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Abstract

We introduce a subclass of multi-sided assignment games that embodies markets with different types of firms that produce different types of homogeneous goods. These markets generalize bilateral Böhm-Bawerk horse markets. We describe the geometric and algebraic structure of the core, which is always nonempty. We also characterize the extreme points of the core and show that they are marginal worth vectors, although this does not hold for arbitrary multi-sided assignment games. Finally we show that prices that are obtained from core allocations are competitive and vice versa.

Keywords: Assignment games, multi-sided markets, homogeneous goods, core

JEL Classification: C71, C78

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1 Introduction

Consider a market in which there are buyers and two types of firms producing respectively two types of indivisible and perfectly complementary goods. Let $S = \{S_1, ..., S_{n_s}\}$ and $H = \{H_1, ..., H_{n_h}\}$ be the two sets of goods. For instance, S could be a set of software products and H a set of hardware products. Suppose there are n_s firms each one of them producing exactly one different unit of S at unitary costs $c_1^S, ..., c_{n_s}^S$ respectively and n_h firms each one of them producing exactly one different unit of H at unitary costs $c_1^H, ..., c_{n_h}^H$ respectively. Each buyer is only interested on buying at most a unit of each type of product, but has no utility on buying separately either a unit of S or a unit of H. If $B = \{B_1, ..., B_{n_b}\}$ is the set of n_b buyers we denote by w_{ij}^k the willingness-to-pay of buyer B_k for the pair (S_i, H_j).

In this market, a transaction can only be carried out when a buyer acquires exactly a unit of S and a unit of H. Let p_i and q_j be the prices the buyer B_k pays for S_i and H_j, respectively. At such prices, her utility is given by $w_{ij}^k - p_i - q_j$, whereas the benefit of the firm producing S_i is $p_i - c_i^S$ and the benefit of the firm producing H_j is $q_j - c_j^H$. If we assume that the utility of the buyer is monetary and transferable the total surplus generated by a transaction is $\left(w_{ij}^k - p_i - q_j\right) + \left(p_i - c_i^S\right) + \left(q_j - c_j^H\right) = w_{ij}^k - c_i^S - c_j^H$. Obviously, if $w_{ij}^k - c_i^S - c_j^H < 0$ no transaction will be carried out because there are no prices that simultaneously give non-negative utility to the buyer and the two producers. Then, let $a_{ijk} = \max\left\{0, w_{ij}^k - c_i^S - c_j^H\right\}$ be the total gain generated when S_i, H_j and B_k are 'assigned', i.e: when buyer B_k buys S_i and H_j. The above market is completely determined by giving the sets of buyers and firms and parameters a_{ijk}. Obviously, the model can be extended to an arbitrary m-sided market, with $m - 1$ different types of goods.

A particular case of the above market is obtained when the willingness-to-pay of any buyer is the same, irrespective which firms she buys products S_i and H_j from. That is, units of S and units of H are respectively *homogeneous* from any buyer's point of view. In such case we can denote $w^k_{ij} = w^k$, and thus the profit generated by buyer B_k and firms S_i and H_j is given by $a_{ijk} = \max\left\{0, w^k - c^S_i - c^H_j\right\}$. This specific market is the natural generalization of the Böhm-Bawerk horse market (Böhm-Bawerk, 1923) to a three-sided market.

The above m-sided market -either with homogeneous goods or not- can be studied in the framework of multi-sided assignment games. Multi-sided assignment games were introduced by Quint (1991) as the generalization of two-sided assignment games (Shapley and Shubik, 1972), and have been studied also in Lucas (1995), Stuart (1997) and Sherstyuk (1998, 1999).

The present paper is devoted to study the subclass of multi–sided assignment games that embodies multi-sided markets with homogeneous goods. Such multi-sided assignment games will be called *truncated additive*. The rest of the paper is organized as follows. In Section 2 we present the necessary notation and background results. In Section 3 we define the truncated additive multi-sided assignment games and describe their core. In Section 4 we characterize the extreme points of the core of such multi-sided assignment games and prove that every extreme core point is a marginal worth vector. In Section 5, we characterize the dimension of the core of any truncated additive multi-sided assignment game. Finally, in Section 6 competitive prices in multi-sided assignment markets with homogeneous goods are analyzed in terms of the results obtained in previous sections.

2 Preliminaries and notation

Consider a m-sided market in which there are m different types (or sectors) of agents. For any type j, $1 \leq j \leq m$, let $\{j\} \times N^j$ be the finite set of agents of type j, where $N^j = \{1, 2, ..., n_j\}$. That is, the i-th agent of type j is the pair $(j, i) \in \{j\} \times N^j$. However, by convention (see Quint, 1991) we will refer to agent (j, i) as j-i. On the other hand, since for every j, $1 \leq j \leq m$ there is a bijection between N^j and $\{j\} \times N^j$, when no confusion can arise we will use the former set instead of the latter. In particular, we call any m-tuple of agents $E = (i_1, ..., i_m) \in N^1 \times ... \times N^m$ an *essential* coalition and we say that agent j-i *belongs* to E

if $E_j = i$.

In a *m-sided assignment problem* (shortly m-SAP) exactly one agent of each type is required to turn a transaction into a value. Thus, for any $(i_1, ..., i_m) \in N^1 \times ... \times N^m$ there is a parameter $a_{i_1 \dots i_m}$ that stands for the worth of the transaction when agents j-i_j, $1 \leq j \leq m$ meet. Hence, a m-sided assignment problem is completely defined by giving the sets N^j, $1 \leq j \leq m$, and the function

$$\begin{aligned} A : N^1 \times ... \times N^m &\longrightarrow \mathbb{R}_+ \\ (i_1, ..., i_m) &\longrightarrow A(i_1, ..., i_m) = a_{i_1 \dots i_m}, \end{aligned}$$

where $\mathbb{R}_+ = [0, +\infty)$. The reader will notice that the function A represents a m-dimensional matrix. A m-sided assignment problem is denoted by $(N^1, ..., N^m; A)$ and, if $n_1 = n_2 = ... = n_m$, we say that the m-SAP is square.

A *matching* $\mu = \{E^1, ..., E^t\}$ among $N^1, ..., N^m$ is a set of essential coalitions such that $|\mu| = t = \min_{1 \leq j \leq m} |N^j|$ and any agent j-i belongs at most to one essential coalition $E^1, ..., E^t$. We say that agent j-i is *unassigned* by μ if she does not belong to E^k for any $1 \leq k \leq t$. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$.

Given a m-SAP $(N^1, ..., N^m; A)$, a matching μ^* is optimal if

$$\sum_{(i_1, ..., i_m) \in \mu^*} A(i_1, ..., i_m) \geq \sum_{(i_1, ..., i_m) \in \mu} A(i_1, ..., i_m),$$

for any $\mu \in \mathcal{M}(N^1, ..., N^m)$. We denote by $\mathcal{M}^*_A(N^1, ..., N^m)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$. Since $n_1, ..., n_m$ are finite, at least one optimal matching always exists and thus $\mathcal{M}^*_A(N^1, ..., N^m)$ is always nonempty.

Following Shapley and Shubik (1972) and Quint (1991), for each multi-sided assignment problem $(N^1, ..., N^m; A)$ the *m-sided assignment game* (m-SAG) is the cooperative game (N, ω_A) with set of players $N = \{j\text{-}i : 1 \leq j \leq m, 1 \leq i \leq n_j\}$ and characteristic function ω_A defined as the smallest superadditive function on 2^N satisfying $\omega_A(\{j\text{-}i\}) = 0$ for all $1 \leq i \leq n_j, 1 \leq j \leq m$ and $\omega_A(\{1\text{-}i_1, ..., m\text{-}i_m\}) = a_{i_1 \dots i_m}$ for all $(i_1, ..., i_m) \in N^1 \times ... \times N^m$.

Formally,

$$\omega_A(S) = \max_{\mu \in \mathcal{M}(N^1 \cap S, \ldots, N^m \cap S)} \left\{ \sum_{(i_1, \ldots, i_m) \in \mu} A(i_1, \ldots, i_m) \right\},$$

where the summation over the empty set is zero.

Notice that for a given m-SAP $(N^1, \ldots, N^m; A)$, there are $n_1 + \ldots + n_m$ different players. Hence, the payoff vectors for the associated m-SAG belong to $\mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_m}$. We will denote any payoff vector $x \in \mathbb{R}_+^{n_1} \times \ldots \times \mathbb{R}_+^{n_m}$ by

$$(x_{11}, \ldots, x_{1n_1}; x_{21}, \ldots, x_{2n_2}; \ldots; x_{m1}, \ldots, x_{mn_m}),$$

where, for all $1 \le j \le m$ and $1 \le i \le n_j$, x_{ji} is the payoff of agent j-i and the semicolons are used to separate payoffs to agents of different types.

Quint (1991) shows that given a multi-sided assignment game $(N; \omega_A)$, its core (Gillies, 1959), that will be denoted by $C(\omega_A)$, is the set of vectors satisfying that $x_{ji} = 0$ if j-i is unmatched under some optimal matching and

$$\sum_{j=1}^{m} x_{ji_j} \ge A(i_1, \ldots, i_m), \tag{1}$$

for any $(i_1, \ldots, i_m) \in N^1 \times \ldots \times N^m$, where (1) holds as equality if $(i_1, \ldots, i_m)$ belongs to some optimal matching.

Finally, if $m = 2$ the setting reduces to the classic Shapley-Shubik assignment market (Shapley and Shubik, 1972). It is well-kown that such games are always *balanced*, i.e. they have a nonempty core. Furthermore, since any subgame of a m-SAG is also a m-SAG, two-sided assignment games are *totally balanced*, i.e. any subgame has always a nonempty core. However, for more than two sides the core of a m-SAG may be empty -see Kaneko and Wooders (1982) in a more general framework or Quint (1991)- and balanced m-SAGs might not be totally balanced (Quint, 1991).

3 Truncated additive m-sided assignment games: the core

We next introduce a subclass of multi-sided assigment problems, namely the *truncated additive*. Roughly speaking, given fixed exogenous vectors one for each type of agents $d_1 \in \mathbb{R}^{n_1}, ..., d_m \in \mathbb{R}^{n_m}$, a m-SAP $(N^1, ..., N^m; A)$ is *truncated additive* if each entry of the matrix A is obtained *adding* exactly one specific element of each vector $d_1, ..., d_m$, whenever the sum is positive. When it is not, the entry is *truncated* to zero. For all j, $1 \leq j \leq m$ we denote the i-th component of the vector $d_j \in \mathbb{R}^{n_j}$ by d_{ji} and we refer to it as the *productivity*[1] of agent j-i.

Definition 1 *Given the sets of agents $N^1, ..., N^m$ and exogenous vectors $d_1 \in \mathbb{R}^{n_1}, ..., d_m \in \mathbb{R}^{n_m}$, the* truncated additive *m-SAP associated to $d = (d_1, ..., d_m)$ is the m-SAP $(N^1, ..., N^m; A^d)$, where*

$$A^d(i_1, ..., i_m) = \max\left\{0, \sum_{j=1}^{m} d_{ji_j}\right\}, \tag{2}$$

for any $(i_1, ..., i_m) \in N^1 \times ... \times N^m$.

Without loss of generality, from now on we will assume that the components of each of these vectors $d_1, ..., d_m$ are arranged in a non-increasing order, that is, $d_{j1} \geq ... \geq d_{jn_j}$ for all $1 \leq j \leq m$. We will also assume that $\sum_{j=1}^{m} d_{j1} > 0$ to avoid the trivial game. On the other hand, we know that an arbitrary m-SAP $(N^1, ..., N^m; A)$ can always be assumed to be square by introducing dummy agents, that is, agents such that any entry of the matrix A related to them is 0. In particular, this can be done for any truncated additive m-SAP $(N^1, ..., N^m; A^d)$ without disturbing its 'truncated additive' nature just by adding dummy agents of low enough productivity, for instance $-\infty$. Hence, from now on we will assume that $d_1, ..., d_m \in \widetilde{\mathbb{R}}^n_{\geq}$, where $\widetilde{\mathbb{R}}^n_{\geq} = \{(x_1, ..., x_n) \in \widetilde{\mathbb{R}}^n : x_1 \geq ... \geq x_n\}$ and $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. Lastly, observe that different vectors d can give rise to the same matrix A^d.

[1] We are aware that referring to d_{ji} as the productivity of agent j_i might not be meaningful when it is negative. We allow negative productivities in order to incorporate costs, like in the Böhm-Bawerk two-sided market.

On the other hand, m-sided markets with homogeneous goods can be represented by truncated additive m-SAGs. Indeed, the market described in the introduction depends only upon $A(i,j,k) = \max\left\{0, -c_i^s - c_j^h + w^k\right\}$, for any firm i producing a unit of S, any firm j producing a unit of H and any buyer k. To better illustrate the above definition consider the following example, which will be used later.

Example 2 *Consider a 3-sided market with homogeneous goods where there are two firms for each of the two different types (of firms) with unitary costs of $c^1 = (1,8)$ and $c^2 = (1,3)$ respectively and two buyers with a vector of willingness-to-pay $w = (7,5)$. The truncated 3-SAP generated by $d = (-1,-8;-1,-3;7,5)$ gives rise to the following matrix A^d, where the optimal matching is marked in bold,*

$$
\begin{array}{c}
\begin{array}{ccc} & d_{21}=-1 & d_{22}=-3 \\ d_{11}=-1 & & \\ d_{12}=-8 & & \end{array}
\end{array}
\qquad
\begin{array}{c} \begin{array}{cc} d_{21}=-1 & d_{22}=-3 \end{array} \\ \left(\begin{array}{cc} \mathbf{5} & 3 \\ 0 & 0 \end{array}\right) \\ d_{31}=7 \end{array}
\qquad
\begin{array}{c} \begin{array}{cc} d_{21}=-1 & d_{22}=-3 \end{array} \\ \left(\begin{array}{cc} 3 & 1 \\ 0 & \mathbf{0} \end{array}\right) \\ d_{32}=5 \end{array}.
$$

Furthermore, other economic situations can be modeled by truncated additive m-SAPs. Indeed, consider a type of indivisible product that has to be sold after some compulsory steps along a (handcrafted) value chain. In every stage, there are agents that receive at most one product, add their own value through an specific procedure and give it to the next stage. Then, the final value of one product, $A(i_1,...,i_m)$, is the sum of its initial value, d_{1i_1}, and what is added to it along the chosen $m-1$ steps of the value chain, d_{ji_j} for all $j = 2,...,m$, whenever the sum of added values to the initial value is positive. When it is not, we assume that the final value is 0, that is, no object is produced.

For any arbitrary truncated additive m-SAP $(N^1,...,N^m;A^d)$ we denote

$$r = \max_{1\le i\le n}\left\{i : A^d(i,...,i) > 0\right\}. \tag{3}$$

For instance, in Example 2 we have that $r = 1$. Due to the 'truncated additive' nature of matrix A^d, the 'diagonal matching' $\{(l, ..., l) : 1 \leq l \leq n\}$ is optimal. Moreover, a matching $\mu \in \mathcal{M}(N^1, ..., N^m)$ that assigns all agents with index not larger than r among them is also optimal. Hence, r represents the number of essential coalitions of an optimal matching that contribute with a positive worth to the worth of the grand coalition, $\omega_{A^d}(N)$.

Now we study the structure of the core of a truncated additive m-SAG and show that it can be expressed as a polyhedron of $\mathbb{R}^{m-1}$. Let (N, ω_{A^d}) be a truncated additive m-SAG. By core conditions -see (1)- a vector $x \in \mathbb{R}^{n \cdot m}$ belongs to $C(\omega_{A^d})$ if and only if it has nonnegative components, for any $(i_1, ..., i_m) \in \{1, ..., r\}^m$

$$\sum_{j=1}^{m} x_{ji_j} = \sum_{j=1}^{m} d_{ji_j}, \tag{4}$$

for any $(i_1, ..., i_m) \notin \{1, ..., r\}^m$

$$\sum_{j=1}^{m} x_{ji_j} \geq \max\left(0, \sum_{j=1}^{m} d_{ji_j}\right), \tag{5}$$

and if $r + 1 \leq i \leq n, 1 \leq j \leq m$

$$x_{ji} = 0. \tag{6}$$

First notice that (4) is a compatible linear system (d is a solution) of r^m linear equations on the $r \cdot m$ variables $x_{11}, ..., x_{1r}, ..., x_{m1}, ..., x_{mr}$. In the Appendix it is shown that in such system there are $m - 1$ degrees of freedom (in Section 6 it is shown that, when truncated m-SAGs are interpreted as assignment markets with $m - 1$ homogeneous goods, each of the $m - 1$ degrees of freedom is related to the price of a different good). In particular, we can recover any vector $x \in C(\omega_{A^d})$ from the first component of the first $m - 1$ types, i.e $x_{11}, x_{21}, ..., x_{(m-1)1}$. Further, from (4) it can be easily deduced that the other components are obtained using the following formulae:

$$x_{ji} = x_{j1} + (d_{ji} - d_{j1}), \quad \textit{for} \quad 1 \leq j \leq m - 1, 1 \leq i \leq r, \tag{7}$$

$$(8) \qquad x_{mi} = d_{mi} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1}, \quad \textit{for} \quad 1 \le i \le r,$$

$$(9) \qquad x_{ji} = 0, \quad \textit{for} \quad 1 \le j \le m,\ r+1 \le i \le n.$$

In particular, observe that the difference between nonzero core payoffs of two agents of the same type is exactly the difference between their productivities. We define the projection of $C(\omega_{A^d})$ onto the space of payoff vectors of agents 1-1, ..., $(m-1)$-1 as

$$C_{m-1}(\omega_{A^d}) = \left\{ (x_{11}, ..., x_{(m-1)1}) \in \mathbb{R}^{m-1} : (x_{11,}..., x_{1n}; ...; x_{m1}, ..., x_{mn}) \in C(\omega_{A^d}) \right\}.$$

Using formulae (7), (8) and (9) it is easy to prove that there is a one-to-one correspondence between $C_{m-1}(\omega_{A^d})$ and $C(\omega_{A^d})$. Therefore, we can describe $C_{m-1}(\omega_{A^d})$ as an specific polyhedron in $\mathbb{R}^{m-1}$. Indeed,

Theorem 3 *Let $(N^1, ..., N^m; A^d)$ be a truncated additive m-SAP. Then (N, ω_{A^d}) is totally balanced and the (nonempty) projection of its core, $C_{m-1}(\omega_{A^d})$, is the polyhedron in $\mathbb{R}^{m-1}$ on the variables $x_{11}, ..., x_{(m-1)1}$ defined by the following constraints:*

$$(10) \qquad x_{j1} \ge d_{j1} - d_{jr}, \quad for \quad j = 1, ..., m-1,$$

$$(11) \qquad \sum_{j=1}^{m-1} x_{j1} \le d_{mr} + \sum_{j=1}^{m-1} d_{j1},$$

$$(12) \qquad \sum_{j=1}^{m-1} x_{j1} \ge d_{m(r+1)} + \sum_{j=1}^{m-1} d_{j1},$$

$$(13) \qquad x_{j1} \le d_{j1} - d_{j(r+1)}, \quad for \quad j = 1, ..., m-1,$$

where (12) and (13) do not apply if $r = n$, being r defined in (3).

Proof. First notice that function $A(\cdot, ..., \cdot)$ is supermodular over the lattice defined by the componentwise order on $N^1 \times ... \times N^m$, i.e.

$$\begin{aligned} A(i_1, ..., i_m) + A(l_1, ..., l_m) \le\ & A(\min\{i_1, l_1\}, ..., \min\{i_m, l_m\}) \\ & + A(\max\{i_1, l_1\}, ..., \max\{i_m, l_m\}). \end{aligned}$$

Then, total balancedness of (N, ω_{A^d}) holds by Sherstyuk (1999).

Second, let $P(d)$ be the polyhedron of $\mathbb{R}^{m-1}$ defined by (10) and (11) (and (12) and (13) whenever $r < n$). We show that $C_{m-1}(\omega_{A^d}) = P(d)$.

On one hand we prove $C_{m-1}(\omega_{A^d}) \supseteq P(d)$. Take $\overline{x} \in P(d)$ and let $x \in \mathbb{R}^{nm}$ be the vector obtained applying (7), (8) and (9) to $\overline{x}$. We want to show that x has nonnegative components and satisfies (4), (5) and (6).

Next observe that nonnegativeness conditions hold from (7), (8), (10) and (11). Indeed, $x_{ji} = x_{j1} + (d_{ji} - d_{j1}) \geq x_{j1} + (d_{jr} - d_{j1}) \geq 0$ for all $1 \leq i \leq r$, $1 \leq j \leq m-1$ and $x_{mi} = d_{mi} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1} \geq d_{mr} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1} \geq 0$ for all $1 \leq i \leq r$ by (11). Additionally (6) trivially holds by (9).

Next, for any $(i_1, ..., i_m) \in \{1, ..., r\} \times ... \times \{1, ..., r\}$ we have by (7) and (8)

$$\sum_{j=1}^{m} x_{ji_j} = \sum_{j=1}^{m-1} x_{j1} + \sum_{j=1}^{m-1} (d_{ji_j} - d_{j1}) + d_{mi_m} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1} = \sum_{j=1}^{m} d_{ji_j}.$$

Then (4) is satisfied. If $r = n$ we are done. If not, i.e. $r < n$, we need to prove that (5) holds. Suppose $(i_1, ..., i_m) \notin \{1, ..., r\}^m$. We must distinguish two cases.

- *Case 1:* $1 \leq i_m \leq r$.

$$\begin{aligned}
\sum_{j=1}^{m} x_{ji_j} &= \sum_{j=1}^{m-1} x_{ji_j} + x_{mi_m} = (by \quad (7),(8),(9)) \\
&= \sum_{j=1, i_j \leq r}^{m-1} \big(x_{j1} + d_{ji_j} - d_{j1}\big) + d_{mi_m} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1} \\
&= -\sum_{j=1, i_j > r}^{m-1} x_{j1} + \sum_{j=1, i_j \leq r}^{m} d_{ji_j} + \sum_{j=1, i_j > r}^{m-1} d_{j1} = (by \quad (13)) \\
&\geq \sum_{j=1, i_j > r}^{m-1} \big(d_{j(r+1)} - d_{j1}\big) + \sum_{j=1, i_j \leq r}^{m} d_{ji_j} + \sum_{j=1, i_j > r}^{m-1} d_{j1} \\
&= \sum_{j=1, i_j \leq r}^{m} d_{ji_j} + \sum_{j=1, i_j > r}^{m-1} d_{j(r+1)} \geq \sum_{j=1}^{m} d_{ji_j}
\end{aligned}$$

  where the last inequality holds since $d_{ji_j} \leq d_{j(r+1)}$ whenever $i_j > r$, and therefore (5) holds.

- *Case 2:* $r+1 \leq i_m \leq n$.

$$\begin{aligned}
\sum_{j=1}^{m} x_{ji_j} &= \sum_{j=1}^{m-1} x_{ji_j} + x_{mi_m} = (by \quad (7),(9)) \\
&= \sum_{j=1, i_j \leq r}^{m-1} x_{ji_j} = \sum_{j=1, i_j \leq r}^{m-1} \left(x_{j1} + d_{ji_j} - d_{j1} \right) \\
&= \sum_{j=1}^{m-1} x_{j1} + \sum_{j=1, i_j \leq r}^{m-1} \left(d_{ji_j} - d_{j1} \right) - \sum_{j=1, i_j > r}^{m-1} x_{j1} = (by \quad (12),(13)) \\
&\geq d_{m(r+1)} + \sum_{j=1}^{m-1} d_{j1} + \sum_{j=1, i_j \leq r}^{m-1} \left(d_{ji_j} - d_{j1} \right) + \sum_{j=1, i_j > r}^{m-1} \left(d_{j(r+1)} - d_{j1} \right) \\
&= d_{m(r+1)} + \sum_{j=1, i_j \leq r}^{m-1} d_{ji_j} + \sum_{j=1, i_j > r}^{m-1} d_{j(r+1)} \geq \sum_{j=1}^{m} d_{ji_j}
\end{aligned}$$

where the last inequality holds since $d_{ji_j} \leq d_{j(r+1)}$ whenever $i_j > r$. Again (5) holds.

On the other hand we prove $C_{m-1}(\omega_{A^d}) \subseteq P(d)$. Take $\overline{x} \in C_{m-1}(\omega_{A^d})$ and let $x \in \mathbb{R}_+^{nm}$ be the corresponding core allocation obtained by (7), (8) and (9). First, by nonnegativeness of x_{jr} for all $1 \leq j \leq m-1$ and using (7), we obtain (10). Second, by nonnegativeness of x_{mr} and using (8), we obtain (11). If $r = n$ we are done. If not, i.e. $r < n$, applying (6) and (5) to $S = \{1\text{-}1, ..., (m-1)\text{-}1, m\text{-}(r+1)\}$ and taking (9) into account we obtain $x(S) = \sum_{j=1}^{m-1} x_{j1} + x_{m(r+1)} = \sum_{j=1}^{m-1} x_{j1} + 0 \geq \max\left(0, d_{m(r+1)} + \sum_{j=1}^{m-1} d_{j1}\right)$ and then (12) holds. Finally, for all $1 \leq j \leq m-1$, by applying (6) and (5) to $S = \{1\text{-}1, ..., (j-1)\text{-}1, j\text{-}(r+1), (j+1)\text{-}1, ..., m\text{-}1\}$ we obtain

$$\begin{aligned}
x_{j(r+1)} + \sum_{k=1, k \neq j}^{m} x_{k1} &= \sum_{k=1, k \neq j}^{m} x_{k1} \geq \max\left(0, d_{j(r+1)} + \sum_{k=1, k \neq j}^{m} d_{k1}\right) \\
&\geq d_{j(r+1)} + \sum_{k=1, k \neq j}^{m} d_{k1} \\
&= d_{j(r+1)} + \sum_{k=1, k \neq j}^{m} x_{k1} + x_{j1} - d_{j1}.
\end{aligned}$$

Lastly, substracting the common terms to both sides of the above inequality and using (9) we obtain that (13) holds, taking into account that $\sum_{k=1}^{m} x_{k1} = \sum_{k=1}^{m} d_{k1}$. ■

It is important to point out that equations (10) and (11) (resp. (12) and (13)) are in fact of the same nature although they appear not to be due to the election of the specific projection space $C_{m-1}(\omega_{A^d})$.[2]

Next we take a look again at Example 2. Recall that the parameter defined in (3) is $r = 1$, whereas $d = (-1, -8; -1, -3; 7, 5)$. From Proposition 3, the core $C(\omega_{A^d})$ of the game can be represented by its projection onto the space $C_2(\omega_{A^d})$, which is the set of pairs (x_{11}, x_{21}) such that

$$\begin{aligned}
0 &= d_{11} - d_{11} \leq x_{11} \leq d_{11} - d_{12} = 7, \\
0 &= d_{21} - d_{21} \leq x_{21} \leq d_{21} - d_{22} = 2, \\
3 &= d_{32} + d_{11} + d_{21} \leq x_{11} + x_{21} \leq d_{31} + d_{11} + d_{21} = 5.
\end{aligned}$$

In Figure 1, the projection of the core, $C_2(\omega_{A^d})$, is the polygon defined by vertices $(1, 2)$, $(3, 0)$, $(3, 2)$ and $(5, 0)$.

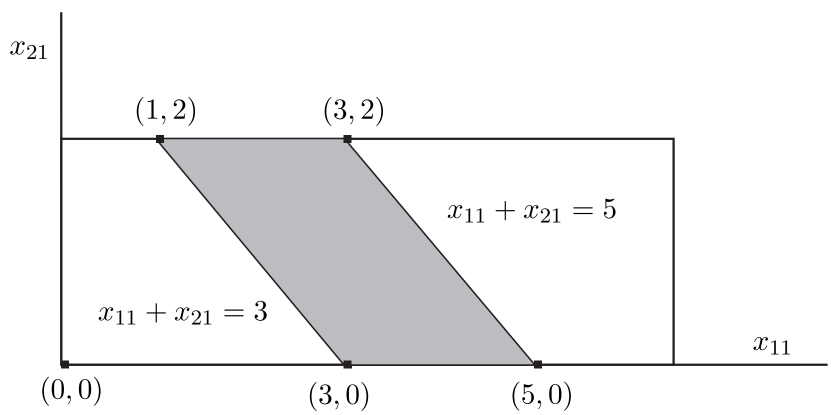

Figure 1: The projected core of the truncated additive m-SAG of Example 2.

[2] In the case of two-sided assignment games the core is usually represented by its projection onto the agents of one sector (either buyers or sellers). This representation is not useful when there are more than two sectors since, unlike the two-sectors case, once fixed the payment to one agent, the payments to other agents of different types that are assigned together to that agent are not completely determined. However, in the specific case of truncated additive m-SAGs, there is still a 'good' representation, namely $C_{m-1}(\omega_{A^d})$.

4 Extreme core points

This section is devoted to characterize the extreme core points of an arbitrary truncated additive multi-sided assignment game. We show that the extreme core points of such games are some specific vectors introduced by Sherstyuk (1999) that generalize to some extent the idea of buyers-optimal and sellers-optimal core allocations from the two-sided case to the general multi-sided case. Additionally, we provide expressions to calculate such vectors and we show that they are marginal worth vectors, generalizing therefore a property that holds for all two-sided assignment games (Hamers et al., 2002). Furthermore, we show that this property fails to hold in the class of arbitrary m-SAGs.

Given the set of types of agents $J = \{1, ..., m\}$, let Π^J denote the set of orderings of J, i.e. the set of bijective maps from J to J. We denote by $(\pi(1), ..., \pi(m))$ an ordering $\pi \in \Pi^J$ and we say that type j appears in the k-th position if $\pi(k) = j$. Then, given a general square m-SAP $(N^1, ..., N^m; A)$, we associate a vector x^π of $\mathbb{R}^{nm}$ to any $\pi \in \Pi^J$. Without loss of generality, suppose that the 'diagonal' assignment is optimal and let $\pi = (1, 2, ..., m)$. We consider the allocation x^π in which all agents of the first type get their marginal contribution, $\omega_A(N) - \omega_A(N \backslash \{1\text{-}i\})$, whereas any agent of the second type gets their 'second' marginal contribution, $\omega_A(N \backslash \{1\text{-}i\}) - \omega_A(N \backslash \{1\text{-}i, 2\text{-}i\})$, that is, her marginal contribution provided that the agent of the first type with her same index (that is assigned to her in the diagonal matching) has left the game. And so on and so forth. Obviously, this can be done for any arbitrary π and any arbitrary matching[3].

In particular, when there are two sectors, i.e. $m = 2$, the set of sectors (types) $J = \{1, 2\}$ can only be ordered in two different ways and the above procedure delivers only two vectors: on one hand, $x^\pi(\omega_A) \in \mathbb{R}^{2n}$ defined by $x^\pi_{1i} = \omega_A(N) - \omega_A(N \backslash \{1\text{-}i\})$ and $x^\pi_{2i} = \omega_A(N \backslash \{1\text{-}i\}) - \omega_A(N \backslash \{1\text{-}i, 2\text{-}i\})$ for all $1 \leq i \leq n$, and, on the other hand, $x^{\pi'}(\omega_A) \in \mathbb{R}^{2n}$ defined by $x^{\pi'}_{2i} = \omega_A(N) - \omega_A(N \backslash \{2\text{-}i\})$ and $x^{\pi'}_{1i} = \omega_A(N \backslash \{2\text{-}i\}) - \omega_A(N \backslash \{2\text{-}i, 1\text{-}i\})$ for all $1 \leq i \leq n$.

[3] Recall that, by relabeling agents for convenience, any optimal matching of an arbitrary m-SAG can always be placed in the diagonal.

Furthermore, it turns out that x^{π} and $x^{\pi'}$are the buyers-optimal and the sellers-optimal vectors of any two-sided assignment game (see for instance Nuñez and Rafels, 2002).

Definition 4 *Given an arbitrary square m-SAP* $(N^1, ..., N^m; A)$ *and* $\pi \in \Pi^J$*, the sector-marginal vector* $x^{\pi}(\omega_A) \in \mathbb{R}^{nm}$ *is defined by*

$$x^{\pi}_{\pi(k)i}(\omega_A) = \omega_A(N\backslash\{\pi(1)\text{-}i, ..., \pi(k-1)\text{-}i\}) - \omega_A(N\backslash\{\pi(1)\text{-}i, ..., \pi(k-1)\text{-}i, \pi(k)\text{-}i\})$$

for all $1 \leq i \leq n$ *and for all* $1 \leq k \leq m$*, provided that* $\{(l, ..., l) : 1 \leq l \leq n\} \in \mathcal{M}^*_A(N^1, ..., N^m)$.

Later we will show that, although the above definition of sector-marginal vectors apparently depends on the optimal matching chosen -that is placed in the diagonal- when $(N^1, ..., N^m; A)$ is truncated additive it actually does not. On the other hand, when no confusion arises we use x^{π} instead of $x^{\pi}(\omega_A)$.

We can give another interpretation of sector-marginal vectors. To this end, let us first introduce the definition of marginal worth vectors for arbitrary games. Given a game (N, v), an ordering θ of N is a bijection from $\{1, ..., |N|\}$ to N, and let $\Theta(N)$ be the set of all orderings of N. Then the *marginal worth vector* $m^{\theta}(v) \in \mathbb{R}^n$ associated with θ is defined (see Shapley, 1971) by $m^{\theta}_i = v(\{j : \theta(j) \leq \theta(i)\}) - v(\{j : \theta(j) < \theta(i)\})$, for all $1 \leq i \leq n$. Now observe that, given some sector-marginal vector x^{π}, if we fix i, $1 \leq i \leq n$ then agents $1\text{-}i, 2\text{-}i, ..., m\text{-}i$ are paid as if they were in the first positions of some marginal worth vector (a different one for each i) associated to some ordering of (all) agents $(\pi(1)\text{-}i, \pi(2)\text{-}i, ..., \pi(m)\text{-}i, ...)$. In particular, their payoffs add up altogether to the marginal contribution of these agents taken as a whole, $\sum_{j=1}^{m} x^{\pi}_{ji} = \omega_A(N) - \omega_A(N\backslash\{1\text{-}i, 2\text{-}i..., m\text{-}i\}) = A(i, ..., i)$. The last equality holds because we assume that $(i, ..., i)$ belongs to an optimal matching.

From now on, given a truncated additive m-SAP $(N^1, ..., N^m; A^d)$ with $r = n$, we will introduce for any type an additional dummy agent with $-\infty$ productivity such that the 'new matrix' A^d restricted to non-dummy agents does not change. Since dummy agents always get a zero payoff in the core, this assumption can be made without loss of generality.

We now turn to characterizing the set of extreme points of the core of a truncated additive m-SAG. Before doing so, for any truncated additive m-SAP $(N^1, ..., N^m; A^d)$, let $\phi(q) = \max\left\{0, \sum_{k=1}^{q} d_{k(r+1)} + \sum_{k=q+1}^{m} d_{kr}\right\}$, for $q = 0, 1, ..., m$. Observe that, because of the way d is constructed, ϕ is a non-increasing function, whereas from (3) and the assumptions just made, we have that $\phi(0) > 0$ and $\phi(m) \leq 0$. It is important to point out that the definition of ϕ is implicitely associated to the 'natural' ordering of types $\pi = (1, ..., m)$. Therefore, we could define ϕ^π for all $\pi \in \Pi^J$. However, in the proofs of this section we will only need ϕ^π when $\pi = (1, ..., m)$, written simply ϕ.

Next we provide expressions to calculate sector-marginal vectors for truncated additive m-SAGs.

Proposition 5 *Let (N, ω_{A^d}) be a truncated additive m-SAG and $\pi \in \Pi^J$. Then, for all sector j, $1 \leq j \leq m$,*

$$x^\pi_{\pi(j)i}(\omega_{A^d}) = d_{\pi(j)i} - d_{\pi(j)r} + \phi^\pi(j-1) - \phi^\pi(j), \quad if \quad 1 \leq i \leq r$$

and $x^\pi_{\pi(j)i}(\omega_{A^d}) = 0$ if $r + 1 \leq i \leq n$.

Proof. Assume without loss of generality that $\pi = (1, 2, ..., m)$. To compute the worth $\omega_{A^d}(S)$ for an arbitrary coalition of agents $S \subseteq N$ it is enough to split S in as many essential coalitions as possible such that each of the 'best' agents of all types in S are matched together, each of the 'second best' agents of all types in S are matched together, and so on and so forth, until an essential coalition with the remaining agents cannot be formed. If we add the worth of all these essential coalitions, we obtain $\omega_{A^d}(S)$. On the other hand, observe that for all $1 \leq j \leq m$, $\omega_{A^d}(N \backslash \{1\text{-}i, ..., (j-1)\text{-}i\}) - \omega_{A^d}(N \backslash \{1\text{-}i, ..., (j-1)\text{-}i, j\text{-}i\})$ is a difference between two similar terms. Then, we next focus on a generic one of such terms. Indeed, if

$1 \leq i \leq r$, observe that for any $k, 0 \leq k \leq m$,

$$\begin{aligned}
& \omega_{A^d}(N \backslash \{1\text{-}i, ..., k\text{-}i\}) \\
= & \sum_{l=1}^{i-1} \left(\max \left(0, \sum_{q=1}^{m} d_{ql} \right) \right) + \sum_{l=i}^{r-1} \left(\max \left(0, \sum_{q=1}^{k} d_{q(l+1)} + \sum_{q=k+1}^{m} d_{ql} \right) \right) \\
& + \max \left\{ 0, \sum_{q=1}^{k} d_{q(r+1)} + \sum_{q=k+1}^{m} d_{qr} \right\} \\
= & \sum_{l=1}^{i-1} \sum_{q=1}^{m} d_{ql} + \sum_{l=i}^{r-1} \sum_{q=1}^{k} d_{q(l+1)} + \sum_{l=i}^{r-1} \sum_{q=k+1}^{m} d_{ql} + \phi(k) \\
= & \sum_{l=1, l \neq i}^{r-1} \sum_{q=1}^{m} d_{ql} + \sum_{q=1}^{k} d_{qr} + \sum_{q=k+1}^{m} d_{qi} + \phi(k).
\end{aligned}$$

where the penultimate equality holds from the way d is arranged and from (3). Then, applying the above expression to $k = j-1$ and $k = j$ we obtain

$$\begin{aligned}
x^{\pi}_{ji} &= \omega_{A^d}(N \backslash \{1\text{-}i, ..., (j-1)\text{-}i\}) - \omega_{A^d}(N \backslash \{1\text{-}i, ..., (j-1)\text{-}i, j\text{-}i\}) \\
&= \sum_{l=1, l \neq i}^{r-1} \sum_{q=1}^{m} d_{ql} + \sum_{q=1}^{j-1} d_{qr} + \sum_{q=j}^{m} d_{qi} + \phi(j-1) \\
&\quad - \left(\sum_{l=1, l \neq i}^{r-1} \sum_{q=1}^{m} d_{ql} + \sum_{q=1}^{j} d_{qr} + \sum_{q=j+1}^{m} d_{qi} + \phi(j) \right) \\
&= d_{ji} - d_{jr} + \phi(j-1) - \phi(j).
\end{aligned}$$

A similar argument proves that if $r+1 \leq i \leq n$ then $x^{\pi}_{ji} = \omega_{A^d}(N) - \omega_{A^d}(N) = 0$, for any $1 \leq j \leq m$. ■

Next we show that, for truncated additive m-SAGs, sector-marginal vectors are core allocations. Although this fact can be deduced from Sherstyuk (1999), an easier argument can be provided for the case of truncated additive m-SAGs. Moreover, such argument gives more insight of sector-marginal vectors. Indeed, let $(N^1, ..., N^m; A^d)$ be a truncated additive m-SAP and assume without loss of generality that $\pi = (1, 2, ..., m)$. On one hand, it is straightforward to check that $x^{\pi}_{ji} \geq 0$ for any agent j-i. On the other hand, we claim that, if $r < i \leq m$, then

$$d_{ji} - d_{jr} + \phi(j-1) - \phi(j) \leq 0 \tag{14}$$

for any $1 \leq j \leq m$. Indeed, depending on whether $\phi(j-1)$ and $\phi(j)$ are positive or not we can distinguish three different cases (recall that ϕ is non-increasing). Consider the case where $\phi(j-1) > 0$ and $\phi(j) = 0$. In such case, (14) reduces to $\phi(j-1) \leq d_{jr} - d_{ji}$, which is equivalent to $\sum_{k=1}^{j-1} d_{k(r+1)} + d_{ji} + \sum_{k=j+1}^{m} d_{kr} \leq 0$. Lastly observe that, because of the way d is ordered, $\sum_{k=1}^{j-1} d_{k(r+1)} + d_{ji} + \sum_{k=j+1}^{m} d_{kr} \leq \phi(j) = 0$. The other two cases can be similarly proved and are left to the reader. Hence, for any $(i_1, ..., i_m) \in N^1 \times ... \times N^m$,

$$\begin{aligned}\sum_{j=1}^{m} x^{\pi}_{ji_j} &= \sum_{j=1,\ 1\leq i_j \leq r}^{m} x^{\pi}_{ji_j} \geq \sum_{j=1}^{m} \left(d_{ji_j} - d_{jr} + \phi(j-1) - \phi(j)\right) \\ &= \sum_{j=1}^{m} d_{ji_j} - \sum_{j=1}^{m} d_{jr} + \phi(0) - \phi(m) = \sum_{j=1}^{m} d_{ji_j},\end{aligned}$$

where the inequality holds as equality if $(i_1, ..., i_m) \in \{1, ...r\}^m$. Thus $x^{\pi}(\omega_{A^d})$ satisfies (7), (8), (9) and therefore it belongs to $C(\omega_{A^d})$. Since the claim has been proved without loss of generality using the natural order, what we have shown is that, for truncated additive m-SAGs, any sector-marginal vector belongs to the core. Furthermore, we next show that, for truncated additive m-SAGs, any sector-marginal vector is a extreme point of the core and conversely.

Theorem 6 *Let (N, ω_{A^d}) be a truncated additive m-SAG. Then, the set of extreme core points of (N, ω_{A^d}) coincides with the set of sector-marginal vectors, i.e.*

$$Ext(C(\omega_{A^d})) = \{x^{\pi}(\omega_{A^d})\}_{\pi \in \Pi^J}.$$

Proof. We prove the two inclusions.

- $Ext(C(\omega_{A^d})) \subseteq \{x^{\pi}(\omega_{A^d})\}_{\pi \in \Pi}$.

  It is easy to check that the extreme points of $C(\omega_{A^d})$ can be mapped one-to-one to the extreme points of $C_{m-1}(\omega_{A^d})$ through (7), (8) and (9). Hence, we only have to find the extreme points of the polyhedron defined in Theorem 3. Let $M^T x \geq b$ be the set of non-trivial[4] inequalities obtained from (10), (11), (12) and (13) that define $C_{m-1}(\omega_{A^d})$.

[4] Observe that, if $d_{m(r+1)} = -\infty$, (12) always holds. The same can be said about (13) for all $1 \leq j \leq m-1$.

For each $x \in C_{m-1}(\omega_{A^d})$, let $tight(x)$ be the set of columns $\{Me_j | \ x^T Me_j = b_j\}$ of M that are tight at x. We know[5] that $x \in C_{m-1}(\omega_{A^d})$ is an extreme point if and only if $tight(x)$ is a complete system of $\mathbb{R}^{m-1}$.

On the other hand notice that for any type j, $1 \leq j \leq m-1$, constraints (10) and (13) cannot hold simultaneously as equality unless $d_{j(r+1)} = d_{jr}$. However in this case both inequalities are in fact the same. For (12) and (11) the same is true.

Now, with some abuse of notation, let $x = (x_{11}, ..., x_{(m-1)1})$ denote the extreme point of $C_{m-1}(\omega_{A^d})$ that corresponds to the extreme point $x = (x_{11}, ..., x_{1n}; \ ...; \ x_{m1}, ..., x_{mn}) \in C(\omega_{A^d})$. Then, $m-1$ linearly independent inequalities of (10), (11), (12) and (13) must be tight at x . We distinguish two cases.

Case 1. *Either (10), for some $1 \leq j \leq m-1$, or (11) are tight at x.*

Recall that the difference between inequalities (10) and (11) comes only from the specific projection chosen to obtain $C_{m-1}(\omega_{A^d})$. Indeed, using (8) we have that (11) is equivalent to $x_{m1} \geq d_{m1} - d_{mr}$. Then, as we can relabel the different types of agents at our best convenience, we suppose without loss of generality that (11) is tight at x. To form a complete system of $\mathbb{R}^{m-1}$ we still need $m-2$ tight linearly independent equations from either (10) or (13). Thus, we can assume that there is $h^* \in \{1, ..., m-1\}$ such that (13) for $j \in \{1, ..., h^*-1\}$ and (10) for $j \in \{h^*+1, ..., m-1\}$ are tight at x, i.e.[6]

$$x_{j1} = d_{j1} - d_{j(r+1)} \quad \text{for} \quad j \in \{1, ..., h^*-1\} \tag{15}$$

and

$$x_{j1} = d_{j1} - d_{jr} \quad \text{for} \quad j \in \{h^*+1, ..., m-1\}. \tag{16}$$

In particular, $d_{j(r+1)}$ must be finite for $1 \leq j \leq h^*-1$.

[5]See for instance Theorem 11.1 in Tijs (2003).

[6]Observe that the case where neither (12) nor (13) hold as equality is considered.

On the other hand, since also (11) is tight at x,

$$\begin{aligned} x_{h^*1} &= d_{mr} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1, j\neq h^*}^{m-1} x_{j1} \\ &= d_{mr} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{h^*-1} \big(d_{j1} - d_{j(r+1)}\big) - \sum_{j=h^*+1}^{m-1} (d_{j1} - d_{jr}) \\ &= d_{h^*1} + \sum_{j=1}^{h^*-1} d_{j(r+1)} + \sum_{j=h^*+1}^{m} d_{jr}. \end{aligned}$$

Observe that the above expression of x_{h^*1} can be rearranged either to

$$x_{h^*1} = d_{h^*1} - d_{h^*r} + \ \phi(h^* - 1) \tag{17}$$

or, whenever $d_{h^*(r+1)} > -\infty$, to

$$x_{h^*1} = d_{h^*1} - d_{h^*(r+1)} + \ \phi(h^*). \tag{18}$$

Being $x \in C_{m-1}(\omega_{A^d})$, x_{h^*1} must satisfy (10), that is $x_{h^*1} \geq d_{h^*1} - d_{h^*r}$, which implies $\phi(h^* - 1) \geq 0$. Moreover, again since $x \in C_{m-1}(\omega_{A^d})$, x_{h^*1} must satisfy (13), i.e. $x_{h^*1} \leq d_{h^*1} - d_{h^*(r+1)}$, which, at its turn, implies that, regardless $d_{h^*(r+1)}$ is finite or not, $\phi(h^*) = 0$. Moreover, since ϕ is non-increasing, we obtain that $\phi(q) \geq 0$ for $0 \leq q \leq h^* - 1$ and $\phi(q) = 0$, for $h^* \leq q \leq m$.

Let now $t^* \in \{0, ..., h^* - 1\}$ be the unique integer such that $\phi(q) > 0$ for $0 \leq q \leq t^*$ and $\phi(q) = 0$ for $t^* + 1 \leq q \leq m$. That is, t^* is the largest value such that $\phi(t^*)$ is positive. Then, if $\pi = (1, ..., m)$, applying Proposition 5 we obtain that

$$x^\pi_{j1} = d_{j1} - d_{j(r+1)} \quad \text{for} \quad 1 \leq j \leq t^*, \tag{19}$$

$$x^\pi_{(t^*+1)1} = d_{(t^*+1)1} - d_{(t^*+1)r} + \phi(t^*), \tag{20}$$

$$x^\pi_{j1} = d_{j1} - d_{jr} \quad \text{for} \quad t^* + 2 \leq j \leq m. \tag{21}$$

To compare vectors x and x^π, since both of them belong to $C(\omega_{A^d})$, it is enough to compare the payoff to agent of index 1 for the first $m-1$ types -applying (7), (8) and (9) to obtain the payoffs to the remaining agents-.

Next, we show that $x_{j1} = x^\pi_{j1}$ for all $1 \leq j \leq m-1$. On one hand, since $x^\pi \in C(\omega_{A^d})$, by (13) and (19), we have that $x^\pi_{j1} \leq d_{j1} - d_{j(r+1)} = x_{j1}$ for $1 \leq j \leq h^*-1$. On the other hand, let us prove that $x^\pi_{j1} = x_{j1}$ for $h^* \leq j \leq m-1$. We distinguish two cases depending on $t^* \in \{0, ..., h^*-1\}$. If $t^* = h^*-1$, by (20) and (17), we know that $x^\pi_{h^*1} = d_{h^*1} - d_{h^*r} + \ \phi(h^*-1) = x_{h^*1}$. Moreover, in this case, by (21) and (16), we also have that $x^\pi_{j1} = d_{j1} - d_{jr} = x_{j1}$ for all $h^*+1 \leq j \leq m-1$. Lastly, if $t^* < h^*-1$, again by (21) and (16), we obtain directly that $x^\pi_{j1} = x_{j1}$ for all $h^* \leq j \leq m-1$.

Since we have proved that $x_{j1} \leq x^\pi_{j1}$ for all $1 \leq j \leq h^*-1$ and $x_{j1} = x^\pi_{j1}$ for all $h^* \leq j \leq m-1$ and x, $x^\pi \in C(\omega_{A^d})$, then $x = x^\pi$, finishing the proof of Case 1.

Case 2. *The $m-1$ inequalities that are tight at x are among (12) and (13) for all $1 \leq j \leq m-1$.*

As we can relabel the different types of agents for convenience, we can assume without loss of generality that, for all $1 \leq j \leq m-1$, inequalities (13) are tight at x. In particular, $d_{j(r+1)}$ must be finite for $1 \leq j \leq m-1$. Since $x^\pi \in C(\omega_{A^d})$, by (13) and (16), we have that $x^\pi_{j1} \leq d_{j1} - d_{j(r+1)} = x_{j1}$ for $1 \leq j \leq m-1$. Moreover, since also $x \in C(\omega_{A^d})$, like in the above case, we obtain $x = x^\pi$.

- $Ext(C(\omega_{A^d})) \supseteq \{x^\pi(\omega_{A^d})\}_{\pi\in\Pi}$.

Without loss of generality suppose that $\pi = (1, ..., m)$. Recall that $x^\pi(\omega_{A^d})$ belongs to the core. From Definition 4 we know that $x^\pi_{1i} = \omega_{A^d}(N) - \omega_{A^d}(N\backslash\{1\text{-}i\})$ for all

$1 \leq i \leq n$. Therefore, for all $1 \leq i \leq n$,

$$\begin{aligned} x^\pi(N\backslash\{1\text{-}i\}) &= x^\pi(N) - x^\pi_{1i} \\ &= \omega_{A^d}(N) - (\omega_{A^d}(N) - \omega_{A^d}(N\backslash\{1\text{-}i\})) = \omega_{A^d}(N\backslash\{1\text{-}i\}). \end{aligned}$$

Next, again from Definition 4 we know that $x^\pi_{2i} = \omega_{A^d}(N\backslash\{1\text{-}i\}) - \omega_{A^d}(N\backslash\{1\text{-}i, 2\text{-}i\})$ for all $1 \leq i \leq n$. Therefore for all $1 \leq i \leq n$.

$$\begin{aligned} x^\pi(N\backslash\{1\text{-}i, 2\text{-}i\}) &= x^\pi(N) - x^\pi_{1i} - x^\pi_{2i} \\ &= \omega_{A^d}(N) - (\omega_{A^d}(N) - \omega_{A^d}(N\backslash\{1\text{-}i\}) - \\ &\quad (\omega_{A^d}(N\backslash\{1\text{-}i\}) - \omega_{A^d}(N\backslash\{1\text{-}i, 2\text{-}i\})) \\ &= \omega_{A^d}(N\backslash\{1\text{-}i, 2\text{-}i\}). \end{aligned}$$

And so on and so forth. In short, within the set of inequalities that define the core $C(\omega_{A^d})$, we have found $n \times m$ linearly independent inequalities that hold as equality. Therefore $x^\pi(\omega_{A^d})$ must be an extreme core point.

■

Notice that, as we had already stated, for truncated additive m-SAGs, the set of sector-marginal vectors, $\{x^\pi(\omega_{A^d})\}_{\pi\in\Pi^J}$, does not depend on the optimal matching chosen, since neither the set of extreme point of the core, $Ext\,\{C(\omega_{A^d})\}$, nor the proof of Theorem 6 do.

To illustrate the above result consider the 3-SAP introduced in Example 2. The set $Ext(C(\omega_{A^d}))$ is obtained from the initial six possible sector marginal worth vectors.

$\boldsymbol{\pi}^J$	$x^\pi \in C(\omega_{A^d})$	$\overline{x}^\pi \in C_2(\omega_{A^d})$
$(1,2,3)$	$(\mathbf{5},0;\mathbf{0},0;0,0)$	$(\mathbf{5};\mathbf{0})$
$(1,3,2)$	$(\mathbf{5},0;\mathbf{0},0;0,0)$	$(\mathbf{5};\mathbf{0})$
$(2,1,3)$	$(\mathbf{3},0;\mathbf{2};0;0,0)$	$(\mathbf{3};\mathbf{2})$
$(2,3,1)$	$(\mathbf{1},0;\mathbf{2},0;2,0)$	$(\mathbf{1};\mathbf{2})$
$(3,1,2)$	$(\mathbf{3},0;\mathbf{0},0;2,0)$	$(\mathbf{3};\mathbf{0})$
$(3,2,1)$	$(\mathbf{1},0;\mathbf{2},0;2,0)$	$(\mathbf{1};\mathbf{2})$

Observe that $x^{(1,2,3)} = x^{(1,3,2)}$. That is, sector-marginal vectors might coincide for different orderings. Also notice that the number $m!$ is an upper bound for the number of extreme points of the core of any truncated additive m-SAG. However, for m-SAGs that are not truncated additive Theorem 6 fails to hold[7].

Hamers et al. (2002) prove that the classical assignment games satisfy the CoMa-property. This property says that any extreme core allocation is a marginal worth vector. However, that result does not extend to arbitrary multi-sided assignment games (even if they are supermodular). Indeed, consider the (supermodular) 3-SAP $(N^1, N^2, N^3; A)$ given by the following matrix, where the optimal matching is marked in bold:

$$A = \begin{pmatrix} \mathbf{3} & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{2} \end{pmatrix}.$$

It can be checked that $x = (1,1;2,0;0,1) \in Ext\left\{C(\omega_A)\right\}$ and that the marginal contributions of agents 1-1, 1-2, 2-1, 2-2, 3-1 and 3-2 are respectively 3, 2, 3, 2, 3 and 2. Then, observe that no agent attains her marginal contribution in x. In particular, x cannot be a marginal worth vector.

Next, using Theorem 6, we prove that any truncated additive m-SAG does satisfy the CoMa-property.

Theorem 7 *Let (N, ω_{A^d}) be a truncated additive m-SAG. Then it satisfies the CoMa-property.*

Proof. The proof consists on associating to each order of types (of agents) $\pi \in \Pi^J$ an ordering of agents $\theta^\pi \in \Theta(N)$ such that x^π and m^{θ^π} coincide. Without loss of generality suppose $\pi = (1, 2, ..., m)$. Then let θ^π be the ordering that is both decreasing in indices (of agents) and also decreasing in types, i.e.

$$\theta^\pi = (m\text{-}n, ..., 1\text{-}n, \dots\dots, m\text{-}(r+1), ..., 1\text{-}(r+1), m\text{-}r, ..., 1\text{-}r, \dots\dots, m\text{-}1, ..., 1\text{-}1).$$

[7]The supermodular and monotonic assignment game where A is defined by $A(1,1) = 8$, $A(2,1) = 2$, $A(1,2) = 3$ and $A(2,2) = 2$ can be taken as a counterexample. It can be checked that the core of such game has dimension 2 and four extreme points, and therefore $\{x^\pi(\omega_A)\}_{\pi\in\Pi} \subsetneq Ext(C(\omega_A))$. See Sherstyuk (1999) for a formal definition of monotonic m-SAGs.

Recall that, for any nonemtpy coalition S of agents, $\omega_{A^d}(S)$ is obtained by partioning S in 'ranking-ordered' essential coalitions.

Next we calculate $m_{ji}^{\theta^\pi}$. On one hand, suppose that $1 \leq i \leq r$. First suppose that $j < m$. In such case, $m_{ji}^{\theta^\pi} = \omega_{A^d}(\{k\text{-}l : \theta(k\text{-}l) \leq \theta(j\text{-}i)\}) - \omega_{A^d}(\{k\text{-}l : \theta(k\text{-}l) > \theta(j\text{-}i)\})$ is equal to

$$\begin{aligned}
& \omega_{A^d}(\{m\text{-}n, ..., 1\text{-}n, ..., m\text{-}i, ..., (j+1)\text{-}i, j\text{-}i\}) \\
& -\omega_{A^d}(\{m\text{-}n, ..., 1\text{-}n, ..., m\text{-}i, ..., (j+1)\text{-}i\}) \\
= & \sum_{l=i}^{r} \max\left\{0, \sum_{q=1}^{j-1} d_{q(l+1)} + \sum_{q=j}^{m} d_{ql}\right\} - \sum_{l=i}^{r} \max\left\{0, \sum_{q=1}^{j} d_{q(l+1)} + \sum_{q=j+1}^{m} d_{ql}\right\} \\
= & \sum_{l=i}^{r-1} \left(\sum_{q=1}^{j-1} d_{q(l+1)} + \sum_{q=j}^{m} d_{ql}\right) + \phi(j-1) - \sum_{l=i}^{r-1} \left(\sum_{q=1}^{j} d_{q(l+1)} + \sum_{q=j+1}^{m} d_{ql}\right) - \phi(j) \\
= & d_{ji} - d_{jr} + \phi(j-1) - \phi(j),
\end{aligned}$$

where again the penultimate equality holds because of (3). Using Proposition 5 we obtain $m_{ji}^{\theta^\pi} = x_{ji}^{\pi}$ for $1 \leq i \leq r$. The case $j = m$ can be analogously proved and it is left to the reader. On the other hand, suppose now that $r+1 \leq i \leq n$. From (3) and the way d is arranged,

$$\begin{aligned}
m_{ji}^{\theta^\pi} &= \omega_{A^d}(\{m\text{-}n, ..., 1\text{-}n, ..., m\text{-}i, ..., (j+1)\text{-}i, j\text{-}i\}) \\
&\quad -\omega_{A^d}(\{m\text{-}n, ..., 1\text{-}n, ..., m\text{-}i, ..., (j+1)\text{-}i\}) \\
&= 0 - 0 = 0.
\end{aligned}$$

Again, by Proposition 5 we obtain $m_{ji}^{\theta^\pi} = x_{ji}^{\pi}$ for $r+1 \leq i \leq n$. That is, we have proved that $\{x^\pi(\omega_{A^d})\}_{\pi \in \Pi} \subseteq \{x^\theta(\omega_{A^d})\}_{\theta \in \Theta(N^1 \cup ... \cup N^m)}$. Theorem 6 completes the proof. ■

Finally, Demange (1982) and Leonard (1983) show that any agent of a classical assignment game attains her marginal contribution in the core. Again, this result does not extend to arbitrary multi-sided assignment games[8]. Now, consider the 3-SAP (observe that it is not a truncated additive m-SAP!) given by the following matrix, where the optimal matching is

[8] Sherstyuk (1999) shows that if a supermodular m-SAP $(N^1, ..., N^m; A)$ is either monotonic or a valuation then there is at least one core allocation such that all agents of one type get their marginal contribution.

marked in bold:

$$A = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{10} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & \mathbf{0} \end{pmatrix}. \tag{22}$$

It can be checked that the marginal contribution of agent *3-2* is 9 and by means of linear programming the maximum payoff that agent *3-2* can get in the core is 7.

5 Dimension of the core

The dimension of the core of two-sided assignment games is an issue that has captured some interest. For instance, Nuñez and Rafels (2008) characterize the dimension of the core of any two-sided assignment game in terms of the number of equivalence classes of the transitive closure of a reflexive and symmetric binary relationship defined on any of the two types of agents. However, as far as we know, nothing is known yet about the dimension of the core of an arbitrary multi-sided assignment game. This section is devoted to study the dimension of the core of the truncated additive m-SAPs, which represents a benchmark from which obtain more general results.

Before we prove the main result of the section, and for the sake of simplicity, we show that the core of an truncated additive multi-sided assignment game can be bijected to a specific region of $\mathbb{R}^m$.

Proposition 8 *Let $(N^1, ..., N^m; A^d)$ be a truncated additive m-SAP. Then, $C(\omega_{A^d})$ can be bijected to the region $C_m(\omega_{A^d})$ of $\mathbb{R}^m$ defined on the variables $x_{11}, ..., x_{m1}$ by the intersection of the m-cube $\prod_{j=1}^{m}[d_{j1} - d_{jr}, d_{j1} - d_{j(r+1)}]$ with the hyperplane $\sum_{j=1}^{m} x_{j1} = \sum_{j=1}^{m} d_{j1}$, where we assume that, for $1 \leq j \leq m$, $d_{j(r+1)} = -\infty$, if $r = n$.*

Proof. From Proposition 3, it is enough to prove that there is a bijection between $C_{m-1}(\omega_{A^d})$ and $C_m(\omega_{A^d})$.

On one hand, let $x_{m-1} = (x_{11}, ..., x_{(m-1)1}) \in C_{m-1}(\omega_{A^d})$. Then let

$$x_m = (x_{11}, ..., x_{(m-1)1}, \overbrace{\sum_{j=1}^{m} d_{j1} - \sum_{j=1}^{m-1} x_{j1}}^{x_{m1}}) \in \mathbb{R}^m.$$

Obviously, there is a bijection between x_{m-1} and x_m. Then we claim that $x_m \in C_m(\omega_{A^d})$. Indeed, from definition we have that $\sum_{j=1}^{m} x_{j1} = \sum_{j=1}^{m-1} d_{j1}$. Then, it only remains to check that $d_{m1} - d_{mr} \leq x_{m1} \leq d_{m1} - d_{m(r+1)}$. However, using that $x_{m1} = \sum_{j=1}^{m} d_{j1} - \sum_{j=1}^{m-1} x_{j1}$ the two later inequalities hold respectively from (11) and (12).

On the other hand, let $x_m = (x_{11}, ..., x_{m1}) \in C_m(\omega_{A^d})$. We claim that $x_{m-1} = (x_{11}, ..., x_{(m-1)1}) \in C_{m-1}(\omega_{A^d})$. Indeed, we only have to prove that (11) and (12) hold. From $d_{m1} - d_{mr} \leq x_{m1} \leq d_{m1} - d_{m(r+1)}$ and $\sum_{j=1}^{m} x_{j1} = \sum_{j=1}^{m-1} d_{j1}$ we have that

$$d_{m1} - d_{mr} \leq d_{m1} + \sum_{j=1}^{m-1} d_{j1} - \sum_{j=1}^{m-1} x_{j1} \leq d_{m1} - d_{m(r+1)},$$

which, rearranging terms, is equivalent to

$$d_{mr} + \sum_{j=1}^{m-1} d_{j1} \geq \sum_{j=1}^{m-1} x_{j1} \geq d_{m(r+1)} + \sum_{j=1}^{m-1} d_{j1}.$$

Hence, (11) and (12) hold. ■

Next, we introduce two definitions.

Definition 9 *Given a truncated additive m-SAP $(N^1, ..., N^m; A^d)$, we say that agent j-i is active*[9] *if $d_{ji} > d_{j(r+1)}$, where, if $r = n$, we assume that $d_{j(r+1)} = -\infty$ for all $1 \leq j \leq m$.*

Let Λ^j be the set of active agents of type j, for all $1 \leq j \leq m$. Active agents are those that, in any optimal matching, contribute to create a strictly positive worth.

Definition 10 *Given a truncated additive m-SAP $(N^1, ..., N^m; A^d)$, we say that type j is degenerate if $|\Lambda^j| < r$. We denote by m_d the number of degenerate types of a m-SAP.*

[9] In Nuñez and Rafels (2005) an agent j-i is said to be *active* if $i \leq r$. Otherwise, the agent is inactive. This definition assumes that 'ties' between agents productivities are broken. However, if $d_{jr} = d_{j(r+1)}$ agents j-r and j-$(r+1)$ should be interchangeable. Thus, in ours both these agents are said to be inactive.

Observe that a type j is degenerate if and only if $r < n$ and $d_{jr} = d_{j(r+1)}$. The following corollary shows that such types of agents are important because agents belonging to them always get a constant payoff in the core.

Corollary 11 *Let $(N^1, ..., N^m; A^d)$ be a truncated additive m-SAP. Then, each agent of a degenerate type j receives a constant payoff in the core. Moreover, if j-i is active her payoff is $d_{ji} - d_{jr}$. Otherwise, she receives 0.*

Proof. Trivial from Proposition 8. ■

That is, although r agents of each type contribute to generate a strictly positive profit, in a degenerate type there are less than r agents that can appropriate of some of this positive profit.

An hyperplane $\pi : a_1x_1 + ... + a_kx_k = b$ of $\mathbb{R}^k$ separates the space $\mathbb{R}^k$ into two half-spaces $\pi^+ : a_1x_1 + ... + a_kx_k \geq b$ and $\pi^- : a_1x_1 + ... + a_kx_k \leq b$. Then, we say that a point $z = (z_1, ..., z_k) \in \mathbb{R}^k$ is above (resp. below) the hyperplane π if it belongs to π^+ (resp. π^-). Lastly, if z belongs to π^+ (resp. π^-) but not to π^- (resp. π^+) then we say it is strictly above (resp. below) π.

Now we are in position to prove the main result regarding the dimension of the core.

Theorem 12 *Let $(N^1, ..., N^m; A^d)$ be a truncated additive m-SAP. Then, either $C(\omega_{A^d})$ is a point or it has dimension*[10] *$m - 1 - m_d$. Moreover, the necessary and sufficient conditions for $C(\omega_{A^d})$ to be a point are either "$r < n$ and $\sum_{j=1}^m d_{j(r+1)} = 0$" or "$m_d = m - 1$".*

From Proposition 8, to find the dimension of $C(\omega_{A^d})$ it is enough to study the region $C_m(\omega_{A^d})$ of $\mathbb{R}^m$. Also observe that a degenerate type reduces the dimension of the m-cube $\prod_{j=1}^m [d_{j1} - d_{jr}, d_{j1} - d_{j(r+1)}]$ in one unit. Let $\pi : \sum_{j=1}^m x_{j1} = \sum_{j=1}^m d_{j1}$. Because of (3), is is straightforward to check that the 'inferior' vertex of the m-cube $(d_{11} - d_{1r}, ..., d_{m1} - d_{mr})$ is strictly below π, whereas the 'superior' vertex of the m-cube $(d_{11} - d_{1(r+1)}, ..., d_{m1} - d_{m(r+1)})$

[10] Following Rockafellar (1970), recall that the dimension of the core as a convex polytope is the dimension of the minimal affine variety in which it is contained.

is above π. On one hand, if $r < n$ and $m_d < m-1$ the 'superior' vertex is below π if and only if $\sum_{j=1}^{m} d_{j(r+1)} = 0$. In such case, the only point of $C_m(\omega_{A^d})$ is the 'superior' vertex itself. Furthermore, when this last condition does not hold, the (nonempty) intersection of π with the m-cube is a space with dimension equal to the dimension of the m-cube minus 1. Lastly, from $\sum_{j=1}^{m} x_{j1} = \sum_{j=1}^{m-1} d_{j1}$ we know that if $m_d = m-1$, i.e. if $m-1$ variables are constant in $C_m(\omega_{A^d})$, the remaining variable is also constant in $C_m(\omega_{A^d})$ and therefore the core must reduce to one point.

Notice that when $m_d = m-1$ the core $C(\omega_A)$ reduces to a point although $\sum_{j=1}^{m} d_{j_{(r+1)}} = 0$ needs not hold. To a better understanding the reader may find the core of the truncated additive 3-SAP generated from $d = (1,0;1,1,1;1,1,1)$ and check that it reduces to $(3,2;0,0,0;0,0,0)$.

Corollary 13 *Let* $(N^1, ..., N^m; A^d)$ *be a truncated additive m-SAP. Then,* $C(\omega_{A^d})$ *has at most dimension* $m-1$.

Proof. Trivial from Theorem 12. Once m fixed, the highest dimensional core is obtained by considering a truncated additive m-SAP with no degenerate types. ■

In particular, we obtain that the core of a 2-sided Böhm-Bawerk horse market is a segment.

Corollary 14 *For any* $0 \leq k \leq m-1$ *there is a truncated additive m-SAP* $(N^1, ..., N^m; A^d)$ *such that* $\dim(C(\omega_{A^d})) = k$.

Proof. Observe that for any given number of types m, by considering a suitable number of degenerate types we can find truncated additive m-SAPs such that the dimension of its core ranges from 0 to $m-1$. ■

6 Competitive prices of multi-sided assignment markets

In this final section, since it is costless to our purpose, we consider arbitrary assignment markets, in which goods may not be homogeneous. The only difference with respect to the case of homogeneous goods is that the willingness-to-pay of any buyer may depend on who she buys the objects from, i.e. we must use $w^{i_m}_{i_1 \ldots i_{m-1}}$ (instead of just w^{i_m}) to refer to the willingness-to-pay of the i_m-th buyer for the $(m-1)$-tuple of goods $(i_1, ..., i_{m-1}) \in N^1 \times ... \times N^{m-1}$. Then, an arbitrary assignment market is denoted by $AM(c_1, ..., c_{m-1}; w)$, where $w \in \mathcal{M}_{n_m \times k}(\mathbb{R}_+)$, $k = n_1 \cdot ... \cdot n_{m-1}$ is the matrix of willingness-to-pay of buyers and, for all $1 \leq j \leq m-1$, $c_j \in \mathbb{R}^{n_j}$ is the vector of unitary costs of firms of type j. We can associate to any m-sided market $AM(c_1, ..., c_{m-1}; w)$ a m-SAP $(N^1, ..., N^m; A)$, where $N^1, ..., N^{m-1}$ are the sets of firms of different sectors, N^m is the set of buyers and $A(i_1, ..., i_m) = \max\left\{0, w^{i_m}_{i_1 \ldots i_{m-1}} - \sum_{j=1}^{m-1} c_{j i_j}\right\}$, for all $(i_1, ..., i_m) \in N^1 \times ... \times N^m$. Without loss of generality, we will assume that there is the same number of buyers and firms of each sector.

In general multi-sided assignment markets, prices are the relevant variables, together with the quantity (0 or 1) of units of a good sold by each firm. Roth and Sotomayor (1990) introduce the notion of *competitive prices* and show that, for any two-sided assignment market, such prices can always be obtained from core allocations. A price that is obtained from a core allocation -in a way that will be shown below- will be called a *core price*. We will prove that, in fact, for arbitrary m-sided assignment markets (i.e. either with homogeneous goods or not), the set of competitive prices coincides with the set of core prices.

Formally, given an arbitrary m-sided assignment market $AM(c_1, ..., c_{m-1}; w)$ and its associated m-SAP $(N^1, ..., N^m; A)$, for each $x \in C(\omega_A) \subset \mathbb{R}^{nm}_+$ we define a unique vector of prices (one price for each firm) $p^x = (p^x_1, ..., p^x_{m-1}) \in \mathbb{R}^{n(m-1)}_+$ by $p^x_{ji} = x_{ji} + c_{ji}$, for $1 \leq j \leq m-1$ and $1 \leq i \leq n$. We say that $P(c_1, ..., c_{m-1}; w) = \{p^x : x \in C(\omega_A)\}$ is the set of *core prices* of $AM(c_1, ..., c_{m-1}; w)$.

On the other hand, we follow Roth and Sotomayor (1990) and define the *demand set* of the i-th buyer at prices $p = (p_1, ..., p_{m-1}) \in \mathbb{R}_+^{n(m-1)}$ by[11]

$$D_i(p) = \arg\max_{(i_1,\dots,i_{m-1})\in N^1\times\ldots\times N^{m-1}} \left\{ w^i_{i_1\ldots i_{m-1}} - \sum_{j=1}^{m-1} p_{ji_j} > 0 \right\}.$$

We say that a price vector $p \in \mathbb{R}_+^{n(m-1)}$ is *competitive* if 1) $p_{ji} \geq c_{ji}$ for $1 \leq j \leq m-1$ and $1 \leq i \leq n$, and there is a matching[12] $\mu = \{(\mu_1(i), ..., \mu_{m-1}(i), i) : 1 \leq i \leq n\} \in \mathcal{M}(N^1, ..., N^m)$ such that 2) if $D_i(p) \neq \varnothing$ then $(\mu_1(i), ..., \mu_{m-1}(i)) \in D_i(p)$ and 3) if $D_i(p) = \varnothing$ then $p_{j\mu_j(i)} = c_{j\mu_j(i)}$, for all $1 \leq j \leq m-1$. Any matching μ satisfying 2) and 3) is said to be *compatible* with p. We next prove that competitive prices are core prices, and vice versa.

Theorem 15 *Let $AM(c_1, ..., c_{m-1}; w)$ be an arbitrary m-sided assignment market. Then, the set of core prices coincides with the set of competitive prices.*

Proof. Let $(N^1, ..., N^m; A)$ be the m-SAP associated to $AM(c_1, ..., c_{m-1}; w)$. First we prove that if $p \in \mathbb{R}^{n(m-1)}$ is a competitive price then it is a core price, that is, $p = p^x$ for some $x \in C(\omega_A)$. Indeed, suppose, without loss of generality, that $\mu = \{(l, ..., l) : 1 \leq l \leq n\}$ is a compatible matching with p. Then define $x^p \in \mathbb{R}^{nm}$ by $x^p_{ji} = p_{ji} - c_{ji}$, for all $1 \leq j \leq m-1$, and $x^p_{mi} = w^i_{i\ldots i} - \sum_{j=1}^{m-1} p_{ji}$ if $D_i(p) \neq \varnothing$ and $x^p_{mi} = 0$ otherwise, for all $1 \leq i \leq n$. Since p is competitive, and by the definition of the demand set, $x^p \geq 0$. Furthermore, we have that, for all $(i_1, ..., i_{m-1}) \in N^1 \times ... \times N^{m-1}$,

$$\sum_{j=1}^{m-1} x^p_{ji_j} + x^p_{mi} \geq \sum_{j=1}^{m-1} \left(p_{ji_j} - c_{ji_j}\right) + w^i_{i_1\ldots i_{m-1}} - \sum_{j=1}^{m-1} p_{ji_j} = w^i_{i_1\ldots i_{m-1}} - \sum_{j=1}^{m-1} c_{ji_j},$$

which, by nonnegativeness of x^p, implies that $\sum_{j=1}^{m-1} x^p_{ji_j} + x^p_{mi} \geq A(i_1, ..., i_{m-1}, i)$, where the inequality is tight if $(i_1, ..., i_{m-1}, i) = (i, ..., i, i)$. Hence, $x^p(N) \geq \sum_{(i_1,\dots,i_m)\in\mu'} A(i_1, ..., i_m)$ for any $\mu' \in \mathcal{M}(N^1, ..., N^m)$ and, therefore, $x^p(N) \geq \omega_A(N)$. Lastly, since p is competitive

[11] Formally, $D_i(p)$ corresponds to those $(i_1, ..., i_{m-1}) \in N^1 \times ... \times N^{m-1}$ maximizing $w^i_{i_1\cdots i_{m-1}} - \sum_{j=1}^{m-1} p_{ji_j}$, whenever this last expression is positive. Otherwise, $D_i(p)$ will be the empty set.

[12] We follow Quint's (1991) notation, $\mu_j(i)$ being the agent of type j that is assigned together with agent m-i under μ, for all $1 \leq j \leq m-1$.

we have that, if $D_i(p)=\varnothing$, then $x^p_{ji}=p_{ji}-c_{ji}=0$, for all $1\leq j\leq m-1$. Hence,

$$x^p(N)=\sum_{\substack{i=1,\\ D_i(p)\neq\varnothing}}^{n} x^p_{ji}=\sum_{\substack{i=1,\\ D_i(p)\neq\varnothing}}^{n} A(i,...,i,i)\leq \omega_A(N).$$

In conclusion, $x^p\in C(\omega_A)$ and p is a core price, since we trivially have that $p^{x^p}=p$.

Second, we prove that if p^x is a core price, i.e. $p^x_{ji}=c_{ji}+x_{ji}$ for all $1\leq j\leq m-1$ and $1\leq i\leq n$, where $x\in C(\omega_A)$, then it is a competitive price. Indeed, suppose without loss of generality that $\mu=\{(l,...,l):1\leq l\leq n\}$ is an optimal matching and that there is $s\in\{1,...,n\}$ such that $A(i,...,i)>0$ for $1\leq i\leq s$ and $A(i,...,i)=0$ otherwise[13]. Since $x\in C(\omega_A)$, we have that $p_x-c=x\geq 0$. Next we prove that μ is a compatible matching with p^x. Indeed, if $1\leq i\leq s$ we have that, for all $(i_1,...,i_{m-1})\in N^1\times...\times N^{m-1}$,

$$\begin{aligned}
w^i_{i_1\dots i_{m-1}}-\sum_{j=1}^{m-1}p^x_{ji_j} &= w^i_{i_1\dots i_{m-1}}-\sum_{j=1}^{m-1}x_{ji_j}-\sum_{j=1}^{m-1}c_{ji_j}-x_{mi}+x_{mi}\\
&\leq w^i_{i_1\dots i_{m-1}}-A(i_1,...,i_{m-1},i)-\sum_{j=1}^{m-1}c_{ji_j}+x_{mi}\\
&= w^i_{i_1\dots i_{m-1}}-\sum_{j=1}^{m-1}c_{ji_j}-\max\left\{0,w^i_{i_1\dots i_{m-1}}-\sum_{j=1}^{m-1}c_{ji_j}\right\}+x_{mi}\\
&\leq x_{mi},
\end{aligned}$$

where the first inequality holds since $x\in C(\omega_A)$ and is tight if $(i_1,...,i_{m-1},i)=(i,...,i,i)$, and the last inequality is tight if $w^i-\sum_{j=1}^{m-1}c_{ji_j}\geq 0$. Since $1\leq i\leq s$, we have that $A(i,...,i)>0$, which implies that $(i,...,i)\in D_i(p)$, that is therefore nonempty. Lastly, if $s+1\leq i\leq n$, we have that $\sum_{j=1}^{m}x_{ji}=A(i,...,i)=0$, which implies that $p^x_{ji}=c_{ji}$ for all i,j, $1\leq j\leq m-1$ and $s+1\leq i\leq n$. ■

Finally, we focus our attention on the particular case of m-sided assignment markets with homogeneous goods. For those markets, their associated m-SAPs are truncated additive (see Section 3). Hence, we are in the position to describe how the set of competitive prices for these type of markets is. Specifically, by Theorem 3 we know that, for any m-sided market with homogeneous goods, the set of core prices -or equivalently, the set of

[13]When the m-SAP is truncated additive, s is precisely defined in (3) and is denoted by r.

competitive prices- is always nonempty. Moreover, suppose without loss of generality that the truncated additive m-SAP associated to an arbitrary assignment market with homogeneous goods $AM(c_1, ..., c_{m-1}; w)$ is $(N^1, ..., N^m; A^d)$, where vector d is equal to $(-c_1, ..., -c_{m-1}, w)$ and w is the vector of willingness-to-pay of buyers -it is a vector rather than a matrix because goods are homogeneous-. Assuming that $d_1, ..., d_m$ are arranged (as usual) in a non-decreasing way, let r be the largest $1 \leq i \leq n$ such that $w^i - \sum_{j=1}^{m-1} c_{ji} > 0$. Then, by (7), (8), (9) and Theorem 3, for each competitive price p we have that 1) all firms with index not greater than r of the same type, j, must charge the same price, i.e. $p_{j1} = ... = p_{jr} = p_j$ for all $1 \leq j \leq m-1$ and 2) any firm with index greater than r must charge its own unitary cost, i.e. $p_{ji} = c_{ji}$ for $1 \leq j \leq m-1$ if $r+1 \leq i \leq n$.[14] Parameter r accounts for the maximum number of benefit-generating transactions that can occur simultaneously in the market. Furthermore, also by Theorem 3, $p_1, ..., p_{m-1}$ must be such that $w_{r+1} \leq \sum_{j=1}^{m-1} p_j \leq w_r$ and $c_j^r \leq p_j \leq c_j^{r+1}$ for all $1 \leq j \leq m-1$.[15]

[14] We have not used $p_1, ..., p_{m-1}$ instead of $x_{11}, ..., x_{(m-1)1}$ throughout the paper because, as we have already shown in Section 3, truncated additive m-SAPs embody more general frameworks than m-sided assignment markets with homogeneous goods.

[15] If $r = n$, the restrictions are $\sum_{j=1}^{m-1} p_j \leq w_r$ and $c_j^r \leq p_j$.

7 Appendix

Equations (4) form a shstem of r^m linear equations on the $r{\cdot}m$ variables $x_{11}, ..., x_{1r}, ..., x_{m1}, ..., x_{mr}$:

$$Hx = z \tag{23}$$

where $H \in M_{r^m \times r \cdot m}(\{0,1\})$ is obtained by joining all the rows $h^k \in \{0,1\}^{r \cdot m}$ satisfying that for any $1 \leq j \leq m$ there is only one non-zero element $h^k_{ji_j}$ and $z^t \in \mathbb{R}^{r \cdot m}$ is defined by $z_k = \sum_{j=1}^m d_{ji_j}$where $\{i_1, ..., i_m\}$ are the non-zero elements of t^k, for all $k = 1...r^m$. Then,

Lemma 16 *The rank of H is $1 + m \cdot (r-1)$.*

Proof. Consider the following $1+m\cdot(r-1)$ vectors (where the semicolon is used to separate the payoffs to the agents of different types):

$$\begin{aligned}
v^{1,1^T} &= (1,0,0,...,0;1,0,...,0;...;1,0,...,0) \\
v^{1,2^T} &= (0,1,0,...,0;1,0,...,0;...;1,0,...,0) \\
v^{1,3^T} &= (0,0,1,...,0;1,0,...,0;...;1,0,...,0) \\
&\quad ... \\
v^{1,r^T} &= (0,0,...,0,1;1,0,...,0;...;1,0,...,0) \\
v^{2,2^T} &= (1,0,0,...,0;0,1,...,0;...;1,0,...,0) \\
&\quad ... \\
v^{2,r^T} &= (1,0,0,...,0;0,0,...,1;...;1,0,...,0) \\
&\quad ... \\
v^{m,1^T} &= (1,0,0,...,0;1,0,...,0;...;0,1,...,0) \\
&\quad ... \\
v^{m,r^T} &= (1,0,0,...,0;1,0,...,0;...;0,0,...,1).
\end{aligned}$$

By definition, every $v^{j,i}$ satisfies that for any $1 \leq k \leq m$ there is only one non-zero element $v^{j,i}_{ki_k}$. Hence, all of them are rows of H.

On one hand, we claim that the above $1 + m \cdot (r-1)$ vectors are linearly independent. Indeed, suppose they are not. Then there must be a vector of constants

$$\overrightarrow{\lambda} = (\lambda_{11}, ..., \lambda_{1r}, \lambda_{22}, .., \lambda_{2r}, ..., \lambda_{m2}, .., \lambda_{mr}) \neq \overrightarrow{0}^T_{1+m\cdot(r-1)}$$

such that

$$\lambda_{11} v^{1,1} + \sum_{k=1}^{m} \sum_{i=2}^{r} \lambda_{ji} v^{j,i} = \overrightarrow{0}^T_{r\cdot m}.$$

Then it is easy to check that, for all $1 \leq j \leq m$ and $2 \leq i \leq r$, $\lambda_{ji} = 0$ whereas

$$\lambda_{11} + \sum_{k=2}^{m} \sum_{i=2}^{r} \lambda_{ki} = 0.$$

Therefore $\lambda_{11} = 0$. Hence we have that $\overrightarrow{\lambda} = \overrightarrow{0}^T_R$, which is a contradiction.

On the other hand, take any other row v of H. From definition, for each type only one coordinate of v related to this type is non-zero. Let $i_1, ..., i_m$ be such that for any $1 \leq j \leq m$ and $1 \leq i_j \leq r$, v_{ji_j} is the only non-zero component of type j. For simplicity of notation we have use $v^{2,1} = ... = v^{m,1}$ to refer to the same vector $v^{1,1}$. Then it is easy to check that

$$\sum_{j=1}^{m} v^{j,i_j} - (m-1) \cdot v^{1,1} = v^T.$$

■

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