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Abstract

This paper deals with the strong constrained egalitarian solution introduced by Dutta and Ray (1991). We show that this solution yields the weak constrained egalitarian allocations (Dutta and Ray, 1989) associated to a finite family of convex games. This relationship makes it possible to define a systematic way of computing the strong constrained egalitarian allocations for any arbitrary game, using the well-known Dutta-Ray's algorithm for convex games. We also characterize non-emptiness and show that the set of strong constrained egalitarian allocations Lorenz dominates every other point in the equal division core (Selten, 1972).

Keywords: Cooperative TU-game, strong constrained egalitarian solution, weak constrained egalitarian solution, equal division core, Lorenz domination.

JEL classification: C71, C78

1 Introduction

The concept of egalitarianism for cooperative TU-games was first introduced by Dutta and Ray (1989) with the aim of combining social values and selfish behavior. Their weak egalitarian solution (from now on the *Dutta-Ray* solution) is defined in a recursive way and it yields, whenever it exists, the unique Lorenz maximal imputation in what they call the *weak Lorenz core*. This is a sharp result because the Lorenz relation generates a partial ordering on the set of allocations. Nevertheless, the Dutta-Ray solution lacks general existence properties. In fact, the class of convex games is the only standard class of TU-games for which an existence result is known. In order to widen the potential class of applications, Dutta and Ray (1991) introduced the *strong constrained egalitarian solution* (SCES for short), a parallel concept that exists for weakly superadditive games. Unfortunately, the SCES loses the uniqueness property, but always produces a finite (but maybe empty) set of outcomes as we prove in this paper. Related studies are Arin and Iñarra (2001), Hougaard et al. (2001), and Arin et al. (2003), which introduced other egalitarian solutions based on the notion of the core.

The SCES selects the Lorenz maximal imputations in the equal division core, a set solution concept introduced by Selten (1972) to explain outcomes of experimental cooperative games, and recently axiomatized by Bhattacharya (2004). This solution, then, combines the natural behavior of the players in order to form claims with a normative rule of egalitarianism. It is also more attractive from the social point of view than its weak counterpart. Indeed, if the Dutta-Ray allocation is not a strong constrained egalitarian allocation, then it is Lorenz dominates by one of them and, in convex games, all the strong constrained egalitarian allocations Lorenz dominate it (Dutta and Ray, 1991). Hence, it seems sensible to focus on this set solution concept. Although the Dutta-Ray solution has been widely studied and axiomatized in the domain of convex games (see, for instance, Dutta, 1990, Klijn et al., 1998, Hokari, 2000 and Arin et al., 2003), a similar analysis for the SCES has not been made. Perhaps one of the reasons is that no algorithm is known to check this solution even for convex games. Moreover, as Dutta and Ray (1991) point out, "the process of constructing strong constrained egalitarian allocations is considerably more complicated since the choice of individuals at each stage cannot be made arbitrarily". Thus, the aim of this paper is provide a methodology for computing all the strong constrained egalitarian allocations for an arbitrary cooperative TU-game.

2 Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. By $N \subseteq \mathbb{N}$ we denote a finite set of players, in general $N = \{1, \ldots, n\}$. A *transferable utility coalitional game (a game)* is a pair (N, v) where $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$ and 2^N denotes the set of all subsets (coalitions) of N. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. Given a finite set $N \subset \mathbb{N}$, Γ^N denotes the set of all games defined on N.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$. We say that $x > y$ if $x \geq y$ and for some $j \in N$, $x_j > y_j$.

The set of *feasible payoff vectors* of a game (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A *solution* on a set of games $\Gamma' \subseteq \Gamma^N$, is a mapping σ which associates with each game $(N, v) \in \Gamma'$ a subset $\sigma(N, v)$ of $X^*(N, v)$. Notice that $\sigma(N, v)$ is allowed to be empty. The *pre-imputation set* of a game (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of *imputations* by $I(N, v) := \{x \in X(N, v) \,|\, x(i) \geq v(i), \text{ for all } i \in N\}$. The core of (N, v) is the set of those imputations in which each coalition gets at least its worth, that is $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game (N, v) is *convex* (Shapley, 1971) if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

For any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-decreasing order, that is, $\hat{x}_1 \leq \hat{x}_2 \leq \ldots \leq \hat{x}_n$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that y *Lorenz dominates* x, denoted by $y \succ_L x$, if $\sum_{j=1}^k \hat{y}_j \geq \sum_{j=1}^k \hat{x}_j$, for all $k \in \{1, \ldots, n\}$, with at least one strict inequality. Given a set $A \subseteq \mathbb{R}^N$, EA denotes the set of allocations that are Lorenz undominated within A. That is, $EA := \{x \in A \mid \text{there is no } y \in A \text{ such that } y \succ_L x\}$.

Given a game (N, v), we say that a payoff vector is in the *strong Lorenz* core, denoted by $L^*(N, v)$, if it is both efficient and not blocked by the equal division allocation for any subcoalition. Formally,

$$L^*(N, v) = \left\{ x \in I(N, v) \mid \text{for all } \emptyset \neq S \subset N, \text{ there is } i \in S \text{ with } x_i \geq \frac{v(S)}{|S|} \right\}.$$

Here the strong Lorenz core coincides with the equal division core when the coalition structure is N and there are no restrictions on coalition formation (see Selten, 1972 for details). The *strong constrained egalitarian solution* of a game (N, v), denoted by $EL^*(N, v)$, selects the vectors that are

Lorenz-undominated within the strong Lorenz core (Dutta and Ray, 1991). On the domain of convex games, the *Dutta-Ray solution* of a game (N, v), denoted by $DR(N, v)$, selects the vectors that are Lorenz-undominated within the core. That is, $DR(N, v) = EC(N, v)$.[1] On this domain, Dutta and Ray (1989) show that it picks the payoff vector that is obtained by the following algorithm.

Let (N, v) be a convex game. **Step 1:** Define $v_1 = v$. Then find the unique coalition $S_1 \subseteq N$ such that for all $S \subseteq N$, (i) $\frac{v_1(S_1)}{|S_1|} \geq \frac{v_1(S)}{|S|}$, and (ii) if $\frac{v_1(S_1)}{|S_1|} = \frac{v_1(S)}{|S|}$ and $S \neq S_1$, then $|S_1| > |S|$. Uniqueness of such a coalition is guaranteed by convexity of (N, v). Then, for all $i \in S_1$, $DR_i(N, v) = \frac{v_1(S_1)}{|S_1|}$. **Step k:** Suppose that $S_1, \ldots, S_{k-1}$ have been defined. Let $N_k = N \setminus (S_1 \cup \ldots \cup S_{k-1})$. Let (N_k, v_k) be the game defined as follows: $v_k(S) := v(S_1 \cup \ldots \cup S_{k-1} \cup S) - v(S_1 \cup \ldots \cup S_{k-1})$, for all $S \subseteq N_k$. It can be shown that (N_k, v_k) is convex. Then find the unique coalition $S_k \subseteq N_k$ such that for all $S \subseteq N_k$, (i) $\frac{v_k(S_k)}{|S_k|} \geq \frac{v_k(S)}{|S|}$, and (ii) if $\frac{v_k(S_k)}{|S_k|} = \frac{v_k(S)}{|S|}$ and $S \neq S_k$, then $|S_k| > |S|$. The uniqueness of such a coalition is guaranteed by the convexity of (N_k, v_k). Then, for all $i \in S_k$, $DR_i(N, v) = \frac{v_k(S_k)}{|S_k|} = \frac{v(S_1 \cup \ldots \cup S_k) - v(S_1 \cup \ldots \cup S_{k-1})}{|S_k|}$.

3 Finding the strong constrained egalitarian allocations

In this section, we provide a systematic procedure for computing strong constrained egalitarian allocations. The key factor is the connection between the SCES and the Dutta-Ray solution. More precisely, we state that the set of strong constrained egalitarian allocations is formed by Dutta-Ray allocations associated to a finite family of convex games. From the computational point of view, this is a nice result which enables the Dutta-Ray algorithm to be used for convex games so that the outcomes of the SCES can be determined for any arbitrary game. In addition, we characterize non-emptiness, and show that the set of strong constrained egalitarian allocations Lorenz dominates the strong Lorenz core.

Before stating our main result, we need to introduce a set of vectors, named *sequential share worth vectors*, one for each ordering of the players. An ordering $\theta = (i_1, \ldots, i_n)$ of N, where $|N| = n$, is a bijection from $\{1, \ldots, n\}$ to N. We denote by Θ_N the set of all orderings of N.

Definition 1 *Let (N, v) be a game and $\theta = (i_1, \ldots, i_n) \in \Theta_N$. The sequential share worth vector*

[1] The original definition of the Dutta-Ray solution, given on a class of games that is larger than the convex class, is different from the one we use here. But both are equivalent on the domain of convex games, as shown by Dutta-Ray (1989).

associated to θ, denoted by $x^\theta(v) \in \mathbb{R}^N$, is defined as follows:

$$x^\theta_{i_k}(v) := \max_{S \in P_{i_k}} \left\{ \frac{v(S)}{|S|} \right\}, \text{ for } k = 1, \ldots, n, \tag{1}$$

where $P_{i_1} := \{S \subseteq N \mid i_1 \in S\}$ and $P_{i_k} := \{S \subseteq N \mid i_1, \ldots, i_{k-1} \notin S,\ i_k \in S\}$, for $k = 2, \ldots, n$.

Given a game (N, v) and an ordering $\theta \in \Theta$, if there is no confusion we write x^θ instead of $x^\theta(v)$. Having determined an ordering $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$, the first player i_1 receives in x^θ the maximal average worth he or she can obtain by choosing coalitions $S \subseteq N$ containing i_1. Now player i_1 leaves the game and the second player i_2 receives the maximal average worth among coalitions $S \subseteq N \setminus \{i_1\}$ containing i_2. Following this procedure, the last player i_n gets his individual worth. Notice that, in general, x^θ cannot be an efficient vector. By $\mathcal{X}^*(v)$ we denote the set of sequential share worth vectors of the game (N, v) that are feasible. That is, $\mathcal{X}^*(v) := \{x^\theta \mid x^\theta(N) \leq v(N)\}$. We say that $x \in \mathcal{X}^*(v)$ is *minimal* if there is no $y \in \mathcal{X}^*(v)$ such that $y < x$. $\mathcal{M}(v)$ denotes the *set of feasible minimal sequential share worth vectors* of the game (N, v).

Remark 1 *In order to find the minimal sequential share worth vectors, it is easy to check that we can restrict attention to orderings such that agents occupying the first place belong to coalitions maximizing average worth. That is, given a game (N, v), any $x \in \mathcal{M}(v)$ can be associated to an ordering $\theta = (i_1, \ldots, i_n) \in \Theta_N$ such that $x = x^\theta$ and $i_1 \in S_1$, where $S_1 \in \arg\max_{\emptyset \neq S \subseteq N} \frac{v(S)}{|S|}$.*

Let (N, v) be a game and $x \in \mathcal{M}(v)$. We define the *almost modular game* associated to x, denoted by (N, v_x), as follows:

$$v_x(S) := \begin{cases} x(S) & \text{if } S \subset N, \\ v(N) & \text{if } S = N. \end{cases} \tag{2}$$

The convexity of (N, v_x) follows straightforwardly from the fact that $x(N) \leq v(N)$.

The next definition introduces the *candidates* to be strong constrained egalitarian payoff vectors.

Definition 2 *Let (N, v) be a game, $x \in \mathcal{M}(v)$, and $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$ such that $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_n}$. Given $k \in \{1, \ldots, n-1\}$, $n \geq 2$, we define the vectors y^k and y_x as follow:*

1. *For $j = 1, \ldots, n$,*

$$y^k_{i_j} := \begin{cases} x_{i_j} & \text{if } j \leq k, \\ \frac{v(N) - x_{i_1} - \ldots - x_{i_k}}{n-k} & \text{otherwise.} \end{cases}$$

2. *For* $k^* = \min\{k \in \{1,\dots,n-1\} \mid y^k \geq x\}$,

$$y_x := y^{k^*}. \tag{3}$$

Notice that for $k = n-1$, $y^{n-1} \geq x$. Therefore, y_x is well-defined. Given $x \in \mathcal{M}(v)$, the associated payoff vector y_x is constructed by using the following procedure: if $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_n}$, then the first player i_1 receives x_{i_1}. The second player i_2 receives the maximum between x_{i_2} and $\frac{v(N)-x_{i_1}}{|N|-1}$. If $\frac{v(N)-x_{i_1}}{|N|-1} \geq x_{i_2}$, then the remaining players also receive $\frac{v(N)-x_{i_1}}{|N|-1}$. If not, the third player i_3 receives the maximum between x_{i_3} and $\frac{v(N)-x_{i_1}-x_{i_2}}{|N|-2}$. As before, if $\frac{v(N)-x_{i_1}-x_{i_2}}{|N|-2} \geq x_{i_3}$, the remaining players also get $\frac{v(N)-x_{i_1}-x_{i_2}}{|N|-2}$, and so on.

The next theorem states that the SCES picks the Lorenz undominated vectors defined in (3).

Theorem 1 *Let (N,v) be an arbitrary game. Then, the set of strong constrained egalitarian allocations is formed by the Lorenz undominated Dutta-Ray allocations associated to the family of convex games $\{(N,v_x\}_{x\in\mathcal{M}(v)}$. Formally,*

$$EL^*(N,v) = E\left\{DR(N,v_x) \mid x \in \mathcal{M}(v)\right\} = E\{y_x \mid x \in \mathcal{M}(v)\}. \tag{4}$$

Proof: Let (N,v) be a game, $x \in \mathcal{M}(v)$, and (N,v_x) be the almost modular game associated to x as defined in (2). We must first prove that

$$L^*(N,v) \subseteq \bigcup_{x\in\mathcal{M}(v)} C(N,v_x). \tag{5}$$

Let $y \in L^*(N,v)$ and $S_1 \in \arg\max_{\emptyset\neq S\subseteq N} \frac{v(S)}{|S|}$ an arbitrary coalition maximizing average worth. Since $y \in L^*(N,v)$, there exists some player $i_1 \in S_1$ such that $y_{i_1} \geq \frac{v(S_1)}{|S_1|}$. Now choose $S_2 \in \arg\max_{\emptyset\neq S\subseteq N\setminus\{i_1\}} \frac{v(S)}{|S|}$. As before, there exists some player $i_2 \in S_2$ such that $y_{i_2} \geq \frac{v(S_2)}{|S_2|}$. Following this process step by step we can construct an order $\theta = (i_1, i_2, \dots, i_n) \in \Theta_N$. Let $x^\theta \in \mathbb{R}^N$ be the associated sequential share worth vector: $x^\theta_{i_j} = \frac{v(S_j)}{|S_j|}$, $j = 1, 2, \dots, n$. By definition we have $x^\theta_i \leq y_i$ for all $i \in N$, and so $x^\theta(N) \leq y(N) = v(N)$. Hence, x^θ is a feasible payoff vector. If $x^\theta \notin \mathcal{M}(v)$, then there is $x' \in \mathcal{M}(v)$ such that $x'_i \leq x^\theta_i \leq y_i$, for all $i \in N$, with at least one strict inequality. Thus, in all cases, there exists $x \in \mathcal{M}(v)$ with $x \leq y$. Now consider the associated convex game (N,v_x). Clearly, $y \in C(N,v_x)$, which proves (5).

Next, we show $EL^*(N,v) \subseteq E\left\{DR(N,v_x) \mid x \in \mathcal{M}(v)\right\}$. Let $y \in EL^*(N,v)$. By definition $y \in L^*(N,v)$ and so, from (5), $y \in C(N,v_x)$ for some $x \in \mathcal{M}(v)$. By convexity, $DR(N,v_x) \in C(N,v_x)$ and it Lorenz dominates every other point in the core of the game (N,v_x) (Dutta and Ray, 1989). Hence, $DR(N,v_x) \succ_L y$, whenever $y \neq DR(N,v_x)$.

Let us see that, for all $x \in \mathcal{M}(v)$,

$$DR(N, v_x) \in L^*(N, v). \tag{6}$$

Indeed, let $x \in \mathcal{M}(v)$ and (N, v_x) be the associated almost modular game. By efficiency, $\sum_{i \in N} DR_i(N, v_x) = v_x(N) = v(N)$. Let $R \subset N$ be a non-empty coalition, $\theta = (i_1, \ldots, i_n) \in \Theta_N$ an ordering associated to $x \in \mathcal{M}(v)$, and $i_k \in R$ the first player in R w.r.t. θ. Since $DR(N, v_x) \in C(N, v_x)$, $DR_{i_k}(N, v_x) \geq v_x(\{i_k\}) = x_{i_k} = \max_{S \in P_{i_k}} \left\{ \frac{v(S)}{|S|} \right\} \geq \frac{v(R)}{|R|}$. These inequalities together with efficiency imply $DR(N, v_x) \in L^*(N, v)$. Since $y \in EL^*(N, v)$ and $DR(N, v_x) \in L^*(N, v)$, we get $DR(N, v_x) = y$. Finally, from (6) we conclude that $EL^*(N, v) \subseteq E\{DR(N, v_x) \mid x \in \mathcal{M}(v)\}$.

To show the reverse inclusion, $E\{DR(N, v_x) \mid x \in \mathcal{M}(v)\} \subseteq EL^*(N, v)$, suppose there is $y \in E\{DR(N, v_x) \mid x \in \mathcal{M}(v)\}$, with $y \notin EL^*(N, v)$. In this situation, $y = DR(N, v_x)$ for some $x \in \mathcal{M}(v)$, and from (6), $y \in L^*(N, v)$. Since $y \notin EL^*(N, v)$, $z \in L^*(N, v)$ must exist with $z \succ_L y$. From (5), there is $x \in \mathcal{M}(v)$ such that $z \in C(N, v_x)$. Thus, either $z = DR(N, v_x)$ or $DR(N, v_x) \succ_L z$. In all cases, since the Lorenz relation is transitive, we obtain $DR(N, v_x) \succ_L y$, which contradicts the fact that $y \in E\{DR(N, v_x) \mid x \in \mathcal{M}(v)\}$. Hence, $E\{DR(N, v_x) \mid x \in \mathcal{M}(v)\} \subseteq EL^*(N, v)$.

To end the proof, we only need to show that $DR(N, v_x) = y_x$, for all $x \in \mathcal{M}(v)$. Let $x \in \mathcal{M}(v)$ and let us assume, w.l.g., $x_1 \geq x_2 \geq \ldots \geq x_n$. Now consider the associated almost modular game (N, v_x) defined by (2). Let $\mathcal{P} = \{S_1, S_2, \ldots, S_m\}$ be the partition of N obtained by means of the Dutta-Ray algorithm to compute their egalitarian allocation for convex games. If $m = 1$, then N is an equity coalition for the game (N, v_x). In this case, $DR_i(N, v_x) = \frac{v_x(N)}{|N|} = \frac{v(N)}{|N|}$ for all $i \in N$. By definition of the sequential share worth vector we obtain $x_1 = \frac{v(N)}{|N|}$, and $DR(N, v_x) = y^1$ (w.r.t. x). If $m \geq 2$, it can be seen that $S_1 = \{i \in N \mid x_i \geq x_k \; \forall \, k \in N\}$ and, for $h = 2, \ldots, m-1$, $S_h = \{i \in N \setminus S_1 \cup \ldots \cup S_{h-1} \mid x_i \geq x_k \; \forall \, k \in N \setminus S_1 \cup \ldots \cup S_{h-1}\}$. That is, the set S_1 is formed by those players with the maximum payoff at x. Then, removing players of S_1, S_2 is formed in a similar way, and so on and so forth until the last but one element of the partition, S_{m-1}. Moreover, $DR_i(N, v_x) = x_i$ for all $i \in S_h$ and all $h = 1, \ldots, m-1$, and $DR_i(N, v_x) = \frac{v(N) - \sum_{i \in N \setminus S_m} x_i}{|S_m|}$ for all $i \in S_m$. Hence, $DR(N, v_x) = y^k$ (w.r.t. x), where $k = |S_1 \cup \ldots S_{m-1}|$. Now suppose that k is not minimal and denote by $k^* = \min\{r \in \{1, \ldots, n-1\} \mid y^r \geq x\}$. Then, $y^{k^*} = \left(x_1, \ldots, x_{k^*}, \frac{v(N) - x_1 - \cdots - x_{k^*}}{n - k^*}, \ldots, \frac{v(N) - x_1 - \cdots - x_{k^*}}{n - k^*}\right)$. By the minimality of k^*, $x_{k^*} > \frac{v(N) - x_1 - \cdots - x_{k^*-1}}{n - (k^* - 1)}$, or equivalently, $x_{k^*} > \frac{v(N) - x_1 - \cdots - x_{k^*}}{n - k^*}$. Thus, $x_1 \geq \ldots \geq x_{k^*} > \frac{v(N) - x_1 - \cdots - x_{k^*}}{n - k^*}$. Hence, for all $i \in \{1, \ldots, k^*, \ldots, k\}$, $DR_i(N, v_x) \leq y_i^{k^*}$. Moreover, for all $i > k$, since $i \in S_m$ and $k \in S_{m-1}$, we have that $DR_i(N, v_x) < DR_k(N, v_x) = x_k \leq y_k^{k^*} = y_i^{k^*}$. But then, $y^{k^*} \succ_L DR(N, v_x)$, which is a contradiction, since $y^{k^*} \in C(N, v_x)$. $\square$

The above result tells us how to compute the strong constrained egalitarian allocations. With the following two examples we illustrate this procedure.

Example 1 *(Dutta and Ray, 1991) Let* (N, v) *be a 3-person game with:* $v(\{1\}) = v(\{2\}) = 1$, $v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,2,3\}) = 2.2$, $v(\{1,3\}) = v(\{2,3\}) = 1.4$.
The set of minimal sequential share worth vectors is $\mathcal{M}(v) = \{x_1, x_2\}$, *where* $x_1 = (1.1, 1, 0)$ *and* $x_2 = (1, 1.1, 0)$. *From Definition 2 we select the candidates to be strong constrained egalitarian allocations:* $y_{x_1} = (1.1, 1, 0.1)$ *and* $y_{x_2} = (1, 1.1, 0.1)$. *Finally, since* y_{x_1} *and* y_{x_2} *are Lorenz undominated, we have* $EL^*(N, v) = \{y_{x_1},\ y_{x_2}\}$.

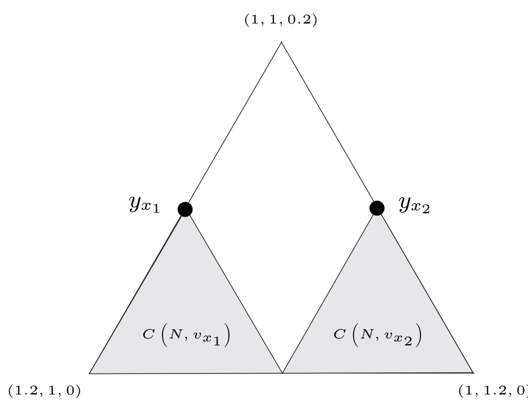

Figure 1: This figure corresponds to Example 1. The two shadowed zones in the triangle form the strong Lorenz core and $EL^*(N, v) = \{y_{x_1}, y_{x_2}\}$.

Example 2 *Let* (N, v) *be a 3-person game with:* $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = 100$, $v(\{2,3\}) = 0$, $v(\{1,2,3\}) = 125$.
The set of minimal sequential share worth vectors is $\mathcal{M}(v) = \{x_1,\ x_2\}$ *where* $x_1 = (50, 0, 0)$ *and* $x_2 = (0, 50, 50)$. *Therefore,* $y_{x_1} = (50, 37.5, 37.5)$ *and* $y_{x_2} = (25, 50, 50)$. *Since* $y_{x_1} \succ_L y_{x_2}$ *we have* $EL^*(N, v) = \{y_{x_1}\}$.

Some interesting and direct consequences can be derived from Theorem 1.

Corollary 1 *Let* (N, v) *be an arbitrary game. Then, the set of strong constrained egalitarian allocations is a finite (but maybe empty) set.*

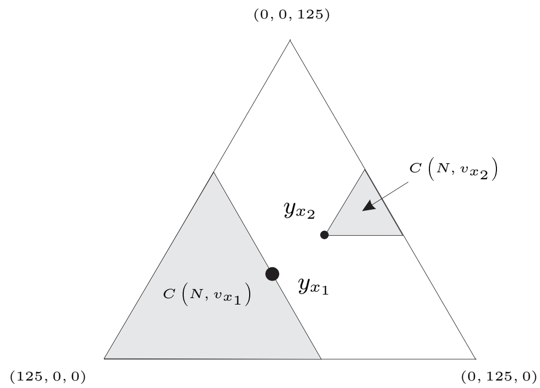

Figure 2: This figure corresponds to Example 2. The union of the two shadowed triangles corresponds to the strong Lorenz core and $EL^*(N,v) = \{y_{x_1}\}$.

Here it is worth to point out that the set of Lorenz maximal elements in a compact set is not, in general, finite (see, for instance, Example 4 in Dutta and Ray, 1989).

Notice that the reasoning used to prove expression (6) can be applied to any arbitrary core element to get the inclusion $\bigcup_{x\in\mathcal{M}(v)} C(N,v_x) \subseteq L^*(N,v)$. This relation, together with (5), implies that the strong Lorenz core of any game (N,v) can be expressed as

$$L^*(N,v) = \bigcup_{x\in\mathcal{M}(v)} C(N,v_x). \tag{7}$$

From this decomposition, and taking into account that the strong Lorenz core is a compact set, the next characterization follows.

Corollary 2 *Let (N,v) be a game. Then, $EL^*(N,v) \neq \emptyset$ if and only if there exists a feasible sequential share worth vector i.e. there exists $\theta \in \Theta$ such that $x^\theta(N) \leq v(N)$.*

Recall that Dutta and Ray (1991) stated a sufficient, but no necessary, condition of existence for this solution.

Finally, we show that the set of strong constrained egalitarian allocations Lorenz dominates every other allocation in the strong Lorenz core. On the domain of convex games, Dutta and Ray (1989) state an equivalent result for their weak egalitarian solution but replace the strong Lorenz core by the core.

Corollary 3 *Let (N,v) be a game and $x \in L^*(N,v) \setminus EL^*(N,v)$. Then, there exists $y \in EL^*(N,v)$ such that $y \succ_L x$.*

PROOF: Let (N,v) be a game and $x \in L^*(N,v) \setminus EL^*(N,v)$. Then, $z \in L^*(N,v)$ must exist with $z \succ_L x$. From (5), $z \in C(N, v_{x'})$ for some $x' \in \mathcal{M}(v)$. Then, either $z = DR(N, v_{x'})$ or $DR(N, v_{x'}) \succ_L z$. In all cases, $DR(N, v_{x'}) \succ_L x$. If $DR(N, v_{x'}) \notin EL^*(N,v)$, then by Theorem 1 there is $DR(N, v_{x''}) \in EL^*(N,v)$, where $x'' \in \mathcal{M}(v)$, and such that $DR(N, v_{x''}) \succ_L DR(N, v_{x'}) \succ_L x$, which gives the result. $\square$

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