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Abstract

Barberà and Coelho (2006) documented six screening rules associated with the rule of k names that are used by diferent institutions around the world. Here, we study whether these screening rules satisfy stability. A set is said to be a weak Condorcet set *à la* Gehrlein (1985) if no candidate in this set can be defeated by any candidate from outside the set on the basis of simple majority rule. We say that a screening rule is stable if it always selects a weak Condorcet set whenever such set exists. We show that all of the six procedures which are used in reality do violate stability if the voters act not strategically. We then show that there are screening rules which satisfy stability. Finally, we provide two results that can explain the widespread use of unstable screening rules.

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1 Introduction

The study of set-valued functions has a long tradition in economics, in general, and in social choice theory, in particular. The Walrasian correspondence is a salient example. More specific to social choice theory is the study of social choice correspondences and of set valued social choice functions.

The specific meaning attached to these rules can be very diverse. But there are two types of competing interpretations, depending on the nature of the objects to be chosen.

In a first interpretation, the chosen sets consist of elements that are not mutually exclusive. Here are some examples:

- The choice of new members for a club, or of several compatible projects, as in Barberà, Sonnenschein and Zhou (1991). In that case, the sets in the range can be of different cardinalities.

- The choice of locations for a fixed number of public facilities, as in Barberà and Beviá (2002).

- The choice of candidates to form a delegation, or to represent a district in a legislative body, as in Dodgson (1884, 1885a, 1885b).

In the last two cases, the cardinality of the sets to be chosen is exogenously given.

Turning now to a second interpretation, the elements chosen by a social choice rule need not be mutually exclusive objects. They may be sets of candidates for office, sets of alternative policies to solve the same social problem, etc... In this case, the set cannot be seen as a full solution of the social choice problem, a further resolution is necessary, and the underlying procedure to solve the remainder of the problem is a necessary reference to complete the interpretation. To give some examples:

- It is sometimes assumed that the final choice will be made through some random procedure. (See Barberà, Dutta and Sen, 2001 and references therein).

- It is sometimes assumed that a new decision process will take place to choose from the pre-selected alternatives. This covers a wide range of possibilities, and it includes the one that motivates our study here.

Under each of these interpretations, and many others, set valued social choices become objects of theoretical and practical interest. What questions to ask, and to eventually solve about them depend very much on our specific interpretations and of the kind of phenomena we want to focus on.

In this paper, we are mainly interested in the characteristics of (set valued) screening rules, which are part of the first stage of rules of k names (Barberà and Coelho, 2006). We refer by this to a widely used class of collective decision procedures, which work as follows. Given a set of candidates, a committee must pre-select a short list consisting of exactly k of them by voting. Then a single decision-maker from outside the committee, the chooser, will select one of the listed candidates. Hence, rules of k names are single valued, but their definition must include a description of the method that the committee will be using to prepare the short list. Our paper concentrates on the properties of these screening methods.

Specifically, we are concerned about the possibility that one candidate who is included in the short list might be defeated in a majority contest by one who is not in it. Should this happen, a majority of a committee members could agree to vote for the removal of the included alternative and to substitute it for the dominating one that is initially excluded. This might be a cause of instability in the proposal. Of course, our statement is conditional. The member of this majority could agree on that, but they don't need to. They might agree, or at least they would be better off with this substitution, under the assumption that the voters' preferences over sets satisfy the following monotonicity axiom: if a voter prefers candidate a to b and b is substituted by a in the elected set then this voter cannot be worse off.[1] Under the rule of k names, this would be a natural assumption in a scenario where no voter has any knowledge whatsoever of the chooser's preferences over the candidates. Thus, for them, each listed name would have the same probability of being the chooser's selected candidate for the office.

We want to distinguish between those screening rules that avoid this possible challenge, which we'll call stable screening rules, and those that may be exposed to it. Notice, however, that we can also easily imagine other scenarios under the rule of k names where the satisfaction of this monotonicity axiom would not necessarily work in favor of the

[1] This axiom was proposed by Kannai and Peleg (1984) and it is used very often in the literature on matching.

members of the committee. For instance, suppose an election for an office under the rule of two names and $\{a, b, c\}$ as being the set of candidates. Let a committee member rank candidate a first, b second and c third, and let the chooser rank b first, a second and c third. So, under the assumption of complete information , this committee member's preferred list would be $\{a, c\}$ instead of $\{a, b\}$, thus violating monotonicity.

Hence, our analysis is inspired by the study the rule of k names but we do not claim that the stability requirement is always equally compelling. But we still feel it worth studying when it can be satisfied. Moreover, the reader may find that some of our results can be used under other circumstances, to discuss issues relating to other types of choice procedures. For example, stability may be attractive for rules that choose sets of representatives. But, here again, it is in some cases and not in other cases. This was already argued by one of the founders of the Social Choice theory, the Rev Charles Lutwidge Dodgson (Lewis Carroll) who wrote in 1884 a pamphlet entitled "The principle of parliamentary representation".[2] In the supplement and postscript of this pamphlet, he rejects the criticism on his method for transferring the spare of district representative candidates that have more votes than they need to be returned. The argument against his method said that it fails to pass the test of always selecting a weak Condorcet set, i.e. a set such that no candidate in it can be defeated by any candidate from outside by majority rule.

Dodgson rejected this criticism by using two different arguments. The first says that this "test" would give too much power to the majority. In some circumstances, 49% of the electors would not return any candidates, and using his own words, 49% of the votes would be wasted. The second says that this test would be valueless by giving an example where there exists no weak Condorcet set with three candidates.

We have two comments about his arguments: first, the fact that sometimes there exists no weak Condorcet set cannot be the basis for a criticism to any particular rule, since this non-existence is prior to it. The most we can ask for a rule is to select such an alternative when it exists. Secondly, his argument on the majority voting power would not apply for the case of screening rules, since only one candidate from its elected outcome will be

[2] William Gehrlein informed us about the existense of this pamphlet. He got it from Bert Levin, an associate of Duncan Black, after he has published a paper about Condorcet winner sets. This pamphlet is reprinted in McLean and Urken (1995) and its supplement and postscript are in Black, McLean, McMillan and Monroe (1995).

chosen for office. This proves that although formally we move in the same framework, the interpretation given to set-valued rules is crucial in order to appreciate the validity of certain axioms or the criticisms to any given rule.

This paper proceeds as follows: In the next section, we formally define screening rules and the stability property. In Section 3, we show that all screening rules documented by Barberà and Coelho (2006) and which are used in reality violate stability. We also discuss the way to adapt several general well known voting rules in order to choose sets, and we study the extent to which the resulting screening rules may or not satisfy stability. Then, in Section 4, we exibit a difficulty that is common to all stable rules. We also present a strategic analysis, suggesting that stability may be recovered when agents act strategically and cooperatively. These two results may help explain the widespread use of rules that are not stable.

2 Notations and definitions

For $n \geq 2$, consider a polity $N = \{1, ..., n\}$, whose members confront a nonempty finite set of candidates A. Writing W for the set of all strict orders (transitive[3]*, asymmetric*[4]*, irreflexive*[5] *and complete*[6]*) on A, each member $i \in N$ has a strict preference $\succ_i \in W$, and we let $\succ_{\mathbf{N}} \equiv \{\succ_i\}_{i\in\mathbf{N}} \in W^{\mathbf{N}}$. Given $k \in \{1, 2, ..., \#A\}$, let $2^{\mathbf{A}}$ be the set of all non empty subsets of A and $A_k \equiv \{B \in 2^{\mathbf{A}} | \#B = k\}$ be the set of all possible subsets of A with cardinality equal to k.*

Our agents will be allowed to influence the choice of the sets by taking actions. *Let $M^N \equiv M \times ... \times M$ where M is the space of actions of a voter in N.* We leave the exact form of the elements of M undefined. But the reader will find it easy to identify the appropriate set of messages for each of the cases we study. For example, if the actions in M^N are casting single votes then $M \equiv A$. If the actions in M^N are submissions of strict preference relation then $M \equiv W$.

Definition 1 *Given $k \in \{1, 2, ..., \#\mathbf{A}\}$, a screening rule for k names is a function $S_k : M^N \longrightarrow \mathbf{A}_k$ associating to each action profile $m_N \equiv \{m_i\}_{i\in N} \in M^N$ the k-element set*

[3] Transitive: For all $x, y, z \in A : (x \succ y$ and $y \succ z)$ implies that $x \succ z$.

[4] Asymmetric: For all $x, y \in A : x \succ y$ implies that $\neg(y \succ x)$.

[5] Irreflexive: For all $x \in A, \neg(x \succ x)$.

[6] Complete: For all $x, y \in A : x \neq y$ implies that $(y \succ x$ or $x \succ y)$.

$S_k(m_N)$.

In words, a screening rule for k names is a voting procedure that selects k alternatives from a given set, on the basis of actions of the voters. These actions may consist of single votes, sequential votes, the submission of preference of rankings, the filling of ballots, etc...

We have defined screening rules as a function of general strategies, in order to allow for methods which go beyond the simple declaration of preferences. However, most of the paper refers to the relationship between the preferences of agents on alternatives and the specific sets that will be chosen. As Gibbard (1973) already pointed out, we can always define a clear cut concept of sincere, or straightforward behavior, by assigning a strategy to each possible preference, even if in principle the agents could fill their ballots in many ways. In what follows, we often limit our statements to the case where agents "act according to their true preferences", in an obvious sense. However, we prefer to keep the larger definition of screening rules, because this is useful for the strategic analysis in Section 4.

Definition 2 (Gehrlein, 1985) *Given* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A}\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$*, a set* $B \in \mathbf{A}_k$ *is a (weak) Condorcet set if for any* $a \in B$ *and* $b \in \mathbf{A} \backslash B$ *we have that* $\#\{i \in \mathbf{N} | a \succ_i b\} (\geq) > \#\{i \in \mathbf{N} | b \succ_i a\}$.

In other words, a set $B \in \mathbf{A}_k$ is a Condorcet set if each candidate in this set defeats any other candidate from outside the set on the basis of simple majority rule. And a set $B \in \mathbf{A}_k$ is a weak Condorcet set if no candidate that belongs to B can be defeated by any other candidate that belongs to $\mathbf{A} \backslash B$ on the basis of simple majority rule.

Remark 1 *Given* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A}\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$*, any Condorcet set, if it exists, is the unique weak Condorcet set. Moreover, when there is an odd number of voters any weak Condorcet set is also Condorcet set.*

Notation 1 *Given* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A}\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$*, denote by* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *the set of all weak Condorcet sets that belong to* $\mathbf{A}_k$.

Example 1 *Consider the following preference profile:*

No. of voters:	1	1
	a	a
	b	c
	c	b
	d	d

We can see that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{a\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, b\}, \{a, c\}\}$ *and* $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, b, c\}\}$ *and* $E(\mathbf{A}, \succ_{\mathbf{N}}, 4) = \{\{a, b, c, d\}\}$.

Remark 2 *Given any* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A}\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, *if* n *is odd then* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *is either empty or singleton.*

Remark 3 *Given any* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, *we have that* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *is never empty whenever the preference profile satisfies single peakedness.*

Ratliff (2003) proposes two procedures that always select Condorcet sets, when such a set exists: these are the Dodgson Method and the Kemeny Method (see definitions 8 and 9). By contrast, we investigate procedures that always select weak Condorcet sets.

Definition 3 *Given any* $\mathbf{A}$, $k \in \{1, 2, ..., \#\mathbf{A}\}$ *and* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, *we say that a screening rule* $S_k : M^{\mathbf{N}} \rightarrow \mathbf{A}_k$ *is stable if* $S_k(m_{\mathbf{N}}) \in E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *whenever* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *is not empty and* m_N *is a profile of sincere actions.*

For example, if S_k is plurality rule then a voter's sincere action is casting a vote for its preferred candidate. If S_k is Borda rule[7] then a voter's sincere action is declaring its true preferences over candidates.

In words, we say that screening rule for selecting k names is stable if it always selects a weak Condorcet set, whenever one exists and voters choose sincere actions.

[7] The Borda rule is defined as follows: Voters are required to rank the candidates, thus giving $\#A - 1$ points to the one ranked first, $\#A - 2$ to one ranked second, and so on. The Borda winner is the candidate with the highest total point score. The Borda loser is the candidate with the lowest total point score.

3 Almost all screening rules are unstable

We will show in this section that different standard voting rules which are actually used in reality do not satisfy stability.

Example 2 *In this example, we provide a single preference profile in which all screening rules documented in Barberà and Coelho (2006) fail simultaneously to select a weak Condorcet set. These screening rules, which are used in reality by different decision bodies around the world, can be described as follows:*
1) Screening 3 names by 3-votes plurality: Each proposer votes for three candidates and the list has the names of the three most voted candidates, with a tie-break when needed. It is used in the election of Irish Bishops and that of Prosecutor-General in most of Brazilian states.
2) Screening 3 names by 1-vote sequential plurality: The list is made with the names of the winning candidates in three successive rounds of plurality voting. It is used in the election of English Bishops.
3) Screening 3 names by 3-vote sequential strict plurality: this is a sequential rule adopted by the Brazilian Superior Court of Justice to select its members. Each proposer votes for three candidates from a set with six candidates, and if there are three candidates with more votes than half of the total number of voters, they will form the list. If there are positions left, the candidate with less votes is eliminated, so as to leave twice as many candidate as there are positions to be filled in the list. The process is repeated until three names are chosen. It may be that, at some stage (including the first one), all candidates have less than half of the total number of voters. Then the voters are asked to reconsider their vote and vote again. Notice that, if they persist in their initial vote, the rule leads to stalemate. Equivalently, we could say that the rule is not completely defined. However, in practice, agents tend to reassess their votes on the basis of strategic cooperative actions. It is used in the election of the members of the Brazilian Superior Court of Justice.
4) Screening 5 names by 3-votes plurality: Each proposer votes for three candidates and the list has the names of the five most voted candidates, with a tie break when needed. It is used in the election of the members of Superior Court of Justice in Chile.
5) Screening 3 names by 2-votes plurality: Each proposer votes for two candidates and the list has the names of the three most voted candidates, with a tie break when needed. It is

used in the election of the members of Court of Justice in Chile.
6) Screening 3 names by 1-vote plurality: Compute the plurality score of the candidates and include in the list the names of the three most voted candidates, with a tie break when needed. It is used in the election of rectors of public universities in Brazil.
Having defined these six rules, let us now propose a case where they all fail to work properly.
Consider the preference profile below with 11 voters and 9 candidates:

Name of the voter:	1	2	3	4	5	6	7	8	9	10	11
	i	d	g	f	a	b	g	g	h	b	i
	e	b	f	a	h	d	c	b	f	c	h
	c	e	i	i	g	g	i	d	e	e	b
	d	h	h	d	e	a	a	a	a	a	a
	a	c	a	c	d	f	f	h	g	f	f
	f	a	c	g	f	e	e	f	c	h	e
	g	f	d	e	c	h	h	e	d	g	d
	h	i	e	h	i	c	d	c	i	d	c
	b	g	b	b	b	i	b	i	b	i	g

Figure 1 below displays the binary relations induced by the preference profile where $\#\{i \in \mathbf{N} | x \succ_i y\} > \#\{i \in \mathbf{N} | y \succ_i x\}$ *if and only if there is a line from* x *to* y *induced by the preference profile above.*

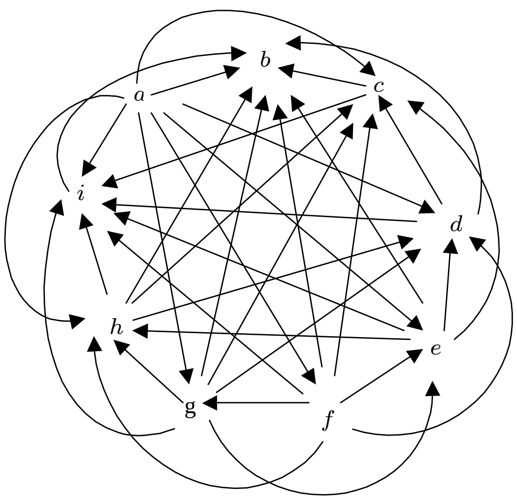

Figure 1

As it can be verified with the help of Figure 1 that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{\{a\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, f\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, f, g\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 4) = \{\{a, f, g, e\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 5) = \{\{a, f, g, e, h\}\}$.

Now let us check whether or not the screening rules listed above select weak Condorcet sets. Assume that the ties are broken according to the following order: $a \succ b \succ c \succ d \succ e \succ f \succ g \succ h \succ i$. *Notice that the first screening rule described above selects* $\{b, g, i\}$ *and the second one* $\{b, d, g\}$. *The table below presents what are the sets selected by the screening rules listed above:*

Screening Rules	*Screened Sets*
1	$\{b, g, i\}$
2	$\{b, d, g\}$
3	*Not defined*
4	$\{b, c, e, g, i\}$
5	$\{b, f, g\}$
6	$\{b, g, i\}$

Therefore all these six rules fail to satisfy stability.

The table below shows the Borda score of each candidate.

Candidates	*Borda score*
a	*56*
f	*49*
g	*47*
h	*45*
e	*44*
d	*43*
c	*39*
i	*37*
b	*36*

Notice that Candidate a *is the Condorcet and the Borda winner candidate, and yet does not belong to the outcomes of those screening rules. Moreover, five of the rules do select candidate* b*, who is the Condorcet and the Borda loser candidate.*

3.1 Scoring rules

Our next proposition states that any screening method based on scoring voting rules fails to satisfy stability.[8]

Definition 4 *A scoring voting rule is characterized by a nondecreasing sequence of real numbers $s_0 \leq s_1 \leq ... \leq s_{\#\mathbf{A}-1}$ with $s_0 < s_{\#\mathbf{A}-1}$. Voters are required to rank the candidates, thus giving $s_{\#\mathbf{A}-1}$ points to the one ranked first, $s_{\#\mathbf{A}-2}$ to one ranked second, and so on. The winner of the election is the candidate with the highest total point score (see Moulin, 1988).*

As first pointed out by Condorcet, there exist some preference profiles in which any scoring voting rule fails to select the Condorcet winner candidate (see Fishburn, 1974, page 544). Moreover, Theorem 1 in Saari (1989) states that the rankings given by the scoring rules over subsets of candidates need not be related to each other in any manner. Thus, our next proposition can be viewed as a natural consequence of this theorem, when applied to the choice of sets. Notice that the screening rules described in Example 2 are based on scoring voting rules.

Proposition 1 *For any $k \geq 1$, screening a list of k names by applying a scoring voting rule, either sequentially or one shot, does not satisfy stability provided that ties are broken according to a fixed ordering over $\mathbf{A}$.*[9]

Proof. For $k = 1$. Consider the following profile with 17 voters and 3 candidates.[10]

[8] Gehrlein (1985) provides estimations of the conditional probability of one-stage constant scoring rules selecting the Condorcet set given that such a set exists, in a context with m candidates and an infinitely large number of voters. One-stage constant scoring rules can be described as follows: Each voter is instructed to vote for q candidates and the k most voted candidates are selected.

[9] A sequential application of a voting rule can be described as follows: given a voting rule, write in the list the name of the winner candidate. A new election is held with the same voting rule on the set of the remaining candidates, then the process is continued until k names are chosen.

[10] This preference profile was used in Fishburn (1984) to prove that the scoring voting rules do not satisfy Condorcet consistency (see Moulin, 1988, page 232).

No. of voters:	3	6	4	4
	c	a	b	b
	a	b	a	c
	b	c	c	a

Here candidate a is the Condorcet winner. However for any scoring method candidate b will be elected. So the elected outcome will not be a weak Condorcet. Let us show why a cannot be elected.

score of $a = 6s_2 + 7s_1 + 4s_0$

score of $b = 8s_2 + 6s_1 + 3s_0$

(score of b) $-$ (score of a) $= (s_2 - s_1) + (s_2 - s_0) > 0$

The inequality above is strict because $(s_2 - s_1)$ is nonnegative and $(s_2 - s_0)$ is strict positive.

For $k = 2$, consider the following preference profile with 17 voters 4 candidates.

No. of voters:	3	6	4	3	1
	d	d	d	d	b
	c	a	b	b	d
	a	b	a	c	c
	b	c	c	a	a

Notice that the only weak Condorcet set is $\{d, a\}$. However for any sequential application of a scoring method the elected set is $\{d, b\}$. This set is not weak Condorcet set, since the majority of the voters prefers a to b. For the case of a simultaneous application of a scoring method the proof need to be a little bit more elaborated. Notice that

score of $a = 6s_2 + 7s_1 + 4s_0$.

score of $b = 1s_3 + 7s_2 + 6s_1 + 3s_0$.

score of $c = 3s_2 + 4s_1 + 10s_0$.

score of $d = 16s_3 + 1s_2$.

(score d+score b) $>$ (score x+score y) for every $x, y \in \{a, b, c, d\}$ such that $(x, y) \neq (b, d)$.

For $k \geq 3$, the proof is similar, we only need to add k-2 candidates at the top of this preference profile. ■

3.2 The Copeland rule

Now we will explore the consequence of trying to use a voting rule that always selects a Condorcet winner candidate whenever one exists.

Definition 5 *Compare candidate a with every other candidate x. Give a score +1 if a majority prefers a to x, -1 if a majority prefers x to a, and 0 if it is a tie. Adding up those scores over all $x \in \mathbf{A} \backslash \{a\}$ yields the Copeland score of a. The winner of the election, called a Copeland winner, is the candidate with the highest total point score (see Moulin 1988).*

Proposition 2 *For any $k \geq 1$, screening a list of k names by applying the Copeland rule, either sequentially or one shot, does not guarantee stability.*

Proof. In the preference profile below we have that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{\{a\}, \{c\}\}$ and $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, c\}\}$. However, applying the Copeland rule sequentially or taking the candidates with highest scores leads to $\{d\}$ when $k = 1$ and $\{d, c\}$ when $k = 2$.

No. of voters:	1	1	1	1
	a	e	d	b
	c	f	h	d
	b	g	g	a
	d	c	f	e
	h	h	c	c
	g	a	e	f
	f	b	a	g
	e	d	b	h

Copeland score

Candidates	1^{st} stage	2^{st} stage
a	1	1
b	-1	-2
c	2	2
d	3	-
e	-1	0
f	-1	0
g	-1	0
h	-2	-1

Therefore the proof is established for $k \in \{1, 2\}$. To prove the result for $k > 2$, we need just to add $k - 2$ candidates at the top of this preference profile. ■

Proposition 3 *For any $k \geq 1$, if there is a Condorcet set with cardinality k then it will be selected by applying the Copeland rule, either sequentially or in one shot.*

Proof. Take any $k \geq 1$, and suppose that B is a Condorcet set. We need to prove that if the Copeland rule is applied sequentially or in one shot then the set B will be the screened set. In order to prove it, it is enough to show that all the candidates in B have higher Copeland scores than any candidate in $\mathbf{A}\backslash B$. Notice that the fact that B is a Condorcet set implies that all candidates of B defeat by majority any candidate in $\mathbf{A}\backslash B$. Let a be the number of candidates in $\mathbf{A}$. Thus, the Copeland score of any candidate in B cannot be smaller than $a-k$. By this same reason, the Copeland score of any candidate in $\mathbf{A}\backslash B$ cannot be higher than $a-1-k$. Since $a-k > a-1-k$, the proof is established. ■

Corollary 1 *For any $k \geq 1$, if the number is odd then screening k names by applying the Copeland rule , either sequentially or in one shot, guarantees stability.*

3.3 The Simpson rule

Now let us check another method that always selects a weak Condorcet winner candidate whenever one exists.

Definition 6 *Compare candidate a with every other candidate x. Let $N(a, x)$ be the number of voters preferring a to x. The Simpson score of a is the minimum of $N(a, x)$ over all $x \in \mathbf{A}\backslash\{a\}$. The winner of the election, called a Simpson winner, is the candidate with the highest total point score (see Moulin, 1988).*

Proposition 4 *Screening a list of two names by selecting the two candidates with highest Simpson score (one shot method) does not guarantee stability. However, applying the Simpson rule sequentially does.*

Proof. First let us prove that making a list of two names by selecting the two candidates with highest Simpson score (one shot method) does not satisfy stability. We will prove it through the example below with 3 voters and 3 candidates.

No. of voters:	2	1
	a	c
	b	a
	c	b

Candidates	Simpson score
a	2
b	0
c	1

Thus the elected outcome is $\{a, c\}$. However $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, b\}\}$.

To prove that applying the Simpson rule sequentially satisfies stability for $k = 2$ just notice that $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{x, y\} \subseteq \mathbf{A} | x$ is a weak Condorcet winner over $\mathbf{A}$ and y is a weak Condorcet winner over $\mathbf{A} \backslash \{x\}\}$. In addition, the set of winning candidates under the Simpson rule is the set of all weak Condorcet winners whenever such candidates exist.

■

Proposition 5 *For any $k \geq 3$, screening k names by applying the Simpson rule, either sequentially or in one shot, does not satisfy stability.*

Proof. This proposition will be proved with an example with 9 voters and 4 candidates.

No. of voters:	3	2	1	3
	a	c	d	d
	b	a	c	b
	c	b	a	c
	d	d	b	a

Candidates	Simpson score
a	3
b	3
c	3
d	4

Notice that $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, b, c\}\}$. However it is easy to see that if we apply the Simpson rule, either sequentially or one shot, the elected set must contain d. To prove this for $k > 3$, we just need to substitute, in the preference profile above, the top cycle of size 3 for another top cycle with size k such that candidate d still is the Simpson winner and does not belong to this top cycle. This completes the proof. ■

3.4 The Dodgson rule

Now let us turn our attention to a method that was proposed specifically to select a Condorcet set provided that one exists.

Definition 7 *The Dodgson method for selecting a set with cardinality k: Compute for each set $B \in \mathbf{A}_k$ the minimum number of adjacency switches on the voters' preferences required for B to become the Condorcet set. The winner is the set with k candidates that requires the fewest adjacency switches (see Ratliff, 2003).*

The proposition below shows that stability is stronger than the requirement of choosing the Condorcet set whenever such a set exists.

Proposition 6 *For any $k \geq 2$, the Dodgson method for selecting a set with cardinality k does not satisfy stability.*

Proof. In the preference profile below we have that $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, c\}\}$. Now let us apply Dodgson method to select a set with cardinality two. Notice that the set $\{d, c\}$ is the Dodgson winner since it requires only four adjacency switches on the voters' preferences. While the weak Condorcet set $\{a, c\}$ requires five switches.

No. of voters:	1	1	1	1
	a	c	d	b
	c	g	g	d
	b	f	f	a
	d	e	e	c
	e	a	c	e
	f	b	a	g
	g	d	b	f

Therefore the proof is established. To prove for $k > 2$, we just need to add $k-2$ candidates at the top of this preference profile. ■

3.5 The Kemeny rule

The following method was proposed by Ratliff (2003). It is a generalization of the procedure proposed by John Kemeny in 1959.

Definition 8 (Ratliff 2003) *The total margin of loss of a set $B \in 2^{\mathbf{A}}$ to the candidates in $\mathbf{A} \backslash B$ induced by a profile of preferences $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$ is denoted by $KE(\mathbf{A}, \succ_{\mathbf{N}}, B)$ and defined over $\boldsymbol{A}$ as follows:*

$$KE(\mathbf{A}, \succ_{\mathbf{N}}, B) = \sum_{y \in \mathbf{A} \backslash S \ and \ x \in S} Max\{0, \#\{i \in \mathbf{N} | y \succ_i x\} - \#\{i \in \mathbf{N} | x \succ_i y\}\}$$

The Kemeny Method (KE_k): compute the KE score for all subsets of candidates with cardinality k. The elected set is the one with the lowest KE score.

The Kemeny method was specifically proposed to select Condorcet sets. The proposition below shows that it also selects weak Condorcet sets whenever such set exists.

Proposition 7 *For any $k \geq 1$, the Kemeny method for selecting a set with cardinality k satisfies stability.*

Proof. Take any $k \geq 1$, suppose that B is a weak Condorcet set with cardinality k. By definition of weak Condorcet set and KE score, we have that $KE(\mathbf{A}, \succ_{\mathbf{N}}, B) = 0$. Thus, it is enough to prove that for any $X \in \mathbf{A}_k$ that is not a weak Condorcet set with cardinality k we have that $KE(\mathbf{A}, \succ_{\mathbf{N}}, X) > 0$. Since X is not a weak Condorcet set then for any $x \in X$ there exists $y \in A \backslash X$ such that $\#\{i \in \mathbf{N} | y \succ_i x\} - \#\{i \in \mathbf{N} | x \succ_i y\} > 0$. It implies that $KE(\mathbf{A}, \succ_{\mathbf{N}}, X) > 0$. Therefore the proof is established. ■

Many other stable screening rules can be conceived. Below we give three examples. We leave to the reader to check that they satisfy stability.

a) Compute for every subset with cardinality k, the total number of pairwise majority defeats of the candidates in the set against the candidates outside the set. The elected set is the one with smallest total number of pairwise majority defeats;

b) Compute for every subset with cardinality k, the highest margin of loss of a candidate in the set against a candidate outside the set. The elected set is the one with smallest margin of loss.

This method can be viewed as an adaptation of the Simpson rule for selecting sets with fixed size.

c) Compute for every subset with cardinality k, the minimum number of adjacency switches on the voters' preferences required for the set to become the weak Condorcet set. The winner is the set with k candidates that requires the fewest adjacency switches.

Notice that this method is an adaptation of the Dodgson method (see Definition 7).

4 Why are unstable screening rules so popular?

We have shown in the previous sections that unstable screening rules are often used. In fact, we do not have any example of a stable screening rules that is actually used by some decision body. Yet, we also have shown that there exist stable and reasonable screening rules. In this section we provide two results that can be viewed as hints to solve this apparent puzzle.

4.1 An impossibility result

The following proposition shows that stability is incompatible with another desirable property that one might expect from screening rules.

Definition 9 *A familiy of screening rules is a function $S : k \longrightarrow S_k$ associating to each $k \in \{1, 2, ..., \#\mathbf{A}\}$ a screening rule for k names S_k.*

Definition 10 *Axiom I: Any listed name should not be excluded if the list is enlarged. In other words, if a candidate is included in the chosen set with k names then he should be also in the chosen set with $k + 1$ names.*

Proposition 8 *There exist no family of stable screening rules satisfying Axiom I .*

Proof. The proof of this proposition is very simple. Let us prove it by contradiction. Suppose that there exist a family of stable screening rules for k satisfying Axiom I . Consider the following preference profile:

No. of voters:	1	1	1	1	1	1
	a	a	a	c	e	d
	b	b	b	d	c	e
	c	e	d	e	d	c
	d	c	e	a	a	a
	e	d	c	b	b	b

Notice that $\{a, b\}$ and $\{c, e, d\}$ are the unique weak Condorcet sets for k=2 and k= 3 respectively. Hence, since the screening rules of this family are stable then we have that for $k = 2$, the selected set has to be $\{a, b\}$ and for $k = 3$, the selected set has to be $\{c, e, d\}$. Therefore, Axiom I is violated since $\{a, b\}$ is not contained in $\{c, e, d\}$. ■

Remark 4 *It turns out that all screening rules based on the sequential application of any voting rules satisfy Axiom I.*

Remark 5 *Notice that in the domain of single peaked preferences, with an odd number of voters, any family of stable screening rules satisfies Axiom I. This last result follows by remarks 1 and 2.*

4.2 A strategic analysis: the Random Chooser Game

We now study the case where the voters act strategically and cooperatively. More specifically, we propose a voting game where the players choose by voting a subset of candidates with a fixed size from a given set of candidates. We call this game by Random Chooser Game because it is inspired in a scenario under the rule of k names where the committee members who are supposed to choose the list with k names by voting, do not have any knowledge whatsoever of the chooser's preferences over the candidates. Thus, each committee member would choose their voting strategies assuming that each listed name would have the same probability of being the chooser's selected candidate for the office. After describing the game, we will show that for some type of unstable screening rules which are used in reality, the chosen set with k names in any pure strong Nash equilibrium of this game is a weak Condorcet set.

Definition 11 *Given $k \in \{1, 2, ..., \#A\}$, a screening rule for k names $S_k : M^{\mathbf{N}} \longrightarrow A_k$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, the Random Chooser Game can be described as follows: it is a simultaneous game with complete information where each voter $i \in \mathbf{N}$ chooses a message $m_i \in M$. Given $m_N \equiv \{m_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$, $S_k(m) \in \mathbf{A}_k$ is the screened set. Each voter $i \in N$ has a payoff function $u_i : M^{\mathbf{N}} \rightarrow \mathbf{R}$ that satisfies the following axioms: (Axiom 1) For any $m_{\mathbf{N}}, m'_{\mathbf{N}} \in M^{\mathbf{N}}$ we have that $u_i(m_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ only if $S_k(m_{\mathbf{N}}) \neq S_k(m'_{\mathbf{N}})$, and (Axiom 2) for any $m_{\mathbf{N}}, m'_{\mathbf{N}} \in M^{\mathbf{N}}$ and any $y, x \in A$ we have that $u_i(m_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ if $x \succ_i y$, $y \in S_k(m'_{\mathbf{N}})$ and $S_k(m_{\mathbf{N}}) = \{x\} \cup (S_k(m'_{\mathbf{N}}) \backslash \{y\})$.*[11]

Let us introduce the solution concept that we will use to analyze this game.

Definition 12 *Given $k \in \{1, 2, ..., \#A\}$, a screening rule for k names $S_k : M^{\mathbf{N}} \longrightarrow A_k$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, a joint strategy $m_{\mathbf{N}} = \{m_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ is a pure strong Nash equilibrium of the Random Chooser Game if and only if, given any coalition $C \subset N$, there exists no $m'_{\mathbf{N}} \equiv \{m'_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ with $m'_j = m_j$ for every $j \in N \backslash C$ such that $u_i(m'_{\mathbf{N}}) > u_i(m_{\mathbf{N}})$ for each $i \in C$.*

The first three screening rules described in Example 2 and documented in Barberà and Coelho (2006) are majoritarian and the others are not.

[11] Axiom 2 is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used among others by Roth and Sotomayor(1990) and Kaymak and Sanver (2003). We have refered to it informally in the introduction.

Definition 13 *We say that a screening rule* $S_k : M^{\mathbf{N}} \longrightarrow \mathbf{A}_k$ *is majoritarian if and only if for every set* $B \in A_k$ *there exists* $m \in M$ *such that for every coalition* $C \subseteq \mathbf{N}$ *with* $\#\mathbf{C} > \mathbf{n/2}$*, and every profile of the complementary coalition* $m_{\mathbf{N}\setminus C} \in M^{\mathbf{N}\setminus C}$ *we have that* $S_k(m_{\mathbf{N}\setminus C}, m_C) = B$ *provided that* $m_i = m$ *for every* $i \in C$.

Proposition 9 *Let* $S_k : M^{\mathbf{N}} \longrightarrow A_k$ *be a majoritarian screening rule. If a set is a pure strong Nash equilibrium outcome of the Random Chooser Game then it is a weak Condorcet set.*

Proof. Suppose that a subset $B \subset \mathbf{A}$ with cardinality k is an outcome of a strong equilibrium of the Random Chooser Game. Thus there exists a strong Nash equilibrium strategy profile $m'_{\mathbf{N}} \equiv \{m'_i\}_{i\in\mathbf{N}} \in M^{\mathbf{N}}$ such that $S_k(m'_{\mathbf{N}}) = B$. Suppose by contradiction that B is not a weak Condorcet set. Then there exists $x \in B$ and $y \in A\setminus B$ such that a strict majority of the voters prefers y to x. Let $D \equiv \{y\} \cup B\setminus\{x\}$ and $C \equiv \{i \in \mathbf{N} | y \succ_i x\}$. Since the screening rule is majoritarian and $\#C > \frac{n}{2}$, there exists $m''_{\mathbf{N}} \equiv \{m''_i\}_{i\in\mathbf{N}} \in M^{\mathbf{N}}$ with $m''_j = m'_j$ for every $j \in N\setminus C$ such that $S_k(m''_{\mathbf{N}}) = D$. By Axiom 2, we have that $u_i(m''_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ for every $i \in C$. This is a contradiction since $m'_{\mathbf{N}}$ is a strong Nash equilibrium. Therefore any Strong Nash equilibrium outcome need to be a weak Condorcet set. ■

This result implies that any majoritarian screening rule tends to be stable if the voters act strategically and cooperatively, provided that the monotonicity axiom holds. Notice also that to be a weak Condorcet set is a necessary but not sufficient condition to be pure strong Nash equilibrium of this game. A sufficient condition would require that a set being a Condorcet set *à la* Fishburn, i.e. a set that cannot be defeated by any other set with the same cardinality on the basis of majority rule.[12] As Kaymak and Sanver (2003) already pointed out, a set being a weak Condorcet set *à la* Gerhlein does not guarantee that it is a weak Condorcet set *à la* Fishburn (1981).[13]

In the example below, we provide a preference profile over candidates in which there is a unique Condorcet set with cardinality two. However, for a given players' payoff function that satisfy both axioms 1 and 2 and a majoritarian screening rule, the set of Strong Nash equilibrium outcome of the Random Chooser Game is empty.

[12] Notice that Fishburn's definition is based on preferences over sets.

[13] See Kaymak and Sanver (2003) studies the connections between this two alternative definitions of weak Condorcet sets.

Example 3 *Consider the following preference profile over candidates:*

$a \succ_1 b \succ_1 c \succ_1 d$

$c \succ_2 b \succ_2 a \succ_2 d$

$d \succ_3 a \succ_3 b \succ_3 c$

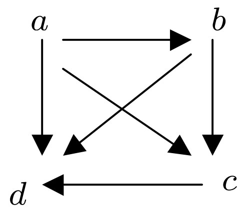

Figure 2

Notice that the any candidate of the set $\{a, b\}$ defeats any other candidate of $\mathbf{A} \backslash \{a, b\}$ on the basis of simple majority rule. Hence $\{a, b\}$ is a Condorcet set.
Consider now the following preference profile over sets of two candidates which is a lexicographic extension of the above preference profile over candidates.

$\{a, b\} \succ_1 \{a, c\} \succ_1 \{a, d\} \succ_1 \{b, c\} \succ_1 \{b, d\} \succ_1 \{c, d\}$

$\{b, c\} \succ_2 \{a, c\} \succ_2 \{c, d\} \succ_2 \{a, b\} \succ_2 \{b, d\} \succ_2 \{a, d\}$

$\{a, d\} \succ_3 \{b, d\} \succ_3 \{c, d\} \succ_3 \{a, b\} \succ_3 \{a, c\} \succ_3 \{b, c\}$

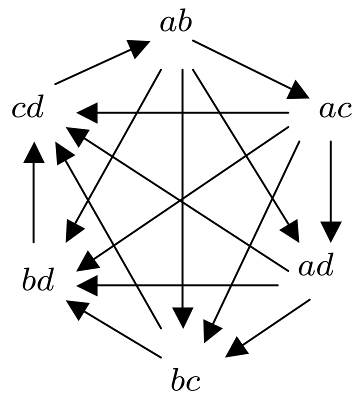

Figure 3

Notice that the players' payoff functions derived from this preference profile over sets satisfy axioms 1 and 2. As we can see in figure above, there exists a strict majority of voters that prefers $\{c, d\}$ to the weak Condorcet set $\{a, b\}$. Hence, $\{a, b\}$ cannot be a strong Nash equilibrium outcome of the Random Chooser Game whenever the screening rule is majoritarian. Therefore, by Proposition 10, the set of strong Nash equilibrium outcomes of the Random Chooser Game is empty. It can be easily check in the preference profile over sets, with the help of Figure 3, that there is no Condorcet set à la Fishburn.

5 Concluding remarks

We have shown that all of the six screening rules documented in Barberà and Coelho (2006) violate stability if the voters do not act strategically. In our search for stable procedures, we have proved that any procedure based on scoring rules or resulting from a sequential use of standard Condorcet consistent methods such as those of Simpson, Copeland and Dodgson rules, also violates this property. We also give there examples of stable screening rules. One example is the Kemeny method proposed by Ratliff (2003).

We provide two results that can explain the widespread use of unstable screening rules. The first one states that there exists no family of stable screening rules satisfying the following natural requirement that any listed name should not be excluded if the list is enlarged. Or, in other words, that if a candidate is included in the chosen list of k names, then he should be also in a larger list. Therefore, leaving stability aside can be seen as a price to pay for a rule to keep an alternative important or desirable property.

The second justification comes from the remark that any majoritarian procedure tends to select weak Condorcet sets if the agents act strategically and cooperatively. More specifically, we propose a voting game where under any majoritarian procedure, a set is a strong Nash equilibrium outcome only if it is a weak Condorcet winner set. Half of the six screening rules documented in Barberà and Coelho (2006) turns out to be majoritarian and would thus generate attractive choices under this form of strategic behavior.

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