, investment decisions, and prices. Technically, this paper also differs in that the proofs used here are more elegant and constructive.

1 The Framework

This section contains the basic methodological contribution of the paper: a model of private risks with non-traded private insurance and a financial market with continuous trading, plus the notion of a competitive insurance market equilibrium. The model and the equilibrium notion provide the basic tools with which the economic results of the paper are demonstrated.

The following two subsections describe in detail all the different components that define the model. The eager reader may jump ahead to subsection 1.3, where the model is summarized briefly and the notion of a competitive insurance market equilibrium is defined, and refer back to sections 1.1 and 1.2 as needed.

1.1 The Basic Economy: Preferences and Risk

The basic model is that of a two-date economy, $\mathcal{E}$, with a finite number of agents who have heterogeneous von Neumann-Morgenstern preferences and risky endowments (extensions are considered in Section 5.3).

Preferences: In this economy there are $n < \infty$ agents with heterogenous preferences described by standard risk-averse expected utility functions:

Assumption 1: For all $i = 1, \ldots, n$,

$$U_i(x) = u_i(x(0)) + \beta_i E[v_i(x(1))],$$

where both u_i and v_i are increasing, strictly concave differentiable functions satisfying the standard Inada conditions (described in Appendix A).

Endowments: Each agent, i, has a constant income at each date $(w_{i,0}, w_{i,1})$. At some point between dates zero and one, each agent may suffer an accident (at most) that translates into a fixed loss, L, of date one income. The probability that agent i has an accident between dates zero and one is equal to p.

Risk: The random variable $N_i(t)$ keeps track of the number of accidents suffered by agent i between date zero and $t \in [0, 1]$. As each agent can have at most one accident, $N_i(t)$ can only take values 0 or 1. The vector $\mathbf{N}(t) \equiv (N_1(t), \ldots, N_n(t))$ keeps track of each agent's accidents, and $N(t) \equiv \sum_{i=1}^{n} N_i(t)$ keeps track of the total number of accidents.

Let $e_i \equiv (e_i(t))_{t \in \{0,1\}}$ denote agent i's endowment and $e \equiv (e(t))_{t \in \{0,1\}}$, $e(t) = \sum_{i=1}^{n} e_i(t)$, the aggregate endowment. Then:

Assumption 2: Agents endowments are:

$$e_i(0) = w_{i,0} \qquad \& \qquad e_i(1) = w_{i,1} - N_i(1)L,$$

where $Pr(N_i(1) = 1) = p \in (0, 1)$, $L > 0$, and $w_{i,0} \geq 0$, $w_{i,1} \geq L$ with strict inequality for at least one i. All agents have the same priors on the distribution of $\mathbf{N}$.

1.2 Insurance and the stock market

In addition to preferences and endowments, the model includes two sectors: a sector with non-traded contracts (private insurance) and a sector with continuously trading contracts (financial market).

Non-traded Contracts: A basic characteristic of private insurance is that it is provided in the form of private, personalized, non-transferable contracts. Insurance contracts are contracts settled via bilateral negotiation between an individual and a corporation (or its representatives) that specifies payments to the individual depending on certain individual-specific occurrences which cannot be transferred to a third party.[4]

Let $\mathcal{I}$ denote the set of non-traded contracts, which is the union of the set of non-traded contracts available to each individual, $\mathcal{I}_i$. Each agent can purchase private insurance, a contract that pays him only in the event that he has an accident (the event: $\{N_i(1) = 1\}$) and can choose what amount he wants reimbursed which can be anything between nothing (no coverage) and all of L (full coverage).[5]

[4] Insurance markets are extremely rich and complex. The description just provided applies quite generally but is highly simplified. A general definition of an insurance contract would include caveats to account for the current richness and complexity of the practice of insurance contracts: contracts who cover more than one individual, whose payments depend on special non-individual specific events, with clauses that allow the transfer of the contract to certain prespecified third parties, etc.

[5] Thus, for all i, $\mathcal{I}_i = \{(\alpha N_i(1))_{\alpha \in [0,L]}\}$.

Insurance companies and Traded Contracts:

An insurance company is an institution which sells private insurance contracts and can issue financial contracts that are tradeable in stock markets (insurance company shares, risk debt, etc.). Financial markets are frictionless and all claims issued by the insurance company will be satisfied, i.e. there is no bankruptcy/insurance companies are fully assessable. In such a setting, the classic result of Modigliani and Miller (1958) applies: the value of the insurance firm is independent of its financing strategy. For simplicity we assume there are J insurance companies who are fully equity financed and each company issues (infinitely divisible) shares at date zero. The total number of shares is normalized to one. A share is a claim on the premia collected by the insurance company (plus any interest earned between dates zero and one) minus any indemnities paid at date one. It is convenient to assume that all premia are invested in riskless bonds.[6] The claims that will be received by the owner(s) of the shares issued by insurance company $j = 1, \ldots, J$ is represented by a random variable, d_j. The price of this claim is determined in financial markets.

Pricing in Financial Markets:

The shares issued by insurance companies can be traded in a frictionless financial market at any time ($t \in [0, 1]$), called a stock exchange. This financial market is an institution in which an auctioneer continuously sets prices to facilitate share trading. We assume there is no private information or agency costs and the auctioneer sets prices such that no arbitrage opportunities exist. Agents can go to the stock market and trade shares at the announced prices at any time without any costs, frictions or constraints. Agents have common priors (described by an objective probability measure, P), so that trades will be motivated purely by the desire to control risk exposures.

No arbitrage in frictionless financial markets implies that there exists a probability measure, Q, and an interest rate process, $r(t)$, such that the price of any (traded) asset can be priced as the present expected discounted value of its future payments (Harrison and Kreps(1979)), where the expectation is taken with respect to available information (for any random variable, x, let $E_{Q_t}[x]$ denote the expectation of x conditional on information avail-

[6] In the current frictionless model, this assumption is without loss of generality as the investment strategy pursued by the insurance company with the premia collected will not add or subtract value to the company.

able at date t using probability measure Q. Thus, the price of a share in company j at date t, $S_j(t)$, which promises claims d_j at date one is given by:

$$\forall t \in [0,1], \qquad S_j(t) = E^Q_t\left[d_j\, e^{-\int_t^1 r(s)\,ds}\right] \qquad P\text{-a.s.} \tag{1}$$

Define the likelihood ratio process, $\xi(t)$, as

$$Q = P\xi(1), \qquad \xi(t) = \frac{dQ_t}{dP_t} \equiv E^P_t[\xi(1)]$$

Note that $\xi(1) = dQ/dP$ is the Radon-Nikodym derivative (the density of Q with respect to P). This object plays a key role in the results of the paper as it is used to describe market prices.

Let $r = \int_0^1 r(t)\,dt$ be the interest rate, so that the price of the riskless bond which, is assumed, promises to pay one unit of consumption at date one, $d_0 = 1$, has price $S_0(t)$ where

$$S_0(t) = e^{-r(1-t)} \equiv \exp\left(-\int_t^1 r(s)\,ds\right).$$

The set of assets traded in stock markets (insurance company shares plus the riskless bond) are described by their no-arbitrage prices and denoted by $\mathcal{D} = \{S_j(t)_{t\in[0,1]} : j = 0, 1, \ldots, J\}$.

Information: Insurance company shares are claims on the performance of those companies, which depends solely on the indemnities they have to pay out. These indemnities themselves depend on who has had accidents. For any random process $\{x(s)\}_{s\in[0,1]}$ and any $t \in (0,1]$, define $x(t-) = \lim_{s\uparrow t} x(s)$. The process of the arrival of accidents to agent i is described by a common hazard rate, $\lambda(t)$, as follows:

Assumption 3: For each i, the dynamics of $N_i(t)$, $t \in [0,1]$ is described by the hazard rate, $\lambda_i(t)$, which is defined by the function $\eta: [0,1] \times \{0, 1, \ldots, n\} \to \mathbb{R}$:

$$\lambda_i(t) = \begin{cases} \eta(t, N(t-)) & \text{if } N_i(t-) = 0 \\ 0 & \text{if } N_i(t-) = 1 \end{cases}$$

To better understand this assumption consider the following three examples:

Example 1 (constant hazard):

If the hazard rate is constant, $\eta(t, N(t-)) = \lambda$, then for any i the distribution of the arrival time of an accident to i is iid exponential with parameter λ and the total number of accidents,

$N(t)$, is similar to a Poisson process. Note that the parameter λ is related to p via the following equation:

$$p = 1 - \exp(-\lambda) \iff \lambda = -\ln(1-p).$$

Example 2 (hurricanes): consider the risk of a hurricane destroying the coconut trees in the example used in the introduction. Describe the risk of a hurricane affecting any one tree using two constants, $\lambda > 0$ and $\gamma > 1$, as follows: at the beginning, if $N(t) = 0$, the hazard rate for every tree is λ. If one agent's tree is hit by a hurricane at time s, so that $N(t) \geq 1$ for $t > s$, then the proximity of a hurricane increases everyone else's hazard rate to $\lambda\gamma$. The hazard rate will then be: $\eta(t, 0) = \lambda$ and $\eta(t, m) = \lambda\gamma$ for $m = 1, \ldots, n$.

Example 3 (aging): consider the risk of coconut trees becoming unproductive and assume all trees are planted at the same time. As time passes a tree may randomly become unproductive. This risk is independent from one tree to another and increases with age. One way to incorporate this risk into the model is by assuming that the probability that a tree becomes unproductive between today and tomorrow is increasing with the time since the tree was planted, e.g., let $\eta(t, N(t-)) = \beta t^{\alpha}$ with $\beta > 0$ and $\alpha > 1$.

Diversified Portfolios: A special construct that will useful throughout the paper is that of a fully diversified portfolio of risky shares. Given a set of prices described by the pair (Q, r), a self-financing portfolio of risky shares, $(\theta_j(t))_{j=1}^{J}{}_{t\in[0,1]}$ is **fully diversified** if there exists a deterministic function g: $\mathbb{R} \to \mathbb{R}$ such that $\sum_{j=1}^{J} \theta_j(1) d_j(1) = g(e(1))$.[7] Then, the portfolio is fully diversified in the sense that dividends from the portfolio only depend on

[7] A portfolio of risky shares is self-financing if

$$\sum_{j=1}^{J} \theta_j(t) S_j(t) = \sum_{j=1}^{J} \theta_j(0) S_j(0) + \sum_{j=1}^{J} \int_0^t \theta_j(s)\, dS_j(s), \quad \forall t \in [0, 1].$$

Further, we assume that a self-financing portfolio satisfies an additional technical condition. Let $S_j(t, N(t)+1)$ be the price of security j, calculated using Condition 1 but changing the current history of events by adding an extra accident (to any of the consumers that have not had an accident yet) at date t. Then, a self-financing portfolio will be assumed to satisfy an additional condition: $\sum_{j=1}^{J} \theta_j(t)(S_j(t) - S_j(t, N(t)+1)) = 0$ P-a.s. on $\{N(t) = n\}$ for all t. This assumption ensures that price of the portfolio always responds to new accidents. This technical condition is generally satisfied by almost all fully diversified portfolios. In fact, a portfolio that does not satisfy this additional condition can be made to satisfy it by a very small change in the portfolio weights.

what happens to the economy on aggregate. This property is very important—it is used, among other things, to characterize efficient risk sharing rules by the mutuality principle (Borch (1962) and Wilson (1968))— and we will refer to it as **the mutuality property.**

This paper will make use of a particular type of fully diversified portfolio: the equally weighted portfolio, a portfolio which puts the same weight in each of the J risky assets (insurance companies), i.e. $\theta_j(t) = 1/J$, $j = 1, \ldots, J$.

Insurance Pricing

A key element of the model proposed here is the presence of non-traded contracts, $\mathcal{I}$. Given that we are interested in insurance contracts which are not traded and are negotiated bilaterally it is not obvious how their prices will be determined.

The sustitutability between consumers and between insurance companies suggests that insurance contracts should be priced "competitively". The buyer (insured) is one of n potential insureds, all with the same risk (p, L). The seller (insurance company) is one of J companies all of which have the same 'technology' for providing insurance, at least for sufficiently large n and J. Therefore, any of the parties, buyer or seller, can be easily substituted and prices should reflect the lack of bargaining power between the parties. Define competitive insurance prices as follows:

Definition 1 *In an economy where the pair (Q, r) prices all traded assets, $\mathcal{D}$, according to equation (1), insurance prices are competitive under (Q, r) if for all i,*

$$S_i = e^{-r} E_Q [N_i(1)] \qquad (2)$$

This condition states, literally, that the competitive price of insurance should be determined as if it were a tradeable financial asset satisfying the no-arbitrage condition. This condition is not defensible using arguments based on arbitrage (as insurance contracts are highly illiquid assets). But, this condition is justified because it arises as the result of competition (e.g., Bertrand-type successive price cutting) among insurance firms. For example, consider the following thought experiment: a company sells k units of coverage at price at S_i to agent i. This contract represents a liability: $k \cdot N_i(1)$ of date one consumption, and an asset: $kS_i e^r$ of date one consumption. The effect of selling this contract at date zero on the price of the insurance company is to change it by $\equiv kS_i - e^{-r} E_Q [kN_i(1)]$. This effect cannot be negative as the insurance company can always reject (not propose) such a contract. But, if it is strictly positive, i.e. if >0, then another insurance company can offer the same contract at a price of $S_i - \triangleleft 2k$ and increase its share price by at least $\triangleleft 2 \triangleright$ The private insurance market is said to be competitive if the possibility to increase a company's share price using such a strategy does not exist. This then implies that the competitive price of insurance contracts will be equal to $e^{-r} E_Q [N_i (1)]$ as claimed.

Insurance and Investment Decisions

An agent in this economy is faced with three basic decisions: how much private insurance to buy, how and how much to invest in financial assets (bond and insurance companies' shares), and how much to consume. Agent's decisions are restricted in the sense that they cannot spend more than they are endowed with, they have access to a restricted set of insurance contracts (they have to be in $\mathcal{I}_i$) and any investment strategy has to be implementable given prices, his wealth, and his insurance decision.

To reduce notation, for each agent i let $B_i (D, S_i)$ denote agent i's budget constraint, i.e., the set of consumption allocations the agent can achieve given his endowment, prices and trading opportunities.[8]

1.3 Model Overview and Equilibrium

There is an economy, $\mathcal{E}$, described by n heterogeneous agents with von Neumann-Morgenstern expected utility, U_i, who are risk averse (Assumption 1) and have incomes at dates one and zero which are subject to a common type of shock, with probability p and of magnitude L (assumption 2). There are non-traded private insurance contracts, $\mathcal{I}$, which can be bought only at date zero. There are also financial assets, $\mathcal{D}$: a bond and shares of J insurance companies, which can be traded at any date, $t \in [0, 1]$. Prices of insurance contracts, S_i, are determined by competition, while share prices, $S_j (t)$, are determined by no-arbitrage. Share prices change over time as new information on the final value of insurance company shares enters the stock market.

For this economy, agent's consumption, insurance and investment decisions together with asset and insurance prices form a competitive insurance market equilibrium if agent's con-

8 A formal and detailed description of $B_i (D, S_i)$ is included in Appendix A.

sumption allocations, x_i, resulting from their investment and insurance decisions are optimal, given their budget constraints ($B_i(D, S_i)$):

Definition 2 A triple $((x_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D})$ is a competitive insurance market equilibrium if:

(i) there exists (Q, r) such that every $S_j \in \mathcal{D}$ satisfies no-arbitrage (Relation (1));

(ii) insurance prices are competitive; and

(iii) for all $i = 1, \ldots, n$, $x_i \in B_i(D, S_i)$ and for all $x \in B_i(D, S_i)$, $U_i(x_i) \geq U_i(x)$.

2 Efficiency and Equilibrium with an Insurance Market

This section studies the properties of a competitive insurance market equilibrium in our model. In particular, it establishes the existence of efficient competitive insurance equilibria in which consumers have the same allocations as in a complete markets equilibrium—markets are complete if the set of available assets are sufficient to reproduce every possible state-contingent consumption allocation. If assets are sufficient to attain Pareto efficient consumption allocations (but not all possible allocations) then the market is said to be effectively complete.

The first notions of equilibria for financial markets are due to Arrow (1964) and Radner (1972). An important question to ask of such equilibria is: when are the allocations from a financial market equilibrium the same as those from a standard (Arrow-Debreu state-contingent commodity) equilibrium? As standard equilibria are Pareto efficient, a financial market equilibrium with the same allocations will also be efficient. The answer usually rests on determining whether financial assets are (effectively) complete.

When considering competitive insurance market equilibria, this paper shows that the combination of private insurance with two dynamically traded assets generates effectively complete markets. In particular, the paper shows that for every state-contingent commodity equilibrium there is a corresponding competitive insurance market equilibrium that decentralizes it, and the set of dynamically traded financial assets that has to be available can be reduced to two: a riskless bond and a fully diversified portfolio (described in Section 1.2: Diversified Portfolios), such as the equally weighted portfolio of insurance company shares.

Theorem 1 *For every state-contingent commodity equilibrium of economy $\mathcal{E}$ with allocations $((x^*_i)_{i=1}^n)$, there exists a competitive insurance market equilibrium with the same allocations, $((x^*_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D})$ and where $\mathcal{D}$ contains only two assets: a zero coupon bond and a fully diversified portfolio.*

The proof of this result is based on the following chain of reasoning: every state-contingent commodity equilibrium is Pareto efficient; every Pareto efficient equilibrium of $\mathcal{E}$ implies optimal risk sharing; optimal risk sharing implies that consumption allocations have the mutuality property (described in Section 1.2: Diversified Portfolios); such consumption allocations can be constructed by purchasing full insurance, which eliminates risk from agents' endowments, and reallocating wealth net of insurance by dynamically trading the bond and the risky portfolio. To conclude, it is shown that the suggested combination of insurance and dynamic trading is feasible and optimal for every agent.

Theorem 1 implies:

Remark 1 *Insurance markets can function efficiently when insurance firms are stock companies, although those stock companies have to reserve in full or, alternatively, be fully assessable (as Lloyd's syndicates used to be).*

The assumption that stock companies are fully assessable is implicit in the definition of equilibrium as equilibrium imposes that the dividends promised by insurance companies are paid/collected in full. The flexibility in modeling the underlying risk in Assumption 3 allows one to have correlated date one risks, and hence to apply this model to natural disasters.

Competitive insurance market equilibria which decentralize state-contingent commodity equilibria (as in Theorem 1) will be referred to as **efficient insurance market equilibria.** As this paper makes repeated use of the combination of the bond and the portfolio of equally weighted insurance company shares, we will denote their prices by $\mathcal{D}^*$.

Given that a state-contingent commodity equilibrium for $\mathcal{E}$ always exists, then it follows that:

Corollary 1 *There exists (Q, r) and $(x_i)_{i=1}^n$ such that $((x_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D}^*)$ is an efficient insurance market equilibrium of $\mathcal{E}$.*

Note that financial markets trade the shares of the J insurance companies. Therefore, the actual securities traded, $\mathcal{D}$, is different from $\mathcal{D}^*$. But, even though the equally weighted portfolio is not traded, it may be synthesized using existing assets.[9] In such cases we say that $\mathcal{D}$ contains $\mathcal{D}^*$. Then, in the economy where only the bond and individual insurance company shares are traded, Theorem 1 says that an optimal investment strategy for any agent is to construct equally weighted portfolios.

3 Demand for Full Insurance

In the previous section, it is established that an economy in which an equally weighted portfolio of insurance company shares and a riskless bond are traded dynamically has an efficient insurance market equilibrium. This section presents the central result of the paper: insurance agents will buy full coverage in this equilibrium. It is also shown that in equilibrium prices will be unfair, i.e. above their actuarial value. This seemingly contradictory result will be analyzed and explained in detail in the following section (Section 4).

3.1 Full Insurance in Equilibrium

In order to determine how much insurance agents are buying in equilibrium, one needs to make sure that the question can be properly framed. The model in its full generality includes economies for which the demand for insurance is not uniquely defined by agents' purchases of private insurance contracts. For example, there could be only two agents and two insurance companies in the economy. If each insurance company insures a different agent it would be impossible to separate insurance coverage decisions from investment decisions as the dividends from the insurance company are perfectly (negatively) correlated with the indemnity payments received by one of the agents. An additional unnatural characteristic of such an economy is that agent's private insurance contracts are explicitly tradeable (although labeled as "shares of insurance company X"). To avoid these extreme cases, assume that insurance contracts are **strictly nontradable**: insurance contracts in economy $\mathcal{E}$ are said to

[9] As one can synthesize puts and calls. Nevertheless, note that an equally weighted portfolio is a portfolio that does not require rebalancing over time, while one that replicates a put or a call would generally require substantial portfolio rebalancing.

be strictly nontradable if there is no way of constructing a portfolio of existing traded assets that acts as a private insurance contract for any agent in the economy.

Definition 3 Insurance contracts in economy $\mathcal{E}(\mathcal{I},\mathcal{D})$ are strictly nontradable if $\nexists\, (\theta_j)_{j=0}^{J}$ such that $\sum_{j=0}^{J} \theta_j d_j(1) = N_i(1)$.

Using this definition we can establish the primary contribution of this paper: the optimal insurance coverage decision can be uniquely determined and for every efficient insurance market equilibrium every agent will optimally choose full insurance coverage.

Theorem 2 In every efficient insurance market equilibrium such that $\mathcal{D}$ contains at least a zero coupon bond and a fully diversified portfolio, and where insurance contracts in economy $\mathcal{E}(\mathcal{I},\mathcal{D})$ are strictly nontradable, then full insurance coverage will be the unique optimal insurance coverage.

3.2 The Loading on Insurance

Economists, specially those of us interested in the economics of insurance are familiar with Mossin (1968)'s result that "if the [insurance] premium is actuarially unfavorable, then it will never be optimal to take full coverage". Thus, the natural reaction is to think that insurance prices in an efficient equilibrium must be fair. Actually, the opposite is true:

Theorem 3 In every efficient insurance equilibrium, for every agent in the economy the price per unit of coverage is the same, $\forall i \in \mathcal{I}\ S_i = S$, and this price has a strictly positive loading, $\exists \gamma > 0$ such that:

$$S = p(1 + \gamma)e^{-r} \tag{3}$$

The positive loading comes from the presence of aggregate risk in the economy. Insurance companies need to convince investors to buy risky shares and the only way to do so is to promise them an expected return that is higher than the riskless rate. This extra return is paid for with the loading.

Note that, as the price is the same for all agents:

Remark 2 The competitive price of private insurance does not depend on the insured's willingness to pay for coverage.

The seeming contradiction between buying full coverage when insurance prices have a positive loading begs the question of what is happening in $\mathcal{E}$and how can these two results be compatible.

4 Explaining Full Insurance with Unfair Prices

Fist of all let us be clear: Mossin did not make a mistake. The main difference between the results in this paper and Mossin's is the context in which the coverage decision is made. Mossin makes a statement about how agents behave when faced with an isolated insurance decision. In this paper, on the other hand, agents make insurance decisions as well as other investment decisions. The question now is what makes the current setup special that leads to such different results.

The answer is that there are three conditions that are satisfied in equilibrium, and which together imply that agents' optimal demand for insurance will include full coverage. The conditions are that:

(i) insurance prices are competitive,

(ii) only aggregate risk matters, and

(iii) financial markets are effectively complete.

The first condition has been formally introduced earlier in the description of the model (Section 1.2). The other two need more explanation.

4.1 Equilibrium Prices

Condition (ii), that only aggregate risk matters, addresses the properties of financial prices. It says that financial prices do not put a premium on idiosyncratic risks which should, in principle, be diversified away. Given that insurance prices are competitive and hence derived from financial markets (as described in Theorem 3), this condition says that the loading on insurance arises only from the undiversifiable component of the private insurance contract. Condition (ii) is stated formally as:

Definition 4 *In economy $\mathcal{E}$, a measure Q is said to price only aggregate risk if there exists a real-valued function g such that the Radon-Nikodim derivative $\xi(1) = g(e(1))$.*

Proposition 1 *For every state-contingent commodity equilibrium of $\mathcal{E}$ with state-contingent prices π, (x^*, π), there exists (Q, r) derived from π such that Q prices only aggregate risk, and for any asset with dividend d_j, there exists a price process $S_j(t)$ satisfying Relation (1).*

Proposition 1 follows naturally from the representative agent representation of prices of every state-contingent equilibrium of $\mathcal{E}$.[10] Suppose that one is presented with a particular state-contingent equilibrium of $\mathcal{E}$, where the representative agent's preferences are characterized by the utility function V:

$$V(x) = v_0(x(0)) + \beta_0 E^P[v_1(x(1))]$$

Then,

$$\xi(1) = \frac{v_1(e(1))}{E^P[v_1(e(1))]} \tag{4}$$

4.2 Effective Completeness and Optimal Investment

Condition (iii) addresses the individual's ability to diversify. In the economy $\mathcal{E}$ all agents are risk averse and hence would like to diversify. The third condition ensures that they have access to financial assets that will allow them to replicate any fully diversified portfolio. The formal statement of Condition (iii) is:

Definition 5 *In economy $\mathcal{E}$, financial markets are effectively complete if for every random dividend d that satisfies the mutuality property, there exists $\theta \in \Theta$ such that*

$$d = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \int_0^1 \sum_{j=0}^{J} \theta_j(t)\, dS_j(t)$$

As Pareto efficient consumption allocations satisfy the mutuality property, a key step in the proof of Theorems 1 and 2 is to show that a bond and a fully diversified portfolio of insurance company shares (such as the equally-weighted portfolio) make financial markets effectively complete. Rather than appeal to a generic result on effective completeness, we

[10] See, for example, Constantinides (1982) or the proof of Proposition 1 in Appendix C.

proceed by characterizing, using deterministic equations, an investment strategy that will replicate any fully diversified portfolio.

Theorem 4 *Suppose that in economy $\mathcal{E}$, the measure Q prices only aggregate risk and there is a fully diversified portfolio with price process, $S_M(t)$, and dividend d_M. For any date 1 dividend, d, that satisfies the mutuality property, there exists deterministic functions $\theta_M(t, m)$, $\theta_0(t, m)$, $S_M(t, m)$, $D^*(t, m)$, $m = 0, 1, \ldots, n$, $t \in [0, 1]$, such that d can be attained with an initial amount of money, $D^*(0)$, and by dynamically trading the bond and a fully diversified portfolio of insurance company shares following the trading strategy:*

- *For any $t \in [0, 1]$, if $N(t-) = m < n$, invest $\theta_0(t, m)$ in the riskless bond and $\theta_M(t, m)$ in the risky asset, where*

$$\theta_M(t, m) = \frac{D^*(t, m+1) - D^*(t, m)}{S_M(t, m+1) - S_M(t, m)}$$

$$\theta_0(t, m) = \frac{D^*(t, m) - \theta_M(t, m) S_M(t, m)}{e^{-r}}$$

- *If $N(t-) = n$, invest $\theta_0(t, n) = D^*(t, n) e^{r}$ in the riskless bond.*

The proof of this result makes use of Duffie and Huang (1985)'s idea of applying martingale representation results to construct trading strategies. The deterministic functions used to describe the strategy arise from the Poisson-like properties of the information represented by the aggregate accident process, $N(t)$. We will describe the basic steps of the proof here and refer the reader to Appendix C for details.

If d satisfies the mutuality property, then for some function f, $d = f(e(1))$. To attain d one needs to start with a certain amount of money, use it to construct an investment portfolio, and follow a trading strategy that will ensure that the value of the portfolio will, in the end, be equal to d.

First, define two processes, $D^*(t)$ and $S_M^*(t)$, where $D^*(t)$ is the date zero discounted value of d given information known at date t, and $S_M^*(t)$ the date zero discounted price of the fully diversified portfolio of shares. Then, $D^*(0)$ will be a constant equal to the expected discounted value of d at date zero, and also the initial amount of money needed to construct the portfolio. $D^*(t)$ gives us the value of the portfolio that the trading strategy has to track. The problem of tracking $D^*(t)$ can be split into two parts: one is maintaining the value of the portfolio, and the other is making sure that any news that change $D^*(t)$ (and security prices) will be matched by changes in the value of the portfolio so that it continues to track $D^*(t)$. The bond helps us with the first part, and the portfolio of insurance shares for the second.[11]

Recall that $e(t)$ is a linear function of $N(t)$. Then, as $N(t)$ jumps randomly (whenever an agent has an accident $N(t)$ jumps up by one unit), $D^*(t)$ and $S^*_M(t)$ will also jump (though not necessarily by one unit). These processes, $D^*(t)$ and $S^*_M(t)$, can be represented in terms of the deterministic functions, $D^*(t, m)$ and $S^*_M(t, m)$ mentioned in Theorem 4 and described in detail in Appendix C.

Now, define the function $\theta_M(t, m)$ as the units of the diversified portfolio needed at date t if there have been m accidents up to (but not including) date t. The amount invested in the diversified portfolio is chosen so as to match changes in the value of $D^*(t)$ if there is an accident:

$$\theta_M(t, m) = \frac{D^*(t, m+1) - D^*(t, m)}{S^*_M(t, m+1) - S^*_M(t, m)}$$

Then, the number of units invested in the riskless asset, $\theta_0(t, m)$, are chosen so as to make sure the value of the portfolio if there is no accident continues to track $D^*(t)$:

$$\theta_0(t, m) = \frac{D^*(t, m) - \theta_M(t, m) S^*_M(t, m)}{e^{-r}}$$

With this strategy, the tracking portfolio will always be equal to $D^*(t)$ and hence its value at date one will be equal to d.

Combining Theorem 4 with Proposition 1 one obtains that Condition (iii) is satisfied in equilibrium:

Remark 3 For every efficient insurance market equilibrium, $((x_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D})$ the combination of a riskless bond and fully diversified portfolio of insurance company shares make financial markets effectively complete.

4.3 Optimality of Full Insurance

With these results all that is needed to complete the proof of Theorem 2 is to show that the above strategy is optimal and ensuring that no partial coverage decision will lead to an

[11] That one portfolio of shares is enough to track the changes can be proven using a martingale representation theorem—in this paper the proof is constructive.

optimal consumption allocation. The next result takes that step:

Theorem 5 In economy $\mathcal{E}$ with prices (D, S_i), agent i's optimal insurance demand includes buying full insurance if:

(i) there exists (Q, r) such that $\mathcal{D}$ satisfy Relation 1 and Q prices only aggregate risk;

(ii) insurance prices, $(S_i)_{i=1}^n$, are competitive under (Q, r), and

(iii) the set of assets $\mathcal{D}$ makes financial markets effectively complete.

If agent i's insurance contract is strictly nontradable then agent i's unique optimal insurance demand is to purchase full coverage.

Theorem 5 establishes why in an efficient insurance market equilibrium the optimal insurance strategy is to purchase full insurance: Conditions (i) and (ii) make the price of insurance "economically fair", i.e. it exactly compensates for the actuarial and the economic risk (the probability and magnitude of indemnity payments plus the market price for undiversifiable risk, minus the time value of premium payments), and Condition (iii) says that having eliminated their idiosyncratic risk, agents can construct their preferred fully diversified consumption allocation by trading in financial markets. Optimality of insurance and the trading strategy described in Theorem 4 then follows from the fact that the budget constraint in the efficient insurance equilibrium is contained in the agent's budget constraint in the state-contingent commodity equilibrium.

Note that a trivial corollary of Theorem 2 is that every agent's optimal investment decision in an efficient insurance market equilibrium is independent of his demand for insurance—independence meant in the classical statistical sense: knowing the demand for insurance in any efficient equilibrium (full coverage) tells us nothing about the individual's optimal investment decision.

5 Discussion and Extensions

5.1 Insurance Equilibrium and Effective Completeness

In Section 2, it was shown that efficient competitive insurance markets equilibria exist with only two dynamically traded assets plus private insurance. A useful illustration of the methodological contribution in this paper is to use the notion of insurance market equilibrium to express existing results on optimal risk sharing and effective completeness with frictionless financial markets. This also helps illustrate the value of the contribution made by Theorem 1.

Arrow Securities: Arrow's pioneering work (Arrow 1964) on the decentralization of state-contingent equilibria using financial assets demonstrates that in general complete markets requires at least one asset for each state of the world. In $\mathcal{E}$ (ignoring the uncertainty on the exact timing of accident arrivals) there are 2^n states of the world, one for each possible realization of $N(1)$. As Arrow did not consider the possibility of dynamically traded assets, his financial equilibrium can be considered as a competitive insurance equilibrium where everyone has access to the same securities, d_A, and can freely contract on them, $\theta_j \in \mathbb{R}$, but (like insurance contracts) they are not traded dynamically: $\mathcal{I} = \mathcal{I}_i = \{(\theta_j d_j)_{\theta_j \in \mathbb{R}}\}_{j=1}^{2^n}$, and $\mathcal{D} = \emptyset$.

Pure and Mutual Insurance: Malinvaud (1973) studied Arrow's economy using insurance contracts rather than Arrow securities. He considers an economy where $\mathcal{I}$ is the set of private insurance contracts. He shows that even if all agents are the same (in terms of preferences, risks and endowments) there is no efficient insurance market for $\mathcal{E}(\mathcal{I}, \emptyset)$ because individual risks generate aggregate risk and n insurance contracts are insufficient to deal both with individual risk and aggregate risk. On the positive side, he shows that as n goes to infinity, aggregate risk (or rather the amount of aggregate risk apportioned to each agent) diminishes down to zero (by the Law of Large Numbers) and an approximate equilibrium can be constructed using only the n insurance contracts.

Cass et al (1996) expand $\mathcal{I}$ to include not just private insurance but also a class of contracts they call "mutual insurance" (contracts that depend on $N_i(1)$ and $N(1)$). They study $\mathcal{E}(\mathcal{I}, \emptyset)$ and conclude that the number of assets needed to effectively complete the market can be reduced (down from Arrow's 2^n). The number depends on the number of types of agents, i.e. amount of symmetry across agents in terms of risks AND endowments and preferences. More precisely, if H is the number of types of agents in the economy $\mathcal{E}$, the number of different types of assets needed is $(n + 1)H$ mutual insurance contracts plus $n + 1$

Arrow-type securities (securities that pay only if a particular aggregate state is reached).[12]

Dynamically Traded Assets: Duffie and Huang (1985) take Arrow's model and allow dynamic trading of all assets ($I = \emptyset \Leftrightarrow \mathcal{D} = \emptyset \Rightarrow \swarrow$ They demonstrate that if agents can change their asset positions over time, then the number of assets needed to complete the market can be much smaller than 2^n. They determine that there exists a number K such that agents need at most $K + 1$ assets (which have to satisfy certain abstract properties). In Appendix B it is demonstrated that for economy $\mathcal{E} \Leftrightarrow K = n$ so that $\mathcal{E} \Leftarrow \emptyset \Leftrightarrow \mathcal{D} \Rightarrow$ has complete financial markets if $\mathcal{D}$ contains n appropriate financial assets.

Insurance and Dynamically Traded Assets: Theorem 1 shows that state contingent equilibria can be decentralized with very few assets: n insurance contracts and two financial assets are sufficient. Furthermore, each agent uses only three contracts: his private insurance and the two financial assets.

Remark 4 It is possible to decentralize a complete market equilibrium allocation of $\mathcal{E}$ with only two dynamically traded financial assets in the economy, plus private insurance for each agent.

5.2 Discussion of Extensions

The primary contribution of the paper is to show that in efficient insurance market equilibria agents' optimal insurance demands are to purchase full coverage despite insurance prices which are actuarially unfair. This result is shown using a quite general model of insurance markets which allows for the presence of dynamically traded financial assets. The efficient equilibrium is achieved using fully assessable stock insurance companies. The model extends existing results on insurance market efficiency as the number of agents is finite and risks can be strongly correlated so that these efficient insurance market equilibria exist even when the Law of Large Numbers does not apply.

The reason that agents will optimally buy full insurance despite loaded prices in equilibrium has been shown to be due to the confluence of three factors: insurance that is priced

[12] Note also, that if two agents i and j are of the same type of agent, the mutual contract that pays depending on the state of agent i and the aggregate state m is 'the same' contract as that depending on the state of agent j and aggregate state m, i.e. these two contracts are counted as one type of contract.

competitively, financial prices that only carry a risk premium for undiversifiable risk, and effectively complete financial markets. Furthermore, one can characterize agent i'strading strategy using deterministic equations.

In this section alternative specifications of the model are considered, together with their consequences on the above results. The first additional element that can be introduced into the model is agents that are not at risk. This would not alter any of the results.

Further, one can change the specification of risk (Assumption 3). This change can have a number of different effects depending on the type of change proposed. For example, the hazard rate could be allowed to depend on the time elapsed since the last accident. All results would continue to hold and the only substantial change is that the equations describing the optimal trading strategy in Theorem 4 would need to include an extra term (to account for the time elapsed since the last accident).

Another possible variation is to allow agents to be exposed to more than one type of risk. In a similar setting, Penalva (2001), it is shown that each additional risk an agent is exposed to will require an additional insurance contract. In this model, it is not necessary to introduce more insurance contracts. It suffices to change the description of the contract and allow insurance companies to write multi-peril contracts. Then, in efficient insurance equilibria, agents would only require one multi-peril insurance contract and they would purchase full coverage for all risks. On the other hand, the number of financial assets needed to make financial markets effectively complete may be greater, as the informational content of accidents of different types may be different.

A third variation is to allow more than one accident per person. The main results stated before will hold true but with some caveats. The full insurance coverage result will continue to hold but, as is the case when accidents are of different types, agents may require additional financial assets. Again what determines the number of financial assets needed is the informational content of accidents. For example, compare the informational content of one agent having two accidents versus two different agents having one accident each. If (from the point of view of aggregate risk) these two events are equivalent, then Condition (iii) will continue to hold, but, if they are not, then once an agent has had an accident, investors need to consider at least three different contingencies (one accident to a different agent, a second accident to this agent, and no more accidents) and this uncertainty may require more than two dynamically traded assets. A minor issue that arises with multiple accidents is the need to ensure that agents not lose more than they have (i.e. agents do not have negative endowments), but this is a technical issue that can be dealt with quite easily in the standard way. A final remark regarding multiple accidents has to do with the insurance loading. The method we have used to prove Theorem 3 is valid only for a single accident. Whether the same result can be proven with multiple accidents requires further research.

5.3 HARA Preferences and Linear Risk Sharing Rules

An interesting special case of the model (mainly due to its extensive use in the literature) is when agents have HARA preferences. Then, agents' optimal investment strategy is to buy assets at date zero and hold on to them, i.e. not trade them at all:

Proposition 2 If agents' preferences in $\mathcal{E}$ are of the form

$$U_i(x) = v_i(x(0)) + \beta_i E(u_i(x(1))),$$

where $-u_i''(x)/u_i'(x) = a_i + bx$ and if agents have access to a bond, full insurance and an equally-weighted portfolio of insurance company shares, then agents' optimal investment strategies are to buy-and-hold the bond and the equally weighted portfolio and purchase full coverage.

This result follows immediately from the linearity of the optimal sharing rule and the fact that the dividends of the equally weighted portfolio are a linear function of the aggregate endowment.

In the HARA case, agents can be exposed to multiple risks and multiple occurrences of the same accident and Proposition 2 would continue to hold verbatim.

5.4 Future Research

A question that remains open is the possibility of the existence of inefficient competitive insurance market equilibria. This problem is not unique to the current model but is shared by any model which has effectively complete (but not complete) markets. The sufficient conditions established in Section 4 imply that for an inefficient equilibrium to exist when the two basic financial assets are present (bonds and a diversified portfolio of insurance company shares) financial prices have to make a distinction across individual risks, but the author is not aware of any results that would generally rule out such prices using only equilibrium conditions.

A final interesting question is what would happen if there were additional costs of capital, such as taxes or costs arising from allowing insurance companies that are not fully assessable. In our competitive setting, insureds pay all the costs (and only the costs) of the insurance activity. Additional capital costs, such as taxes (as in Harrington and Niehaus (2003)) or default risk[13] would trickle down to insurance premia and it would be interesting to determine what effect these will have on agents' insurance decisions.

[13] The main existence results for general equilibrium with default (Dubey, Geanakoplos, Zame (2000), Geanakoplos and Zame (2002), Araujo, Pascoa and Torres-Martinez (2002)) prove existence, but their detailed implications for insurance have not been explored, to the best of my knowledge.

Appendix

The order of the results in the Appendix is slightly different than in the text. This order makes the logic of the proofs more transparent. Appendix A contains basic mathematical definitions that were unsuitable for the presentation but are used in the proofs. Appendix B proves the remark made in the text that the number of financial assets needed to decentralize a state-contingent commodity equilibrium, K, is equal to the number of agents at risk, n. Appendix C includes the proofs of Theorem 1, 2, 4 and 5 together with some auxiliary lemmas. Finally, Appendix D contains the proof of Theorem 3.

A Formal Definitions

The Inada conditions on an increasing function u: $\mathbb{R}_+ \to \mathbb{R}$ are: $\inf_x u(x) = 0$ and $\sup_x u(x) = +\infty$. Note that this condition is sufficient though not necessary for the results in the paper–they are used to guarantee existence and representative agent characterization of prices. Existence and representative agent characterization of prices can be extended to economies with preferences that do not satisfy the Inada conditions in the standard way.

Stochastic Processes: There is a canonical probability space on which the stochastic process, N, is defined and is denoted Ω (for more details on jump processes see Bremaud (1981)). The information generated by N is formally described by the filtration $(\mathcal{F}_t)_{t\in[0,1]}$ generated by N. Let $\sigma(x)$ denote the sigma-algebra generated by the random variable x, then for each $t \in [0, 1]$, $\mathcal{F}_t = \bigcap_{s\leq t} \sigma(N(s))$. A process $x(t)$ is said to be adapted to $(\mathcal{F}_t)_{t\in[0,1]}$ if for all t, $x(t)$ is $\mathcal{F}_t$-measurable. A process $x(t)$ is said to be $\mathcal{F}_t$-predictable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-} \equiv \bigcap_{s<t} \sigma(N(s))$. Let $x(t-)$ denote $\lim_{s\uparrow t} x(s)$. A process $x(t)$ is said to be $(P, (\mathcal{F}_t)_{t\in[0,1]})$-integrable if $x(t)$ is measurable with respect to $\mathcal{F}_{t-}$ and for all t

♣§⇐⊔⇒♣$\mathcal{P}$(dω) $< \infty$.

Allowable Trading Strategies: In an economy with J assets, $\mathcal{D} = ((S_j(t))_{t\in[0,1]})_{j=0}^{J}$, let Θ denote the vector of allowable trading strategies given $\mathcal{D}$. Feasible trades satisfy the usual restrictions: an allowable trading strategy on asset j is an $\mathcal{F}_t$-predictable and $(P, \mathcal{F}_t)$-integrable stochastic process $\theta_j^i(t)$, where $\theta_j^i(t)$ is the number of units of asset j agent i plans to hold going into date t. As real activity (endowments and consumption) takes place only at dates zero and one, feasible trades also require that allowable trading strategies satisfy the following self-financing condition: for all $\theta \in \Theta$:

$$\sum_{j=0}^{J} \theta_j(t) S_j(t) = \sum_{j=0}^{J} \theta_j(0) S_j(0) + \sum_{j=0}^{J} \int_0^t \theta_j(s)\, dS_j(s), \quad \forall t \in [0, 1].$$

<u>The budget constraint</u>

The budget constraint is the set of consumption allocations the agent can achieve given his endowment (and trading opportunities). In a competitive insurance market, agents can only alter their consumption by buying insurance and trading financial assets so that the set of attainable consumption allocations is determined by asset prices, $\mathcal{D} \equiv ((S_j(t))_{t \in [0,1]})_{j=0}^{J}$, allowable trading strategies, Θ (as defined above), and the availability of private insurance.

Keeping the model simple, the agent's initial asset holdings are assumed not to affect his total wealth (so that the distribution of wealth is uniquely determined by agent's risky income). Thus, if any agent has an initial endowment of insurance company shares, either the price of the shares is zero (which is the case if insurance companies are fully assessable and only issue equity) or any initial share holdings are compensated with a corresponding negative holding of the bond (i.e. initial share holdings are fully leveraged). Let $\theta_{j,0}$ denote the number of units of asset j agent i is endowed with at date zero. The complete description of agent i's endowment is $e_i(0) = w_{i,0} + \sum_{j=0}^{J} \theta_{j,0} S_j(0)$, $e_i(1) = w_{i,1} - N_i(1)L$.

Agent i's budget constraint, $B_i(D, S_i)$, is determined by asset prices, the price of his insurance coverage for him, S_i, and the set of allowable trading strategies Θ:

$$B_i(D, S_i) = \Big\{ x = (x(0), x(1)) : \exists \theta_i \in \Theta,\ \alpha_i \in [0, L] : x(0) = e_i(0) - \alpha_i S_i - \sum_{j=0}^{J} \theta_j(0) S_j(0),\ x(1) = e_i(1) + \alpha_i N_i(1) + \sum_{j=0}^{J} \theta_j(1) d_j \Big\}$$

B Martingale Dimension

Duffie and Huang (1985) show how to decentralize a state-contingent equilibrium as a Radner equilibrium if you have one riskless asset plus K appropriate risky assets—we refer the reader to the original for full details, such as the exact definition of 'appropriate'. The number K is equal to the dimension of the space of $(Q, (\mathcal{F}_t)_{t \in [0,1]})$-martingales, where the measure Q is derived from the equilibrium price of the state-contingent commodity equilibrium via the Radon-Nikodym derivative, $\xi(1)$ (see the discussion after Relation (1) in the main text).

As $\xi(1) > 0$ P-a.s. then Q is absolutely continuous relative to P. Fortunately, the martingale dimension of the space of martingales is invariant to an absolutely continuous change of measure so that to prove the claim in the text (that $K = n$) it suffices to show:

Lemma 1 The space of martingales on $(\Omega, \mathcal{F}_1, (\mathcal{F}_t)_t, P)$ has martingale dimension of n.

Proof This follows from the properties of $M_i(t) \equiv N_i(t) - \int_0^t \lambda_i(s)ds$. Namely, the stochastic processes $M_1, \ldots, M_n$ are $(P, \mathcal{F}_t)$-martingales and pairwise orthogonal. Thus, applying the martingale representation theorem for marked point processes (see Last and Brandt (1991, pp. 342-346)), the martingales represent a minimal basis (if you eliminate any M_i, the set $\{M_1, \ldots, M_n\} \setminus \{M_i\}$ is not a basis) for the space of martingales so that the martingale dimension $K = n$. ■

C Competitive Insurance Market Equilibria and Insurance Coverage: Proofs of Theorems 1, 2, 4 and 5 and Proposition 1

We proceed as follows:

1. Determine the properties of a state-contingent equilibrium;
2. Relate state-contingent equilibria with the Radon-Nikodym derivative and Condition (ii) (Proposition 1);
3. Determine effective completeness (Theorem 4);
4. Establish the insurance equilibrium (Theorem 1).
5. Establish the uniqueness of full insurance demand (Theorems 2);
6. Relate Condition (ii) and the mutuality property of optimal consumption;
7. Relate the optimality of full insurance to Conditions (i)-(iii) (Theorem 5);

±∞¬The main properties of state-contingent equilibria for this economy are summarized in the following result (see Constantinides (1982)):

Proposition 3 A state-contingent equilibrium of $\mathcal{E} = ((e_i, U_i)_{i=1}^n)$ exists. Every state-contingent equilibrium of $\mathcal{E}$, $((x_i^*)_{i=1}^n, \tilde{\pi})$, is Pareto efficient and there exists a representative agent representation of prices with strictly concave vonNeumann-Morgenstern preferences

$$v_0(x(0)) + \beta_0 E_P[v_1(x(1))]$$

such that

$$\tilde{\pi} = \frac{\beta_0 v_1'(e(1))}{v_0'(e(0))} \tag{5}$$

where $v_t'(x)$ is the first derivative of the representative agent's utility function at date $t = 0, 1$. For all i there exists $f_i : R \to R$ such that $x_i^*(1) = f_i(e(1))$.

±∈↙Proof of Proposition 1: (in Section 4.1) Given a state-contingent equilibrium as described in Proposition 3 above, construct Q using Equation (4). Then, let $e^{-r} = E_P[v_1'(e(1))]$ and use it to define $r(t)$ using

$$S_0(t) = e^{-r(1-t)} \equiv \exp\left(-\int_t^1 r(s)\,ds\right).$$

Then, for any asset with dividends described by d_j define $S_j(t) = E_{Q_t}[d_j S_0(t)]$. ■

±∋↙Proof of Theorem 4: To establish that a dividend with the mutuality property can be attained using the strategy specified in the theorem, one first has to construct the functions $D^*(t)$ and $S^*(t)$. The following lemma will be crucial:

Lemma 2 Assumption 3 implies there exists a function $\mathcal{P}$: $[0, 1] \times \{1, \ldots, n\}^2 \to [0, 1]$ such that

$$P(N(1) = m \mid N(t) = n, t) = \mathcal{P}(t, n, m) \tag{6}$$

Proof: This follows immediately from the characterization of the distribution function of arrival times by their hazard rates. $\mathcal{P}$ is constructed recursively. Let $R_m(s, t)$ be the cumulative hazard rate from the hazard rate $(n - m)\eta(t, m)$, $R_m(s, t) = \int_s^t (n - m)\eta(r, m)\,dr$.

Then define the function H: $[0,1]^2 \times \{1, \ldots, n\}^2 \to [0,1]$ as follows:

$$H(s,t,k,m) = 0, \quad s \geq t$$

$$H(s,t,k,m) = 0, \quad k > m$$

$$H(s,t,m,m) = \exp(-R_m(s,t))$$

$$H(s,t,k,m+1) = (n-m)\int_s^t H(s,r,k,m)\eta(r,m)\exp(-R_{m+1}(r,t))dr,$$

$$m = k, \ldots, n-1$$

Then, define $\mathcal{P}$ using H. $\mathcal{P}(t,k,m) = H(t,1,k,m)$. ■

As d and d_M have the mutuality property, there exists f and g such that $d_M = g(e(1))$ and $d = f(e(1))$. Abusing notation slightly, and using the fact that for $m = 0, \ldots, n$, $e(1)$ and $\xi(1)$ are constant on $\{N(1,\omega) = m\}$, define $f(m) = f(e(1))$, $g(m) = g(e(1))$, and $\xi(1,m) = \xi(1)$ where $e(1)$ and $\xi(1)$ are evaluated on the set $\{N(1,\omega) = m\}$, $m = 0, \ldots, n$. Then, using the $\mathcal{P}(t,n,m)$ function constructed in the proof of Lemma 2 define the functions

$$D^*(t,k) = \sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(t,k,m) f(m)\, e^{-r}, \quad k = 0, \ldots, n;$$

$$S_M^*(t,k) = \sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(t,k,m) g(m)\, e^{-r}, \quad k = 0, \ldots, n;$$

$$\xi(t,k,r) = \sum_{m=k}^{n} \mathcal{P}(t,k,m)\xi(1,m), \quad k = 0, \ldots, n, \; r = k, \ldots, n,$$

Note that with these functions one can construct the processes $D^*(t, N(t))$ and $S_M^*(t, N(t))$ where $D^*(t, N(t)) = E^{Q_t}[e^{-r} d]$ and $S_M(t, N(t)) = e^{-rt} S_M^*(t)$.[14]

The strategy to attain d at date one (as in Duffie and Huang (1985)) is to solve the stochastic differential equation:

$$dD^*(t, N(t)) = \theta_M(t, N(t))\, dS_M^*(t, N(t)), \tag{7}$$

Using the above equations, the solution to this stochastic differential equation can be described using a deterministic function, $\theta_M(t,m)$: $[0,1] \times \{1, \ldots, n-1\} \to \mathbb{R}$ which is con-

[14] Note, that both processes are Q-martingales and, as they are functions of N (t), they can be shown to come from a one-dimensional space of martingales using the same arguments used in Appendix B.

structed as follows: as equation (7) has to hold and $\theta_M(t, N(t))$ has to be predictable (investment strategies cannot anticipate surprise changes in stock prices) the value of $\theta_M(t, N(t))$ will be determined by what happens at accident times. This is because, $\theta_M(t, N(t))$ solves

$$D^*(t, N(t-) + 1) - D^*(t, N(t-)) = \theta(t, N(t)) S_M(t, N(t-) + 1) - S_M(t, N(t-)$$

Define the function θ_M:

$$\theta_M(t, m) = \frac{D^*(t, m+1) - D^*(t, m)}{S_M(t, m+1) - S_M(t, m)} \quad m = 0, 1, \ldots, n-1$$

Then, to reconstruct x, define

$$\theta_0(t, m) = \frac{D^*(t, m) - \theta_M(t, m) S_M(t, m)}{e^{-r}} \quad m = 0, 1, \ldots, n-1$$

Then, following the strategy proposed in the Theorem using the two functions (θ_0, θ_M) and an initial amount of money $D^*(0)$, the value of the portfolio $\theta_0(t)e^{-r} + \theta_M(t)S_M(t)$ will be equal to $D^*(t)$ and hence equal to d at date one. ■

Proof of Theorem 1: We need to construct a triple $((x_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D})$ that satisfies the conditions in Definition 2.

(i) By Proposition 3, every state-contingent commodity equilibrium is efficient as has a representative agent representation, so that for every i, x_i^* has the mutualization property.

By Proposition 1, construct (Q, r) using π so that $\mathcal{D}$ satisfy Relation 1 and Q prices only aggregate risk.

(ii) Let $S_i = E_Q[N_i(1)e^{-r}]$.

(iii) For each i we first need to show $x_i^* \in B_i(D, S_i)$. Assume the agent buys full insurance. This changes i's allocation from the risky $(w_{i,0}, w_{i,1} - N_i(1)L)$ to the riskless $(w_{i,0} - S_i, w_{i,1})$.

Then, apply Theorem 4 using $d = x_i^*(1) - w_{i,1}$ (recall from Proposition 3 that $x_i^*(1)$ has the mutuality property). This makes i's allocation equal to $(w_{i,0} - S_i - D^*(0), w_{i,1} + d) = (w_{i,0} - S_i L - D^*(0), x_i^*(1))$.

We now use the (Q, r) as constructed above to show that the new allocation is equal to $(x_i^*(0), x_i^*(1))$: as x_i^* is in the state-contingent budget constraint then, using the definition of Q and r, $x_i^*(0) + E_Q[x_i^*(1)]e^{-r} = w_{i,0} + E_Q[w_{i,1} - N_i(1)L]e^{-r}$. From equilibrium pricing of insurance, $S_i = E^Q[N_i(1)]e^{-r}$. By construction, $D^*(0) = E^Q[de^{-r}] = E^Q[x^*_i(1) - w_{i,1}]e^{-r}$.

So that,

$$w_{i,0} - S_i L - D^*(0) = w_{i,0} - E^Q[N_i(1)]e^{-r} L - E^Q[x^*_i(1) - w_{i,1}]e^{-r}$$

$$= w_{i,0} + E^Q[w_{i,1} - N_i(1)L]e^{-r} - E^Q[x^*_i(1)]e^{-r}$$

$$= x^*_i(0)$$

which implies $x^*_i \in B_i(D, S_i)$.

Finally, we need to show that $\forall x_i \in B_i(D, S_i)$, $U_i(x_i) \leq U_i(x^*_i)$. As (Q, r) are derived from state-contingent prices, $\forall x_i \in B_i(D, S_i)$, $x_i(0) + E^Q[x_i(1)e^{-r}] \leq w_{i,0} + E^Q[(w_{i,1} - LN_i(1))e^{-r}]$ so that $B_i(D, S_i)$ is contained in the budget constraint in the state-contingent equilibrium. As x^*_i was optimal in the state-contingent equilibrium, it is also optimal restricted to the smaller budget constraint, $B_i(D, S_i)$. ■

Proof of Theorem 2: Let $((x^*_i)_{i=1}^n, (S_i)_{i=1}^n, \mathcal{D})$ be an efficient insurance market equilibrium. Consider the strategy used in the proof of Theorem 1 to show $x^*_i \in B_i(D, S_i)$. Suppose instead that agent i optimally chooses partial insurance coverage, i.e. $\alpha_i \in [0, 1)$ and there is strategy $(\theta_j(t))_{t\in[0,1]}$ such that $x^*_i \in B_i(D, S_i)$. Then, $(\theta_j(t))_{t\in[0,1]}$ achieves the date one net trade, denoted $X_i(1)$. $X_i(1) = x^*_i(1) - w_{i,1} + (1 - \alpha_i)L1_{N_i(1)=1}$, i.e. $\sum_{j=0}^{J} \theta_j(1) d_j(1) = X_i(1)$. However, note that (by Theorem 4) one can also construct a trading strategy $(\hat{\theta}_j(t))_{t\in[0,1]}$ such that $\sum_{j=0}^{J} \hat{\theta}_j(1) d_j(1) = x^*_i(1) - w_{i,1}$. Taking the difference between the two strategies $(\theta_j - \hat{\theta}_j)$ and dividing by $(1 - \alpha_i)L$ one obtains a new strategy whose dividend is equal to $N_i(1)$. But, as insurance is strictly non-tradable, such a trading strategy does not exist—a contradiction. ■

The next lemma shows that if Condition (ii) is satisfied, and markets are complete, optimal consumption satisfies the **mutuality property:**

Lemma 3 For agents whose preferences satisfy Assumption 1, if there exists (Q, r) such that $\mathcal{D}$ satisfy Relation 1 and the probability measure Q prices only aggregate risk then the optimal consumption in the following problem:

$$\text{[Problem B]} \qquad \max_x U_i(x) \text{ s.t. } \quad x(0) + E_Q[x(1)]e^{-r} = e_i(0) + E_Q[e_i(1)]e^{-r}$$

has the mutuality property.

Proof: Using $E_Q[z] = E_P[\xi(1)z]$, the Lagrangian for [Problem B] is

$$L = U_i(x) - \lambda\left(x(0) - e_i(0) + e^{-r} E_P[\xi(1)(x(1) - e_i(1))]\right)$$

The necessary and sufficient first order conditions are:

$$v_i(x(0)) = \lambda$$

$$\forall \omega \in \Omega \quad \beta u_i(x(1, \omega)) = \lambda \xi(e(1, \omega))e^{-r}$$

where λ is the Lagrange multiplier in the constrained maximization problem. The properties of u_i ensure that u_i^{-1} is a well-defined function so that the optimal consumption allocation, x^*, at date one is equal to $f_i(e(1))$ where:

$$f_i(e(1)) \equiv u_i^{-1}\left(\frac{v_i(x^*(0))\xi(e(1))e^{-r}}{\beta}\right)$$

and $x_i^*(0)$ is the constant that solves

$$x^*(0) + E_P[\xi(e(1))f_i(e(1))]e^{-r} = e_i(0) + e^{-r} E_P[\xi(1)e_i(1)]$$

The properties of the problem (preferences satisfying Assumption 1 plus the linearity of the constraint) imply that such a $x_i^*(0)$ exists. ■

Proof of Theorem 5: To prove this theorem one can follow a similar strategy as in the proof of Theorem 1: consider (Q, r) given by Condition (i) and $S_i = E_Q[N_i e^{-r}]$ from Condition (ii). Then, use Condition (iii) and Lemma 3 to determine that the the solution to [Problem B] is attainable using full insurance and asset trading. As the budget constraint in [Problem B] includes the budget constraint of an agent in economy $\mathcal{E}$ with prices (D, S_i), then full insurance could be agent i's optimal insurance demand.

Uniqueness of full insurance coverage as the optimal insurance strategy follows by the same arguments used in the Proof of Theorem 2. ■

D Proof of Theorem 3

The proof make use of the concept of exchangeability: for any arbitrary event C, let 1_C denote the indicator function of C. The function $\iota : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$ is a permutation function on $\{1, 2, \ldots, n\}$ if ι is bijective. The events $A_1, \ldots, A_n$ are exchangeable if for all permutations of the indexes, i.e., for all permutation functions on $\{1, 2, \ldots, n\}$,

$$P(A_1, A_2, \ldots, A_n) = P(A_{\iota(1)}, A_{\iota(2)}, \ldots, A_{\iota(n)})$$

Let B_i represent an event of the kind $\{N_i(1) = 0\}$ (agent i did not have an accident between dates zero and one) or $\{N_i(1) = 1\}$ (agent i did have an accident between dates zero and one).

Remark 5 If $\lambda_i(t)$ satisfies Assumption 3 then any set of events $\{B_1, B_2, \ldots, B_n\}$ is a set of exchangeable events.

That this is true can be seen from the way the function $\mathcal{P}$ was constructed in the proof of Lemma 2.

Proof of Theorem 3: Let A_i denote events of the kind $\{N_i(1) = 1\}$. Exchangeability implies that for $i, j, k \in \{1, 2, \ldots, n\}$, $P(A_i) = P(A_k) = p$ and

$$P(N = j, N_i = 1) = P(N = j, N_k = 1) = P(N = j) \frac{\binom{n-1}{j-1}}{\binom{n}{j}} = P(N = j) \frac{j}{n}$$

Let $q_i = E_Q[A_i]$, $\xi = dQ/dP$ and recall n is finite, then

$$q_i = \sum_{\omega \in A_i} P(\omega)\xi(\omega)$$

$$q_i = \sum_{j=0}^{n} P(N = j|A_i) P(A_i) \xi(N = j)$$

$$q_i = p E_P[\xi(N(1))|A_i]$$

and by exchangeability, $q_i = pE_P[\xi|A_k]$, $\forall k \in \{1, 2, \ldots, n\}$. Thus, for all $i = 1, \ldots, n$, $q_i = q$. Let $p_j \equiv P(N = j)$, then

$$P(N = j|A_i) - P(N = j) = \frac{p_j j}{pn} - p_j = (j - np)\frac{p_j}{pn}$$

Also, $p = \sum_{k=0}^{n} P(N = k|A_i)$ and $P(N = k|A_i) = p_k\, k/n$ so that

$$P(N = j|A_i) - P(N = j) = \frac{p_j k}{\sum_{k=0}^{n} j - np_k\, npn} - p_j = \frac{p_j}{np}(j - E[N])$$

As $E[(j - E[N])] = 0$ and N is increasing, then for all $k \leq n$

$$F_{N|A_i}(k) \equiv \sum_{j=0}^{k} P(N = j|A_i) \leq F_N(k) \equiv \sum_{j=0}^{k} P(N = j)$$

and the inequality is strict at least for $N = 0$. That is, $N(1)|A_i$ first-order stochastically dominates[15] $N(1)$. The economy has a representative agent representation with strictly increasing and concave utility (see Proposition 3) so that the equilibrium $\xi(N)$ will be strictly increasing. By definition $E[\xi] = 1$. Stochastic dominance of $F_{N|A_i}$, ξ increasing, and $E_P[\xi] = 1$ imply $E_P[\xi|A_i] > 1$ and $q > p$. Putting everything together:

$$S_i = E_Q[N_i(1)]e^{-r} = qe^{-r} = p(1 + \gamma)e^{-r}$$

with $\gamma > 0$.

■

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