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A Comment on Jackson and Watts JET (2002)[*]

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Abstract

We extend Jackson and Watts's (2002) result on the coincidence of S-stochastically stable and core stable networks from marriage problems to roommate problems. In particular, we show that the existence of a side-optimal core stable network, on which the proof of Jackson and Watts (2002, Theorem 2) hinges, is not crucial for their result.

JEL classification: C62, C78.

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1 Introduction

Jackson and Watts (2002), from now on JW, model network formation as a dynamic process in which individuals form and sever links based on the improvement that the resulting network offers them relative to the current network. JW's contribution is twofold. First, they introduce and characterize a dynamic solution concept for networks – stochastic stability. Second, for marriage problems (Gale and Shapley, 1962) they show that the set of S-stochastically stable networks coincides with the set of core stable networks (JW, Theorem 2).

In their seminal paper, Gale and Shapley (1962) also generalize (two-sided) marriage problems to (one-sided) roommate problems.[1]

Two features of the core (the set of core stable networks in JW's terminology) do not extend from marriage to roommate problems. First, not every roommate problem is solvable, *i.e.*, a

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[1]There is a large literature on the marriage problem; see, for instance, Roth and Sotomayor (1990) and the two-sided matching bibliography on Al Roth's game theory, experimental economics, and market design page. In comparison relatively few papers and books deal with roommate problems; some of the key references concerning roommate problems are Chung (2000); Diamantoudi et al. (2004); Gusfield and Irving (1989); Tan (1991).

core stable network does not always exist. Second, while in marriage problems side-optimal core stable networks always exist and play a crucial role (*e.g.*, in JW's proof of Theorem 2), no similarly prominent core stable networks exist for solvable roommate problems.

Our contribution has two dimensions. First, we extend JW's Theorem 2 to a larger class of network problems (from marriage to roommate problems).[2] Second, we provide a short proof that does not use the existence of side-optimal core stable networks, on which the proof of JW hinges. In particular, JW's result is not a consequence of the "polarization structure" of the set of core stable networks for marriage problems (which after all is absent for some roommate problems).

2   Roommate Networks

We refer to JW (pp. 270-281) for the notation and definitions related to networks.

In a roommate problem, a set of players N has to be partitioned in pairs (roommates) and singletons. Thus, a network is a roommate network if each player $i \in N$ is linked to at most one other player. Hence, the set of roommate networks is $G = \{g \mid ij \in g \text{ implies } ik \notin g \text{ for } k \neq j\}$. For roommate networks the value function is defined by $v(g) := \sum_{i \in N} u_i(g)$ and the allocation rule by $Y_i(v, g) := u_i(g)$, where for each $i \in N$, $u_i : G \to \mathbb{R}$ only depends on the player to which i is linked (possibly himself). From now on we assume that networks are roommate networks. Core stability, simultaneous improving paths, and S-stochastic stability are defined as in JW (pp. 282-285). A roommate problem is solvable if the set of core stable networks is non empty.

Theorem 1. *Consider any solvable roommate problem with strict preferences. Then, the set of S-stochastically stable networks coincides with the set of core stable networks.*

Proof: Let g' be a core stable network. Given a network g, let $n(g, g')$ denote the number of links and single players that are common to g and g'. Interpreting $n(\cdot, \cdot)$ as distance measure we say g_1 is closer to g' than g_2 if $n(g_1, g') > n(g_2, g')$.

Diamantoudi et al. (2004, Claim, p. 21) show the following: For any $\tilde{g}$ that is not core stable, there exists a finite simultaneous improving path $\tilde{g} = \tilde{g}_0, \ldots, \tilde{g}_L$ such that $n(\tilde{g}_L, g') \geq n(\tilde{g}, g') + 1$. Repeated application of Diamantoudi et al.'s (2004) Claim leads to the following result.

Lemma 1. *For any non core stable network $\tilde{g}$ there exists a finite simultaneous improving path $\tilde{g} = \tilde{g}_0, \ldots, \tilde{g}_L = g''$ of distinct networks such that g'' is core stable and $n(g'', g') \geq n(\tilde{g}, g') + 1$.*

Claim 1. *There exist no closed cycles.*

This follows from Lemma 1.

Claim 2. *If there is a singleton core stable network, then it is the unique S-stochastically stable network.*

This follows from Claim 1.

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[2]This answers in the affirmative a research question posed by Diamantoudi et al. (2004). They showed that Roth and Vande Vate's (1990) convergence result for marriage problems also holds for solvable roommate problems. Given that JW used Roth and Vande Vate's (1990) convergence result to prove their Theorem 2, Diamantoudi et al. (2004) asked whether the extended convergence result could be used to extend JW's Theorem 2 as well.

Claim 3. *If there are $k > 1$ core stable networks, then the resistance of any core stable network is $k - 1$.*

Let $\{g_1, g_2, \ldots, g_k\}$ denote the set of core stable networks. We prove Claim 3 by constructing a g_1-tree with resistance $k - 1$.

First, we show how to connect the core stable networks $g_2, \ldots, g_k$ with one mutation each to construct a subtree of the final g_1-tree. We order core stable networks according to their distance to g_1: without loss of generality, $n(g_1, g_1) > n(g_2, g_1) \geq \ldots \geq n(g_k, g_1)$.

Let $\tilde{g}_2$ be an adjacent network of g_2 such that $n(\tilde{g}_2, g_1) \geq n(g_2, g_1)$.[3] Since g_2 is core stable and $\tilde{g}_2$ is adjacent, $r(g_2, \tilde{g}_2) = 1$ and $\tilde{g}_2$ is non core stable. By Lemma 1, there exists a finite simultaneous improving path $\hat{p} = \{\tilde{g}_2, \ldots, \hat{g}\}$ of distinct networks such that $\hat{g}$ is core stable and $n(\hat{g}, g_1) \geq n(\tilde{g}_2, g_1) + 1$. Recall that no core stable network is strictly closer to g_1 than g_2. Thus, $\hat{g} = g_1$. By definition of $\hat{p}$ the directed path

$$g_2, \underbrace{\tilde{g}_2, \ldots, g_1}_{\text{path } \hat{p}}$$

is a g_1-subtree of resistance 1 that contains g_1 and g_2 together with the non core stable networks on path $\hat{p}$.

Let $l \in \{3, \ldots, k\}$ and assume that we have constructed a g_1-subtree of resistance $l - 2$ that contains $g_1, g_2, \ldots, g_{l-1}$ together with some of the non core stable networks. Let $\tilde{g}_l$ be an adjacent network of g_l such that $n(\tilde{g}_l, g_1) \geq n(g_l, g_1)$. Since g_l is core stable and $\tilde{g}_l$ is adjacent, $r(g_l, \tilde{g}_l) = 1$ and $\tilde{g}_l$ is non core stable. By Lemma 1, there exists a finite simultaneous improving path $\bar{p} = \{\tilde{g}_l, \ldots, \bar{g}\}$ of distinct networks such that $\bar{g}$ is core stable and $n(\bar{g}, g_1) \geq n(\tilde{g}_l, g_1) + 1$. Recall that no core stable network in $\{g_{l+1}, \ldots, g_k\}$ is strictly closer to g_1 than g_l. Thus, $\bar{g} = g_i \in \{g_1, \ldots, g_{l-1}\}$.

Case 1: Except for $\bar{g}$, none of the networks on path $\bar{p}$ is connected to the existing g_1-subtree. We add the directed path

$$g_l, \underbrace{\tilde{g}_l, \ldots, g_i}_{\text{path } \bar{p}}$$

to the existing g_1-subtree. By definition of $\bar{p}$ the resulting graph is a g_1-subtree of resistance $l - 1$ that contains $g_1, g_2, \ldots, g_l$ together with some non core stable networks.

Case 2: Some network on path $\bar{p}$ different from $\bar{g}$ is connected to the existing g_1-subtree. Let g^* be the first network on path $\bar{p}$ that is connected to the existing g_1-subtree. We now only add the directed path $g_l, \tilde{g}_l, \ldots, g^*$ to the existing g_1-subtree. Since $\bar{p}$ is a path of distinct networks we obtain a g_1-subtree of resistance $l - 1$ that contains $g_1, g_2, \ldots, g_l$ together with some non core stable networks.

After $k - 1$ steps we have constructed a g_1-subtree of resistance $k - 1$ that contains all core stable networks together with some non core stable networks.

Finally, by Lemma 1, we can connect any remaining (non core stable) network that is not connected to the g_1-subtree through a zero resistance path (as in Case 2, paths have to be adjusted if they contain already connected networks).

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[3]Such a $\tilde{g}_2$ can be obtained by breaking a link in g_2 that does not exist in g_1 or adding a link in g_2 that exists in g_1 but not in g_2.

Hence, we have constructed a g_1-tree with resistance $k-1$. As at least one mutation is needed to connect two core stable networks, g_1 has resistance $k-1$.

Claims 1, 2, and 3 establish the theorem. $\qquad\Box$

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