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Von Neumann–Morgenstern stable-set solutions in the assignment market

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Abstract

Existence of von Neumann–Morgenstern solutions (stable sets) is proved for any assignment game. For each optimal matching, a stable set is defined as the union of the core of the game and the core of the subgames that are compatible with this matching. All these stable sets exclude third-party payments and form a lattice with respect to the same partial order usually defined on the core.

Key words: assignment game, core, dominance, von Neumann-Morgenstern stable set
PACS: C71

1 Introduction

Von Neumann and Morgenstern (1944) introduced a main solution concept for cooperative games with transferable utility. Their solution is based on a dominance relation between imputations, that is allocations of the worth of the grand coalition. One imputation dominates another if there exists a coalition such that each of its members gets more in the first imputation than in the second one, and this payoff is feasible for this coalition since it does not exceed the worth it can obtain by its own. With this definition, a von Neumann-Morgenstern solution (a stable set) V is a set of imputations satisfying (i) internal stability, no two imputations in the set dominate one

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another, and (ii) external stability, every imputation outside V is dominated by some imputation in V. These two conditions can be put together by saying that V is a stable set if it is the set of imputations not dominated by any element in V.

Nevertheless, the core and not the stable sets has become the most commonly used set-solution concept applied to TU cooperative games. The reason may be that while the core of a cooperative game can be easily defined by a set of linear inequalities, what has prevented the theory of the stable sets from being more successful is that it is so difficult to work with. Aumann (1985) wrote that "finding stable sets involves a new *tour de force* of mathematical reasoning for each game or class of games that is considered. And because stable sets do not always exist, you cannot even be sure that you are looking for something that is there". The first examples of a game with no stable set were provided by Lucas (1968, 1969). And some of these examples are in fact related to market situations. We will show in this paper that bilateral assignment markets do not present this drawback.

The core was also initially defined by means of the above dominance relation (Gillies, 1959). Provided it is nonempty, the core is the set of undominated imputations. Thus, when we choose the core as a set-solution concept we exclude from the solution an imputation that is dominated by some other imputation, although this other imputation may also be out of the core. In that case, the argument for excluding the first imputation from our solution is rather weak. Of course, this drawback is saved if the core is a stable set, since then the core coincides with the set of imputations undominated by any other core imputation. But, except for some particular classes of games, for instance convex games, the core is not a stable set.

The aim of this paper is to prove the existence of von Neumann-Morgenstern solutions for the class of assignment games. Assignment games were introduced by Shapley and Shubik (1972) as a cooperative model for two-sided markets. In this market each seller wants to sell one unit of an indivisible good, and each buyer wants to buy at most one unit. Units are non-homogeneous and thus buyers may value differently the units of the different sellers. From these valuations, and the reservation prices of the sellers, an assignment matrix is deduced which represents the profit that each mixed-pair can attain by the trade of the object between the members of the pair. Then, the worth of the grand coalition is the total profit that can be obtained by optimally matching buyers to sellers, and the worth of any other coalition is similarly obtained, just restricting to the corresponding submatrix.

In their paper "The Assignment Game I: The Core", Shapley and Shubik prove that the core of the assignment game is nonempty. Moreover, together with efficiency, only coalitional rationality for mixed-pair coalitions is needed to define the core of the assignment game. As a consequence of that, although the core definition allows side payments among all the participants, it turns out that in the core of the assignment games transfers of money are only made between optimally matched agents and thus third-party payments are excluded. Then, the core is a lattice with respect to a defined partial ordering on the set of imputations, and there are two special core elements: the buyers-optimal core allocation, where each buyer gets her maximum core payoff and each seller his minimum one, and the sellers-optimal core allocation where the situation is reversed.

At the end of their paper, Shapley and Shubik point out that the core does not recognize the bargaining power of all the agents, and a simple example of this weakness of the core is provided by the glove market. A glove market is an assignment game where units are homogeneous and agents on each side are symmetric and thus the assignment matrix is a constant matrix. In any core allocation all the profit is given to the short side of the market. For instance in a glove market with one seller and two buyers, the unique core allocation gives all the profit to the seller, while buyers get zero. Thus, the core does not take into account the fact that the seller needs the cooperation of at least one buyer to make any profit.

In 1959, Shapley describes most of the von Neumann–Morgenstern stable sets of a symmetric market game (glove market). There exist stable sets where only extra players on the large side of the market get zero, all matched pairs share the profit in the same way and third-party payments are excluded. But there also exist (infinitely many) stable sets where agents on the large side of the market cooperate to get their bargaining power recognized by means of side-payments. Again, the core gives no hint of this.

It is said that Shapley and Shubik planned to work on a second paper about the assignment game, probably focused on the stable sets. At the end of Shapley and Shubik (1972) the authors write: "It may not be possible to realize the bargaining potentials described above within a given institutional form. When we are conducting a general analysis of the abstract model, as here, it behooves to us to explore and correlate a number of different solution concepts. This we hope to do in subsequent papers."

Although this second paper never appeared, they probably worked on the subject since in Shubik (1984), and also in some personal notes of Shapley, a set of imputations is proposed as a stable set for the assignment game. However, this claim is not accompanied by a complete proof. As far as we know, proving the existence of von Neumann–Morgenstern stable sets for the assignment game is still an open problem (see for instance Solymosi and Raghavan, 2001), and this is what we aim to close with the present paper.

The set defined by Shubik consists of the union of the core of the assignment market and the core of some selected submarkets that are compatible with an optimal matching that has been fixed beforehand. Although our definition of compatible subgame slightly differs from that of Shubik, the set we prove to be a von Neumann and Morgenstern stable set is the same that he proposes.

In 2001, Solymosi and Raghavan characterized those assignment games with a stable core. They proved that the core of an assignment game is a stable set if and only if the assignment matrix has a dominant diagonal (each diagonal entry is a row and column maximum). When the matrix did not satisfy this property, it still remained to see what imputations should be added to the core in order to obtain a stable set.

In Núñez and Rafels (2009) we show how to relate, to any assignment game, another one (the related exact assignment game) with a dominant diagonal matrix and a core that is a translation of the core of the initial game. Thus, the related exact assignment game has a stable core, and this will be used in this paper to dominate the imputations outside the core but still inside the limits of the core bounds. The strategy to dominate the imputations outside these limits will be totally different, but also makes use of other techniques developed in the recent literature on assignment games.

Any payoff vector in the stable set we construct can be achieved without any side payments other than the direct payments from each buyer to her assigned seller. Thus, as it happens with the core, third-party payments are excluded. The lattice structure is another property that the core shares with the stable set.

In the last years, and for different cooperative settings, attention has turned to von Neumann and Morgenstern stable sets. For instance, Ehlers (2007) gives a characterization of stable sets in one-to-one matching problems (the marriage problem). In this setting, as far as we know, the existence of a stable set is still an open problem.

In Section 2 the basic definitions regarding cooperative games and assignment games are given. For a given optimal matching μ, the μ-compatible subgames and their properties are introduced in Section 3. In Section 4, and for each optimal matching μ, the union of the core of the assignment market and the cores of the μ-compatible subgames is proved to be a von Neumann and Morgenstern stable set. To conclude this section, we prove the lattice property of the stable set and remark that this stable set is the only one that excludes third party payments.

2 Cooperative games and the assignment game

In a cooperative game with transferable utility (a game), a finite set of agents N must share the profit of cooperation, $v(N) \in \mathbb{R}$, by agreeing in a payoff vector x, where x_i stands for the payoff to agent $i \in N$. To reach an agreement, what each coalition $S \subseteq N$ can attain by itself, $v(S) \in \mathbb{R}$ (with $v(\emptyset) = 0$), can be taken into account. The set of all coalitions of N is denoted by 2^N. Thus, the game is defined by the pair (N, v) formed by the set of players and the characteristic function.

If (N, v) is a game and $S \subseteq N$, the subgame $(S, v_{|S})$ is the game with player set S and characteristic function $v_{|S}(T) = v(T)$ for all $T \subseteq S$. Then we will denote by $x_{|S}$ the restriction of the payoff vector $x \in \mathbb{R}^N$ to the agents in coalition S and by x_{-S} its restriction to the agents in $N \setminus S$. Moreover, $|S|$ will stand for the cardinality of coalition S.

An *imputation* is a payoff vector that is efficient, $x(N) = \sum_{i \in N} x_i = v(N)$, and individually rational, $x_i \geq v(i)$ for all $i \in N$. The set of imputations is denoted by $I(v)$. The *core* of the game, is the set of imputations that are coalitionally rational, $C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S),\ \text{for all } S \subseteq N\}$. A game is *exact* if the worth of any coalition is attained at some core allocation: for all $S \subseteq N$ there exists $x \in C(v)$ with $x(S) = v(S)$.

An imputation x *dominates* another imputation y *via coalition* S, $x\, dom_S^v\, y$, if $x(S) \leq v(S)$ and $x_k > y_k$ for all $k \in S$. Then, a binary relation is defined on the set of imputations: given $x, y \in I(v)$, we say x *dominates* y, and write $x\, dom^v\, y$, if it does so via some coalition (we simply write $x\, dom\, y$, if no confusion arises regarding the game). With this definition, the core, whenever it is nonempty, is proved to coincide with the set of undominated imputations. This means that all allocations outside the core are dominated, although not necessarily dominated by a core allocation.

A subset V of imputations is a *stable set* (von Neumann and Morgenstern, 1944) if it is *internally stable* (for all $x, y \in I(v)$, x does not dominate y) and *externally stable* (for all $y \in I(v) \setminus V$, there exists $x \in V$ such that $x\, dom\, y$).

Since the core is the set of undominated imputations, all the stable sets of a given game (N, v) contain its core. And when the core is a stable set, then it is the unique stable set. But existence of stable sets is not guaranteed, as shown in Lucas (1968, 1969).

The assignment game is a well-known cooperative game introduced by Shapley and Shubik (1972) from the following market situation. In a *two-sided assignment market*, the set of agents is partitioned into a finite set of buyers M and a finite set of sellers M', since the product on sale in this market comes in indivisible units and each agent either supplies or demands exactly one unit. The units need not be alike, so let $h_{ij} \geq 0$ be how much buyer i valuates the unit of seller j, and let $c_j \geq 0$ be the reservation price of this seller. The profit this mixed-pair coalition can attain is $a_{ij} = \max\{0, h_{ij} - c_j\}$, and let us denote by $A = (a_{ij})_{(i,j)\in M\times M'}$ the assignment matrix.

A *matching* for the two-sided assignment market (M, M', A) is a bijection μ between a subset of M and a subset of M'. We denote by $\mathcal{M}(M, M')$ this set of matchings. An *optimal matching* is $\mu \in \mathcal{M}(M, M')$ such that $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all other $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}^*_A(M, M')$ the set of optimal matchings for the market (M, M', A). If $(i, j) \in \mu$ we say that i and j are *matched* by μ and we also write $j = \mu(i)$ and $i = \mu^{-1}(j)$. If for some buyer $i \in M$ there is no $j \in M'$ such that $(i, j) \in \mu$ we say that i is *unmatched* by μ (and similarly for sellers).

Given $S \subseteq M$ and $T \subseteq M'$, we denote by $\mathcal{M}(S, T)$ and $\mathcal{M}^*_A(S, T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{|S\times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matching is $\mu = \emptyset$ and by convention $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

The cooperative model for the assignment market is defined by the set of players $N = M \cup M'$ and the characteristic function $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(M \cap S, M' \cap T)\}$. Shapley and Shubik prove that an assignment game $(M \cup M', w_A)$ always has a non-empty core. Moreover, fixed any optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, a non-negative payoff vector $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ is in the *core* if and only if $u_i + v_j \geq a_{ij}$ for all $(i, j) \in M \times M'$, $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$, and all agents unmatched by μ get a null payoff. This means that only coalitional rationality for mixed-pair coalitions is necessary to define the core of this game, and thus the core can be obtained from the assignment matrix, without needing to compute the whole characteristic function [2].

The core of the assignment game $(M \cup M', w_A)$ is a *lattice* with respect to the partial order defined by $(u, v) \leq_M (u', v')$ if and only if $u_i \leq u'_i$ for all $i \in M$ (and also with the dual order defined by comparing the sellers' payoffs). The maximum element in this lattice is the *buyers-optimal core allocation* $(\overline{u}^A, \underline{v}^A)$ and the minimum element is the *sellers-optimal core allocation* $(\underline{u}^A, \overline{v}^A)$.

It is known from Demange (1982) and Leonard (1983) that the maximum core payoff of an agent in the core of the assignment game is his or her marginal contribution, that is,

[2] Often in the literature, the core elements of an assignment game are called stable allocations and the constraints $u_i + v_j \geq a_{ij}$ are referred to as pairwise stability conditions. In this paper we will avoid this denominations, not to be confused with von Neumann and Morgenstern stability.

$$\overline{u}_i^A = w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M'), \text{ for all } i \in M, \quad (1)$$
$$\overline{v}_j^A = w_A(M \cup M') - w_A(M \cup (M' \setminus \{j\})), \text{ for all } j \in M'. \quad (2)$$

Solymosi and Raghavan (2001) characterize some properties of the core of the assignment game in terms of the assignment matrix. Among these properties we point out stability of the core and exactness of the game.

An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has a *dominant diagonal*[3] if and only if, for all $\mu \in \mathcal{M}^*_A(M, M')$, and all $i \in M$ assigned by μ, $a_{i\mu(i)} \geq a_{ik}$ for all $k \in M'$ and $a_{i\mu(i)} \geq a_{k\mu(i)}$ for all $k \in M$. It is straightforward to see that having a dominant diagonal is equivalent to $\underline{u}_i^A = 0$ for all $i \in M$ and $\underline{v}_j^A = 0$ for all $j \in M'$. It turns out that an assignment game has a stable core if and only if it has a dominant diagonal. Notice that, given an assignment game where the sides of the market have a different size, once its assignment matrix has been made square by adding dummy agents to the short side of the market, it does not generally have a dominant diagonal and consequently its core is not stable.

An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has a *doubly dominant diagonal* if and only if, for all $\mu \in \mathcal{M}^*_A(M, M')$, all $(i, j) \in M \times M'$ and all $k \in M$ assigned by μ, we have $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}$. Then, Solymosi and Raghavan (2001) prove that an assignment game is exact if and only if it has a dominant diagonal and a doubly dominant diagonal.

The alone property of having a doubly dominant diagonal characterizes those assignment games that are *buyer-seller exact*, that is, that satisfy exactness for mixed-pair coalitions: for all $(i, j) \in M \times M'$, there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$. In Núñez and Rafels (2002) it is shown that given an assignment game $(M \cup M', w_A)$, there exists another (and unique) assignment game $(M \cup M', w_{A^r})$ that is buyer-seller exact and has its same core, $C(w_A) = C(w_{A^r})$. This game is defined by $a^r_{ij} = \min_{(u,v)\in C(w_A)} u_i + v_j$, for all $(i, j) \in M \times M'$ but, when there are as many buyers as sellers, it can also be obtained only in terms of the matrix entries: $a^r_{ij} = \max\{a_{ij}, \tilde{a}_{ij}\}$ where

$$\tilde{a}_{ij} = \max_{\substack{k_1,\ldots,k_r \in M\setminus\{i,\mu^{-1}(j)\} \\ different}} \{a_{i\mu(k_1)} + a_{k_1\mu(k_2)} + \cdots + a_{k_r j} - a_{k_1\mu(k_1)} - \cdots - a_{k_r\mu(k_r)}\}, \quad (3)$$

and μ is an optimal matching that does not leave any agent unassigned.

Moreover, to any given assignment game we can associate an *exact assignment*

[3] The name is justified by the fact that, when the optimal matching is placed on the diagonal of the matrix, the assignment game has a dominant diagonal if and only if every diagonal entry is a row and column maxima.

game with a translated core (Núñez and Rafels, 2009). This related exact assignment game is $(M \cup M', w_{A^e})$ where, for all $(i,j) \in M \times M'$,

$$a^e_{ij} = a^r_{ij} - \underline{u}^A_i - \underline{v}^A_j \,, \tag{4}$$

and it satisfies $C(w_A) = \{(\underline{u}^A, \underline{v}^A)\} + C(w_{A^e})$. Notice that, since $(M \cup M', w_{A^e})$ is exact, it has a dominant diagonal, and thus the core of this game is stable. The stability of $C(w_{A^e})$ will be used to dominate some of the allocations in the proof of the external stability of our stable set.

Finally, before proposing a stable set for the assignment game, we must analyze some particularities of the dominance relation when applied to these games. It is straightforward to see that, in an assignment game, only mixed-pair coalitions are to be taken into account for domination. That is, if $x, y \in I(w_A)$, we have $x \, dom \, y$, if and only if $x \, dom \, y$ via some coalition $S = \{i, j\}$ where $i \in M$ and $j \in M'$, that is to say, $x_i > y_i$, $x_j > y_j$ and $x_i + y_j \leq a_{ij}$.

3 The compatible subgames

We are interested in those imputations where, as it happens to the core, transfers of money are only made between optimally matched agents and thus third-party payments are excluded. Given an assignment game $(M \cup M', w_A)$ and once fixed an optimal matching $\mu \in \mathcal{M}^*_A(M, M')$, the *$\mu$-principal section* $B^\mu(w_A)$ is defined by, given $(u,v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$, $(u,v) \in B^\mu(w_A)$ if and only if $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$, while unmatched agents are paid zero.

Notice that, the μ-principal section is in between the core and the imputation set: $C(w_A) \subseteq B^\mu(w_A) \subseteq I(w_A)$. The stable set we are going to introduce is included in the μ-principal section, once fixed an optimal μ. So, in fact, we are obtaining one stable set for each optimal matching of the market.

Remark: The domination between imputations in the μ-principal section is preserved if we make the market square by adding dummy agents on the short side. Assume $|M| < |M'|$ and define $\hat{M} \supseteq M$ with $|\hat{M}| = |M'|$ and $\hat{A}$ by $\hat{a}_{ij} = a_{ij}$ if $(i,j) \in M \times M'$ and $\hat{a}_{ij} = 0$ if $(i,j) \in (\hat{M} \setminus M) \times M'$. Take $\mu \in \mathcal{M}^*_A(M, M')$ and any $\hat{\mu} \in \mathcal{M}^*_{\hat{A}}(\hat{M}, M')$ such that the restriction of $\hat{\mu}$ to $M \times M'$ coincides with μ. Then $(u,v) \in B^\mu(w_A)$ if and only if $(\hat{u}, v) \in B^{\hat{\mu}}(w_{\hat{A}})$, where $\hat{u}_i = u_i$ for all $i \in M$ and $\hat{u}_i = 0$ for all $i \in \hat{M} \setminus M$. We have that, for all $(u,v), (u',v') \in B^\mu(w_A)$, $(u,v) dom^{w_A} (u',v')$ if and only if $(\hat{u}, v) dom^{w_{\hat{A}}} (\hat{u}', v')$.

We are now in disposition of defining the compatible subgames, which were already introduced in Shubik (1984).

Definition 1 *Let* $(M \cup M', w_A)$ *be an assignment game,* $\mu \in \mathcal{M}^*_A(M, M')$ *and let* $I \subseteq M$ *and* $J \subseteq M'$. *The subgame* $((M \setminus I) \cup (M \setminus J)), w_{A_{|(M\setminus I)\times(M\setminus J)}})$ *is a* μ*-compatible subgame of* $(M \cup M', w_A)$ *if and only if*

$$w_A((M \setminus I) \cup (M' \setminus J)) + \sum_{\substack{i \in I \\ i \text{ assigned by } \mu}} a_{i\mu(i)} + \sum_{\substack{j \in J \\ j \text{ assigned by } \mu}} a_{\mu^{-1}(j)j} = w_A(M \cup M') . \quad (5)$$

Notice that taking $I = J = \emptyset$ the game $(M \cup M', w_A)$ can be looked at as a μ-compatible subgame. Notice also that taking $I = M$ and $J = \emptyset$ (or $J = M'$ and $I = \emptyset$) we always have a μ-compatible subgame that is the null game.

In the sequel, and to simplify notation, we will write $w_{A_{-I\cup J}}$ for the characteristic function of the subgame with player set $(M \setminus I) \cup (M' \setminus J)$. To say that $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame is equivalent to saying that the restriction of μ to $(M \setminus I) \times (M' \setminus J)$ is still optimal for the resulting submarket and moreover $I \cap \mu^{-1}(J) \subseteq \{i \in M \mid a_{i\mu(i)} = 0\}$ (or equivalently $\mu(I) \cap J \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$). As a result, when $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame, all agents in $(M \setminus I) \cap \mu^{-1}(J)$ or $(M' \setminus J) \cap \mu(I)$ are unassigned by $\mu_{|(M\setminus I)\times(M'\setminus J)}$.

All this means that any payoff vector (u, v) in the μ-principal section of the compatible subgame[4] can be lifted to a payoff vector $(\hat{u}, \hat{v})$ in the μ-principal section of the initial market, only by paying $\hat{u}_i = a_{i\mu(i)}$ to all $i \in I \cap \mu^{-1}(M')$, $\hat{v}_j = a_{\mu^{-1}(j)j}$ to all $j \in J \cap \mu(M)$, and zero to all agents in $I \cup J$ not assigned by μ. In other words, $B^\mu(w_{A_{-I\cup J}})$ can be seen as a facet of $B^\mu(w_A)$.

The next example illustrates the definition of a compatible subgame. The assignment matrix is taken from Shapley and Shubik (1972), where a picture of the core of this game is provided.

Example 2 Let $M = \{1, 2, 3\}$ be the set of buyers, $M' = \{1', 2', 3'\}$ be the set of sellers, and the assignment matrix be

	1'	2'	3'
1	5	**8**	2
2	7	9	**6**
3	**2**	3	0

[4] Throughout the paper, and with some abuse of notation, when referring to the $\mu_{|(M\setminus I)\times(M'\setminus J)}$-principal section of the μ-compatible subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ we will simply write $B^\mu(w_{A_{-I\cup J}})$

The only optimal matching is $\mu = \{(1,2'),(2,3'),(3,1')\}$ and all the non-trivial μ-compatible subgames $w_{A_{-I\cup J}}$ are the ones defined by the following pairs (I,J):

$$\begin{array}{l|l} I=\{2\}\,,\ J=\emptyset & I=\emptyset\,,\ J=\{1'\} \\ I=\{2,3\}\,,\ J=\emptyset & I=\emptyset\,,\ J=\{1',2'\} \\ I=\{2\}\,,\ J=\{1'\} & \end{array}$$

To obtain a μ-compatible subgame, usually I and J cannot be simultaneously non-empty. The reason is that if agents of both sides of the market are removed (and these are not unassigned agents), then their optimal partners by μ tend to become matched in the submarket and this in general contradicts that the restriction of μ is an optimal matching of the submarket. There are, nevertheless, exceptions that occur either when the market is not square and one of the removed agents is an unassigned agent on the large side of the market, or also when $i \in I \cap \mu^{-1}(M')$, $j \in J \cap \mu(M)$ and $a_{\mu^{-1}(j)\mu(i)} = 0$. This is what happens in this example, where $I = \{2\}$ and $J = \{1'\}$ define a μ-compatible subgame, since $a_{\mu^{-1}(1')\mu(2)} = a_{33'} = 0$.

Notice that while the core of $(M \cup M', w_A)$ lies in $\mathbb{R}^M \times \mathbb{R}^{M'}$, the core of any of its compatible subgames $((M \setminus I) \cup (M \setminus J), w_{A_{-I\cup J}})$ lies in a lower dimensional space. However, any core allocation of a μ-compatible subgame can be extended to a payoff vector in the μ-principal section of the initial game. Let us denote by $\hat{C}(w_{A_{-I\cup J}})$ this *extended core*,

$$\hat{C}(w_{A_{-I\cup J}}) := \left\{ (u,v) \in B^{\mu}(w_A) \;\middle|\; \begin{array}{l} (u_{-I}, v_{-J}) \in C(w_{A_{-I\cup J}})\,, \\ u_i = a_{i\mu(i)} \text{ for all } i \in I \text{ assigned by } \mu\,, \\ v_j = a_{\mu^{-1}(j)j} \text{ for all } j \in J \text{ assigned by } \mu\,. \end{array} \right\} \quad (6)$$

Taking (6) into account, the reader will easily check that

$$\hat{C}(w_{A_{-I\cup J}}) \supseteq \left\{ (u,v) \in C(w_A) \;\middle|\; \begin{array}{l} u_i = a_{i\mu(i)} \text{ for all } i \in I \text{ assigned by } \mu \text{ and} \\ v_j = a_{\mu^{-1}(j)j} \text{ for all } j \in J \text{ assigned by } \mu \end{array} \right\} \quad (7)$$

In order to compare the above Definition 1 with Shubik's definition of compatible subgame, we need to assume for a moment that $(M \cup M', w_A)$ has as many buyers as sellers and $\mu \in \mathcal{M}_A^*(M,M')$ does not leave agents unassigned. Then the extended core of a μ-compatible subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$

turns out to coincide with the core of another game $(M \cup M', v_Q)$ defined by

$$v_Q(S) = w_A(S \cap Q) + \sum_{i \in S \cap I} a_{i\mu(i)} + \sum_{j \in S \cap J} a_{\mu^{-1}(j)j}, \text{ for all } S \subseteq M \cup M', \quad (8)$$

with $Q = (M \setminus I) \cup (M' \setminus J)$. This game $(M \cup M', v_Q)$, satisfying $v_Q(M \cup M') = w_A(M \cup M')$, is what Shubik (1984) names a compatible subgame, although it is not strictly speaking a subgame of $(M \cup M', w_A)$, since it is defined over the same set of agents, and in general it is no more an assignment game. Trivially, if for some $Q \subseteq M \cup M'$, $(M \cup M', v_Q)$ is compatible *à la* Shubik, then $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$, where $I \subseteq M$ and $J \subseteq M'$ satisfy $Q = (M \setminus I) \cup (M' \setminus J)$, is a μ-compatible subgame according to Definition 1.

This aforementioned core coincidence is stated in the next proposition, together with a relationship between the extended core of the compatible subgame and the core of the initial market that strengthes that of equation (7).

Proposition 3 *Let $(M \cup M', w_A)$ be an assignment game with as many buyers as sellers and $\mu \in \mathcal{M}_A^*(M, M')$ such that $\mu(M) = M'$. Let $I \subseteq M$ and $J \subseteq M'$ be such that $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$ is a μ-compatible subgame. Then*

(1) $\hat{C}(w_{A_{-I \cup J}}) = C(v_Q)$ where v_Q, with $Q = (M \setminus I) \cup (M' \setminus J)$, is defined as in (8).
(2) If $(M \cup M', w_A)$ has a dominant diagonal, then

$$\hat{C}(w_{A_{-I \cup J}}) = \left\{ (u,v) \in C(w_A) \;\middle|\; \begin{array}{l} u_i = a_{i\mu(i)} \text{ for all } i \in I \text{ and} \\ v_j = a_{\mu^{-1}(j)j} \text{ for all } j \in J \end{array} \right\}.$$

PROOF. 1) We first prove the inclusion $C(v_Q) \subseteq \hat{C}(w_{A_{-(I \cup J)}})$. Take $(u,v) \in C(v_Q)$ and recall the definition of v_Q in (8). For all $j^* \in J$, if we take $S = M \cup (M' \setminus \{j^*\})$ we get $(u,v)(S) \geq v_Q(S) = w_A((M \setminus I) \cup (M' \setminus J)) + \sum_{i \in I} a_{i\mu(i)} + \sum_{j \in J \setminus \{j^*\}} a_{\mu^{-1}(j)j} = w_A(M \cup M') - a_{\mu^{-1}(j^*)j^*} = \sum_{j \in M' \setminus \{j^*\}} a_{\mu^{-1}(j)j}$, where the second equality follows from the μ-compatibility of the subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$. Together with efficiency of (u,v), this implies $v_{j^*} \leq a_{\mu^{-1}(j^*)j^*}$. Since we also have $v_{j^*} \geq v_Q(\{j^*\}) = a_{\mu^{-1}(j^*)j^*}$, we get $v_{j^*} = a_{\mu^{-1}(j^*)j^*}$. Similarly we get $u_{i^*} = a_{i^*\mu(i^*)}$ for all $i^* \in I$.

Moreover, taking $S = \{i,j\}$ with $i \notin I$ and $j \notin J$, we have $u_i + v_j \geq v_Q(\{i,j\}) = w_A(\{i,j\}) = a_{ij}$. Since

$$\sum_{i \in M} u_i + \sum_{j \in M'} v_j = v_Q(M \cup M') = w_A((M \setminus I) \cup (M' \setminus J)) + \sum_{i \in I} a_{i\mu(i)} + \sum_{j \in J} a_{\mu^{-1}(j)j}$$

and $u_i = a_{i\mu(i)}$ for all $i \in I$ and $v_j = a_{\mu^{-1}(j)j}$ for all $j \in J$, we have that $\sum_{i\in M\setminus I} u_i + \sum_{j\in M'\setminus J} v_j = w_A((M\setminus I)\cup(M'\setminus J))$ and together with $u_i+v_j \geq a_{ij}$ for all $(i,j)\in \mu\cap (M\setminus I)\times(M'\setminus J)$, implies $u_i+v_j=a_{ij}$ for all $(i,j)\in\mu\cap(M\setminus I)\times(M'\setminus J)$ and $u_i=v_j=0$ for all i or j that are unassigned by $\mu_{|(M\setminus I)\times(M'\setminus J)}$. Therefore, $(u_{-I},v_{-J})\in C(w_{A_{-I\cup J}})$ and, since $(u,v)\in B^{\mu}(w_A)$ also holds, we obtain $(u,v)\in\hat{C}(w_{A_{-(I\cup J)}})$.

To prove the converse inclusion, take $(u,v)\in\hat{C}(w_{A_{-I\cup J}})$ and notice first that, by (6) and (8), $(u,v)(M\cup M') = v_Q(M\cup M')$. Also, for all $S\subseteq M\cup M'$,

$$
\begin{aligned}
(u,v)(S) &= \sum_{i\in S\cap M} u_i + \sum_{j\in S\cap M'} v_j = \sum_{i\in S\cap(M\setminus I)} u_i + \sum_{i\in S\cap I} u_i + \sum_{j\in S\cap(M'\setminus J)} v_j + \sum_{j\in S\cap J} v_j \\
&\geq w_A(S\setminus(I\cup J)) + \sum_{i\in S\cap I} a_{i\mu(i)} + \sum_{j\in S\cap J} a_{\mu^{-1}(j)j} = v_Q(S)
\end{aligned}
$$

and thus $(u,v)\in C(v_Q)$.

2) One inclusion has already been stated in (7). Assume now that $(M\cup M',w_A)$ has a dominant diagonal and let us prove the converse inclusion. We only need to see that if $(u,v)\in\hat{C}(w_{A_{-I\cup J}})$, then $(u,v)\in C(w_A)$. Notice first that from $(u,v)\in\hat{C}(w_{A_{-I\cup J}})$ we already have by definition that $(u,v)\in B^{\mu}(w_A)$.

If $(i,j)\notin\mu$, then we consider three cases. If $i\in M\setminus I$ and $j\in M'\setminus J$, then from $(u_{-I},v_{-J})\in C(w_{A_{-I\cup J}})$ we get $u_i+v_j\geq a_{ij}$. If $j\in J$, then $v_j=a_{\mu^{-1}(j)j}$ and, taking into account that $(M\cup M',w_A)$ has a dominant diagonal, we get $u_i+v_j\geq v_j = a_{\mu^{-1}(j)j}\geq a_{ij}$ for all $i\in M$. Similarly, if $i\in I$ we have that $u_i+v_j\geq u_i = a_{i\mu(i)}\geq a_{ij}$ for all $j\in M'$. □

Solymosi and Raghavan (2001) prove that the core of an assignment game is a von Neumann and Morgenstern stable set if and only if the assignment game has a dominant diagonal. When this is not the case, we wonder which imputations must be added to the core to obtain a stable set. We prove in the next section that what we must add are the extended cores of the compatible subgames, for some fixed $\mu\in\mathcal{M}^*_A(M,M')$. Notice now that, by the above proposition, when the assignment game has a dominant diagonal, the extended cores of all the compatible subgames are already included in the core of the initial market, and thus we are not really adding any new imputation to the set of core imputations.

The next proposition provides a sufficient condition for a subgame to be μ-compatible. When only one agent is removed, this condition is also necessary.

Proposition 4 *Let $(M\cup M',w_A)$ be an assignment game and $\mu\in\mathcal{M}^*_A(M,M')$.*

(1) *If* $I \subseteq M$ *and* $J \subseteq M'$ *are such that there exists* $(x,y) \in C(w_A)$ *with* $x_i = a_{i\mu(i)}$ *for all* $i \in I$ *assigned by* μ *and* $y_j = a_{\mu^{-1}(j)j}$ *for all* $j \in J$ *assigned by* μ, *then* $((M \backslash I) \cup (M' \backslash J), w_{A_{-I\cup J}})$ *is a* μ*-compatible subgame of* $(M \cup M', w_A)$.

(2) *If* $i^* \in M$ *is assigned by* μ, *the subgame* $((M \setminus \{i^*\}) \cup M', w_{A_{-\{i^*\}}})$ *is* μ*-compatible if and only if* $\overline{u}^A_{i^*} = a_{i^*\mu(i^*)}$.

(3) *If* $j^* \in M'$ *is assigned by* μ, *the subgame* $((M \cup M' \setminus \{j^*\}), w_{A_{-\{j^*\}}})$ *is* μ*-compatible if and only if* $\overline{v}^A_{j^*} = a_{\mu^{-1}(j^*)j^*}$.

PROOF. 1) Since $(x,y) \in C(w_A)$ and $x_i = a_{i\mu(i)}$ for all $i \in I$ assigned by μ, we have that $y_j = 0$ for all $j \in \mu(I)$. Similarly, from $y_j = a_{\mu^{-1}(j)j}$ for all $j \in J$ assigned by μ, we get $x_i = 0$ for all $i \in \mu^{-1}(J)$. Moreover, $x_i = 0$ and $y_j = 0$ for any $i \in M$ or $j \in M'$ unmatched by μ. Then,

$$\sum_{(i,j)\in\mu_{|(M\setminus I)\times(M'\setminus J)}} a_{ij} = \sum_{i\in M\setminus I} x_i + \sum_{j\in M'\setminus J} y_j \geq w_A((M \setminus I) \cup (M' \setminus J))$$

which implies that $\sum_{(i,j)\in\mu_{|(M\setminus I)\times(M'\setminus J)}} a_{ij} = w_A((M \setminus I) \cup (M' \setminus J))$. Moreover, if $i \in I \cap \mu^{-1}(J)$ (and similarly if $j \in J \cap \mu(I)$), we have $x_i = a_{i\mu(i)}$ and $y_{\mu(i)} = a_{i\mu(i)}$. Since $(x,y) \in C(w_A)$, $x_i + y_{\mu(i)} = 2a_{i\mu(i)} = a_{i\mu(i)}$ implies that $a_{i\mu(i)} = 0$. As a consequence, $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame.

2) We only need to prove the "only if" implication since the other one is a particular case of part 1). So assume that $((M \setminus \{i^*\}) \cup M', w_{A_{-\{i^*\}}})$ is a μ-compatible subgame and $\mu(i^*)$ exists. Then, by Definition 1, $w_A(M \cup M') = w_A((M \setminus \{i^*\}) \cup M') + a_{i^*\mu(i^*)}$, and, as a consequence, taking into account expression (1),

$$a_{i^*\mu(i^*)} = w_A(M \cup M') - w_A((M \setminus \{i^*\}) \cup M') = \overline{u}^A_{i^*}.$$

Finally, part 3) is proved analogously. □

An immediate consequence is that for a given assignment game $(M \cup M', w_A)$ and a fixed optimal matching μ, a proper μ-compatible subgame always exists.

Corollary 5 *Let* $(M \cup M', w_A)$ *be an assignment game and* $\mu \in \mathcal{M}^*_A(M, M')$.

(1) *If* $|M| \leq |M'|$ *and* $M = \mu^{-1}(M')$, *there exists* $i^* \in M$ *such that* $\overline{u}^A_{i^*} = a_{i^*\mu(i^*)}$.

(2) *If* $|M| \leq |M'|$, *there exists* $\emptyset \neq I \subseteq M$ *such that* $((M \setminus I) \cup M', w_{A_{-I}})$ *is a* μ*-compatible subgame.*

(3) *If* $|M'| \leq |M|$ *and* $M' = \mu(M)$, *there exists* $j^* \in M'$ *such that* $\overline{v}^A_{j^*} = a_{\mu^{-1}(j^*)j^*}$.

(4) If $|M'| \leq |M|$, there exists $\emptyset \neq J \subseteq M'$ such that $(M \cup (M' \setminus J), w_{A_{-J}})$ is a μ-compatible subgame.

PROOF. To prove statement 1) notice that if $\overline{u}_i^A < a_{i\mu(i)}$ for all $i \in M$, then $\underline{v}_j^A > 0$ for all $j \in \mu(M)$ and then the payoff vector $(u', v') \in \mathbb{R}^M \times \mathbb{R}^{M'}$ defined by $u'_i = \overline{u}_i^A + \varepsilon$ for all $i \in M$, $v'_j = \underline{v}_j^A - \varepsilon$ for all $j \in \mu(M)$ and $v'_j = 0$ for all $j \in M' \setminus \mu(M)$ is in $C(w_A)$ for $\varepsilon > 0$ small enough. This contradicts $\overline{u}_i^A$ being the maximum core payoff of buyer i. All this means that there exists $i^* \in M$ such that $\overline{u}_{i^*}^A = a_{i^*\mu(i^*)}$.

Let us now prove statement 2). If $M = \mu^{-1}(M)$, then by 1) and part 2) of Proposition 4 we have that, taking $I = \{i^*\}$ with $\overline{u}_{i^*} = a_{i^*\mu(i^*)}$, $((M \setminus I) \cup M', w_{A_{-I}})$ is a μ-compatible subgame. Otherwise, that is if there exists $i \in M$ unmatched by μ, then it follows straightforwardly from Definition 1 that taking $I = \{i\}$ the subgame $((M \setminus I) \cup M', w_{A_{-I}})$ is μ-compatible. □

In general, it is not true that a μ-compatible subgame of a μ-compatible subgame is in its turn a μ-compatible subgame of the initial game, unless we impose some additional condition. For instance, in Example 2, $((M \setminus \{2\}) \cup (M' \setminus \{3'\}), w_{A_{-\{2,3'\}}})$ is a μ-compatible subgame of $((M \setminus \{2\}) \cup M', w_{A_{-\{2\}}})$ but not of $(M \cup M', w_A)$.

Remark 6 *Let $(M \cup M', w_A)$ be an assignment game, $\mu \in \mathcal{M}_A^*(M, M')$ and $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$ a μ-compatible subgame of $(M \cup M', w_A)$, where $I_1 \subseteq M$ and $J_1 \subseteq M'$. Let it also be $I'_2 \subseteq M \setminus I_1$ and $J'_2 \subseteq M' \setminus J_1$ such that $\mu(I_1) \cap J'_2 \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$ and $I'_2 \cap \mu^{-1}(J_1) \subseteq \{i \in M \mid a_{i\mu(i)} = 0\}$, and take $I_2 = I_1 \cup I'_2$ and $J_2 = J_1 \cup J'_2$. If $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$ is a μ-compatible [5] subgame of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$, then $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$ is a μ-compatible subgame of $(M \cup M', w_A)$.*

To see that, let us first write $\mu_1 = \mu \cap (M \setminus I_1) \times (M' \setminus J_1)$. Then, $w_A((M \setminus I_2) \cup (M' \setminus J_2)) + \sum_{i \in I_2 \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J_2 \cap \mu(M')} a_{\mu^{-1}(j)j} = w_A((M \setminus I_2) \cup (M' \setminus J_2)) + \sum_{i \in I'_2 \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J'_2 \cap \mu(M)} a_{\mu^{-1}(j)j} + \sum_{i \in I_1 \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J_1 \cap \mu(M)} a_{\mu^{-1}(j)j} = w_A(((M \setminus I_1) \setminus I'_2) \cup ((M' \setminus J_1) \setminus J'_2)) + \sum_{i \in I'_2 \cap \mu_1^{-1}(M' \setminus J_1)} a_{i\mu_1(i)} + \sum_{j \in J'_2 \cap \mu_1(M \setminus I_1)} a_{\mu_1^{-1}(j)j} + \sum_{i \in I_1 \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J_1 \cap \mu(M)} a_{\mu^{-1}(j)j} = w_A((M \setminus I_1) \cup (M' \setminus J_1)) + \sum_{i \in I_1 \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J_1 \cap \mu(M)} a_{\mu^{-1}(j)j} = w_A(M \cup M')$, where the first equality follows from the definition of I'_2 and J'_2 and the assumptions $\mu(I_1) \cap J'_2 \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$ and $I'_2 \cap \mu^{-1}(J_1) \subseteq \{i \in M \mid a_{i\mu(i)} = 0\}$, the third equality follows from the fact that $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$ is

[5] With some abuse of notation, when saying that $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$ is a μ-compatible subgame of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$, we mean that it is a $\mu_{|(M \setminus I_1) \times (M' \setminus J_1)}$-compatible subgame of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$.

a μ-compatible subgame of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$, and the forth one from the μ-compatibility of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$.

Now that the existence of μ-compatible subgames is guaranteed, we must analyze the relative position of the cores of a game and those μ-compatible subgames where only agents of one side of the market have been removed. When passing from $((M \setminus I) \cup M', w_{A_{-I}})$ to $(M \cup M', w_A)$, a new set of buyers enters the market and thus, by the type-monotonicity of the sellers-optimal core allocation (and also of the buyers-optimal core allocation), none of the already existing buyers can be better off, and none of the sellers can be worse off (Roth and Sotomayor, 1990):

$$\underline{u}_i^{A_{-I}} \geq \underline{u}_i^A \text{ and } \overline{u}_i^{A_{-I}} \geq \overline{u}_i^A \text{ for all } i \in M \setminus I, \tag{9}$$
$$\underline{v}_j^{A_{-I}} \leq \underline{v}_j^A \text{ and } \overline{v}_j^{A_{-I}} \leq \overline{v}_j^A \text{ for all } j \in M'. \tag{10}$$

Similarly, if $J \subseteq M'$, then

$$\underline{u}_i^{A_{-J}} \leq \underline{u}_i^A \text{ and } \overline{u}_i^{A_{-J}} \leq \overline{u}_i^A \text{ for all } i \in M, \tag{11}$$
$$\underline{v}_j^{A_{-J}} \geq \underline{v}_j^A \text{ and } \overline{v}_j^{A_{-J}} \geq \overline{v}_j^A \text{ for all } j \in M' \setminus J. \tag{12}$$

The following statement can be deduced as a consequence of a more general result in Mo (1988) but we include a simpler proof for our particular framework. What this proposition shows is that the cores of a game and some of its μ-compatible subgames are connected in a particular way, and this fact will be used in the proof of Lemma 15.

Proposition 7 *Let $(M \cup M', w_A)$ be an assignment game and $\mu \in \mathcal{M}_A^*(M, M')$.*

(1) If $I \subseteq M$ and $\overline{u}_i^A = a_{i\mu(i)}$ for all $i \in I$, then the payoff vector (u, v), where $u_i = \underline{u}_i^{A_{-I}}$ for $i \in M \setminus I$, $u_i = a_{i\mu(i)}$ for all $i \in I$ and $v_j = \overline{v}_j^{A_{-I}}$ for $j \in M'$, belongs to $C(w_A)$.

(2) If $J \subseteq M'$ and $\overline{v}_j^A = a_{\mu^{-1}(j)j}$ for all $j \in J$, then the payoff vector (u, v), where $u_i = \overline{u}_i^{A_{-J}}$ for $i \in M$, $v_j = a_{\mu^{-1}(j)j}$ for $j \in J$ and $v_j = \underline{v}_j^{A_{-J}}$ for $j \in M' \setminus J$, belongs to $C(w_A)$.

PROOF. We prove part 1) since 2) is proved analogously. Since $\overline{u}_i^A = a_{i\mu(i)}$ for all $i \in I$ (notice that by assumption all agents in I are assigned by μ), we have from Proposition 4 that $((M \setminus I) \cup M', w_{A_{-I}})$ is a μ-compatible subgame of $(M \cup M', w_A)$ and, by (6) and (7), $(\overline{u}_{-I}^A, \underline{v}^A) \in C(w_{A_{-I}})$. As a consequence, since $(\underline{u}^{A_{-I}}, \overline{v}^{A_{-I}})$ is the sellers-optimal core allocation of the subgame $((M \setminus I) \cup M', w_{A_{-I}})$, it holds $\overline{v}_j^{A_{-I}} \geq \underline{v}_j^A$ for all $j \in M'$.

Let us now check that $(u,v) \in C(w_A)$. From $(\underline{u}^{A_{-I}}, \overline{v}^{A_{-I}}) \in C(w_{A_{-I}})$ we have $u_i + v_j \geq a_{ij}$ if $(i,j) \in (M \setminus I) \times M'$, $u_i + v_{\mu(i)} = a_{i\mu(i)}$ if $(i,j) \in \mu \cap (M \setminus I) \times M'$. Also, from the μ-compatibility of $((M \setminus I) \cup M', w_{A_{-I}})$ we have $v_j = \overline{v}_j^{A_{-I}} = 0$ for all $j \in \mu(I)$ and thus, since $u_i = a_{i\mu(i)}$ for all $i \in I$, we get $u_i + v_{\mu(i)} = a_{i\mu(i)}$ for all $i \in I$. Moreover, if $i \in M \setminus I$ or $j \in M'$ are unassigned by μ, they are also unassigned by $\mu_{|(M \setminus I) \times M'}$ and thus $u_i = v_j = 0$. Finally, for $i \in I$ and $j \in M'$ we have

$$u_i + v_j = a_{i\mu(i)} + \overline{v}_j^{A_{-I}} \geq \overline{u}_i^A + \underline{v}_j^A \geq a_{ij} \,. \qquad \square$$

There is one more property of the set of μ-compatible subgames that will be crucial to prove the stability of the set proposed in the next section: under the assumption that $a_{ij} > 0$ for all $(i,j) \in \mu$, if $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$ and $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$, with $I_1 \supseteq I_2$ and $J_1 \supseteq J_2$, are μ-compatible subgames of $(M \cup M', w_A)$, then the subgame $((M \setminus I_1) \cup (M' \setminus J_2), w_{A_{-I_1 \cup J_2}})$ is also μ-compatible for $(M \cup M', w_A)$.

The proof of this rather technical result has been consigned to the Appendix (see Lemma 14). We are now prepared to prove the existence of stable sets for the assignment game.

4 The stable set in the μ-principal section

Since a core allocation cannot be dominated by any imputation, a stable set must always contain the core. From Solymosi and Raghavan (2001) we know that the core of an assignment game forms a stable set if and only if the assignment matrix has a dominant diagonal. Since the core is always internally stable, when the matrix has not a dominant diagonal the core is not externally stable. In this case we must add some imputations to the core in such a way that the new set preserves the internal stability and at the same time the additional imputations are enough to guarantee external stability.

For a fixed optimal matching μ of a given assignment game $(M \cup M', w_A)$, Shubik (1984) suggests to join to the core of the game the extended cores of all its μ-compatible subgames, in order to obtain a stable set. If

$$\mathcal{C}_A^\mu = \{(I,J) \in 2^M \times 2^{M'} \mid ((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}}) \text{ is a } \mu\text{-compatible subgame}\} \quad (13)$$

then consider

$$V^\mu(w_A) = \bigcup_{(I,J) \in \mathcal{C}_A^\mu} \hat{C}(w_{A_{-I \cup J}}) \,. \qquad (14)$$

Notice first that taking $I = J = \emptyset$, $\hat{C}(w_{A_{-I\cup J}}) = C(w_A)$ and thus the set $V^\mu(w_A)$ contains the core of the initial game. Moreover, by definition, $V^\mu(w_A)$ is included in the μ-principal section. Before proving the stability of $V^\mu(w_A)$ let us illustrate this set by means of an example.

Example 8 Let it be the assignment market with set of buyers $M = \{1, 2\}$, set of sellers $M' = \{1', 2', 3'\}$ and defined by the matrix

	1'	2'	3'
1	**6**	2	1
2	4	**3**	1

Notice first that there exists only one optimal matching and it is $\mu = \{(1, 1'), (2, 2')\}$. Thus, $v_{3'} = 0$ in each core allocation and Figure 8 represents in dark grey the projection of $C(w_A)$ to the space of the buyers' payoffs. Since there are agents with positive minimum core payoff, $(\underline{u}, \underline{v}) = (1, 1; 1, 0, 0)$, this core is not a stable set. But there are several μ-compatible subgames, those obtained by removing all the agents in one (and only one) of the following sets: $K_1 = \{2\}$, $K_2 = \{1, 2\}$, $K_3 = \{3'\}$, $K_4 = \{2', 3'\}$, $K_5 = \{1', 3'\}$, $K_6 = \{1', 2', 3'\}$ and $K_7 = \{2, 3'\}$. The extended core of the compatible subgame $((M \setminus K_1) \cup M', w_{A_{-K_1}})$ is the segment with extreme points $(2, 3; 4, 0, 0)$ and $(6, 3; 0, 0, 0)$, while the extended core of $(M \cup (M' \setminus K_3), w_{A_{-K_3}})$ adds to $C(w_A)$ the area shadowed in light grey. The extended cores of the remaining μ-compatible subgames are included in $\hat{C}(w_{A_{-K_1}}) \cup \hat{C}(w_{A_{-K_3}})$ and thus the union of all these cores is the set $V^\mu(w_A)$ that is a stable set for the initial assignment market.

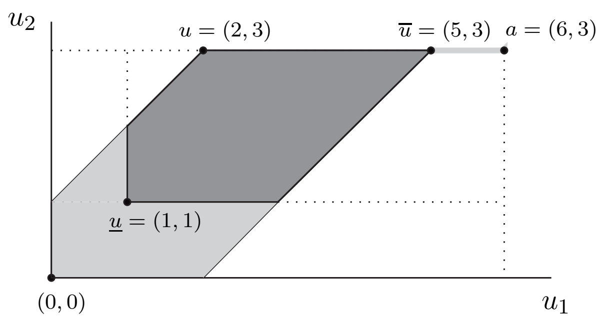

Fig. 1.

As also mentioned in Shubik (1984), it is straightforward to see that, given an assignment game $(M \cup M', w_A)$, a payoff vector (u, v) in the μ-principal section belongs to $V^\mu(w_A)$ if and only if for every pair $(i, j) \in \mu^{-1}(M') \times \mu(M)$

one of the following holds:

$$\begin{aligned} &(i) \quad u_i + v_j \geq a_{ij} \, , \\ &(ii) \quad u_i = a_{i\mu(i)} \ \text{ or} \\ &(iii) \ v_j = a_{\mu^{-1}(j)j} \, . \end{aligned} \tag{15}$$

The internal stability of the set $V^\mu(w_A)$ is proved in Shubik (1984). What is in fact proved is that $V^\mu(w_A)$ is undominated by any imputation in the μ-principal section. We include it here for the sake of comprehensiveness.

Proposition 9 (Shubik, 1984) *Let* $(M \cup M', w_A)$ *be an assignment game and* $\mu \in \mathcal{M}_A^*(M, M')$ *. The set* $V^\mu(w_A)$ *is internally stable.*

PROOF. Take (u, v), $(u', v') \in V^\mu(w_A)$ and assume $(u', v') dom^{w_A}_{\{i,j\}}(u, v)$ for some $(i, j) \in M \times M'$. Then , $u_i < u'_i$, $v_j < v'_j$ and $u'_i + v'_j \leq a_{ij}$. It must be that $i \in \mu^{-1}(M)$ because if $i \notin \mu^{-1}(M)$ then from $(u', v') \in B^\mu(w_A)$ we get $u'_i = 0$, which contradicts $u_i < u'_i$. A similar argument shows that $j \in \mu(M)$. Again from $(u', v') \in B^\mu(w_A)$ we obtain $u_i < a_{i\mu(i)}$, $v_j < a_{\mu^{-1}(j)j}$ and $u_i + v_j < a_{ij}$, in contradiction with $(u, v) \in V^\mu(w_A)$ by (15). □

The external stability of $V^\mu(w_A)$ will be proved in several steps. Notice first that the two extreme imputations $(a, 0)$, where $a_i = a_{i\mu(i)}$ for all $i \in \mu^{-1}(M')$ and $a_i = 0$ if $i \in M \setminus \mu^{-1}(M')$, and $(0, a)$, where $a_j = a_{\mu^{-1}(j)j}$ for all $j \in \mu(M)$ and $a_j = 0$ if $j \in M' \setminus \mu(M)$, must belong to any stable set contained in a principal section. The reason is that such an imputation will never be dominated by another imputation $(u, v) \in B^\mu(w_A)$ since this would imply $u_i > a_{i\mu(i)}$ for some $i \in M$, or $v_j > a_{\mu^{-1}(j)j}$ for some $j \in M'$. Next proposition proves that in fact this two points are connected through a curve in $V^\mu(w_A)$. Moreover, the imputations in this curve suffice to dominate all the imputations outside the μ-principal section.

Proposition 10 *Let* $(M \cup M', w_A)$ *be an assignment game with* $|M| = |M'|$ *and* $\mu \in \mathcal{M}_A^*(M, M')$ *that does not leave agents unassigned. Then,*

(1) There exists a connected piecewise linear curve L *in* $V^\mu(w_A)$ *passing through* $(a, 0)$ *and* $(0, a)$ *.*
(2) For all $(u, v) \in I(w_A) \setminus B^\mu(w_A)$ *there exists* $(u', v') \in L$ *such that* $(u', v') \, dom^{w_A}(u, v)$ *.*

Part 2) of the above proposition is proved in Shubik (1984), based on part 1) that is also stated there but without a proof. For the sake of comprehensiveness we include the complete proof of Proposition 10 in the Appendix.

In order to continue proving the external stability of $V^\mu(w_A)$, and since $C(w_A) \subseteq V^\mu(w_A)$, we begin by analyzing which imputations are dominated by some core allocation. To this end we define the subset $R^\mu(w_A)$ of the μ-principal section which is comprised inside the core bounds: given an assignment game $(M \cup M', w_A)$ and $\mu \in \mathcal{M}^*_A(M, M')$,

$$R^\mu(w_A) = \{(u, v) \in B^\mu(w_A) \mid \underline{u}^A_i \le u_i \le \overline{u}^A_i \text{ for all } i \in M\}. \tag{16}$$

Consequently, for all $(u, v) \in R^\mu(w_A)$ we also have $\underline{v}^A_j \le v_j \le \overline{v}^A_j$ for all $j \in M'$.

Notice that,

$$C(w_A) \subseteq R^\mu(w_A) \subseteq B^\mu(w_A) \subseteq I(w_A).$$

We find out that the only imputations in $V^\mu(w_A)$ that are inside the core bounds are the core imputations, and that these imputations dominate any imputation in $R^\mu(w_A)$.

Proposition 11 *Let $(M \cup M', w_A)$ be an assignment game and $\mu \in \mathcal{M}^*_A(M, M')$.*

(1) $V^\mu(w_A) \cap R^\mu(w_A) = C(w_A)$.
(2) For all $(u, v) \in R^\mu(w_A) \setminus C(w_A)$, there exists $(u', v') \in C(w_A)$ such that $(u', v')\, dom^{w_A} (u, v)$.

PROOF. 1) By (14) and (16), $C(w_A) \subseteq V^\mu(w_A) \cap R^\mu(w_A)$. To prove the converse inclusion, take $(u, v) \in V^\mu(w_A) \cap R^\mu(w_A)$ and assume $(u, v) \notin C(w_A)$. Since (u, v) belongs to the μ-principal section $B^\mu(w_A)$ it must be that $u_i + v_j < a_{ij}$ for some $(i, j) \in M \times M'$. Then, since $(u, v) \in V^\mu(w_A)$ there exists a μ-compatible subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ such $(u, v) \in \hat{C}(w_{A_{-I\cup J}})$. Then, either $i \in I$ or $j \in J$. Assume without loss of generality that $i \in I$. If i is matched by μ, by (6) we have $u_i = a_{i\mu(i)}$ and thus, from $(u, v) \in R^\mu(w_A)$, $u_i = \overline{u}^A_i$ and together with $\underline{v}^A_j \le v_j$ leads to the contradiction $\overline{u}^A_i + \underline{v}^A_j < a_{ij}$. On the other hand, if i is not matched by μ, and taking into account that $(u, v) \in B^\mu(w_A)$, we also obtain $u_i = \overline{u}^A_i$ and reach the same contradiction.

2) We may assume without loss of generality that A is a square matrix (see page 8), and that μ does not leave agents unassigned.[6] Let $(M \cup M', w_{A^e})$ be the related exact assignment game. Remember also from (4) that $a^e_{ij} = a^r_{ij} - \underline{u}^A_i - \underline{v}^A_j$ for all $i \in M$, $j \in M'$. Since μ is also an optimal matching for

[6] Once A is square, if μ does leave agents $i \in M$ and $j \in M'$ unassigned, it must be that $a_{ij} = 0$. Then we consider $\mu' = \mu \cup \{(i, j)\}$ and notice that $B^\mu(w_A) = B^{\mu'}(w_A)$ and thus $R^\mu(w_A) = R^{\mu'}(w_A)$.

(M, M', A^e) and for (M, M', A^r), and $C(w_A) = \{(\underline{u}^A, \underline{v}^A)\} + C(w_{A^e})$, given $(u, v) \in R^\mu(w_A) \setminus C(w_A)$, we have $(u - \underline{u}^A, v - \underline{v}^A) \in B^\mu(w_{A^e}) \setminus C(w_{A^e})$.

Since $(M \cup M', w_{A^e})$ has a dominant diagonal, by Solymosi and Raghavan (2001), there exists $(u', v') \in C(w_{A^e})$, and there exists $(i^*, j^*) \in M \times M'$ such that $(u', v')\, dom^{w_{A^e}}_{\{i^*, j^*\}} (u - \underline{u}^A, v - \underline{v}^A)$.

This means that

$$\begin{aligned} &u'_{i^*} > u_{i^*} - \underline{u}^A_{i^*}\,, \\ &v'_{j^*} > v_{j^*} - \underline{v}^A_{j^*}\,, \text{ and} \\ &u'_{i^*} + v'_{j^*} \leq a^e_{i^* j^*}\,. \end{aligned} \tag{17}$$

Let us now define

$$(\tilde{u}, \tilde{v}) = (u', v') + (\underline{u}^A, \underline{v}^A)\,. \tag{18}$$

Notice that $(\tilde{u}, \tilde{v}) \in C(w_A) = C(w_{A^r})$ and, by (17),

$$\begin{aligned} &\tilde{u}_{i^*} = u'_{i^*} + \underline{u}^A_{i^*} > u_{i^*}, \quad \tilde{v}_{j^*} = v'_{j^*} + \underline{v}^A_{j^*} > v_{j^*} \text{ and} \\ &\tilde{u}_{i^*} + \tilde{v}_{j^*} = u'_{i^*} + v'_{j^*} + \underline{u}^A_{i^*} + \underline{v}^A_{j^*} \leq a^e_{i^* j^*} + \underline{u}^A_{i^*} + \underline{v}^A_{j^*} = a^r_{i^* j^*}\,, \end{aligned} \tag{19}$$

which implies $(\tilde{u}, \tilde{v})\, dom^{w_{A^r}}_{\{i^*, j^*\}} (u, v)$.

We must prove that (u, v) is also dominated by a core allocation in terms of the game w_A instead of w_{A^r}. From (3), we have that either $a^r_{i^* j^*} = a_{i^* j^*}$, and we are done, or

$$\tilde{u}_{i^*} + \tilde{v}_{j^*} \leq a^r_{i^* j^*} = a_{i^* \mu(i_1)} + a_{i_1 \mu(i_2)} + \cdots + a_{i_r j^*} - a_{i_1 \mu(i_1)} - a_{i_2 \mu(i_2)} - \cdots - a_{i_r \mu(i_r)}$$

for some $i_1, i_2, \ldots, i_r \in M \setminus \{i^*, \mu^{-1}(j^*)\}$ and different.

In this case, since $(\tilde{u}, \tilde{v}) \in B^\mu(w_A)$, $\tilde{u}_{i_l} + \tilde{v}_{\mu(i_l)} = a_{i_l \mu(i_l)}$ for $l \in \{1, 2, \ldots, r\}$ and we obtain

$$\tilde{u}_{i^*} + \tilde{v}_{j^*} + \tilde{u}_{i_1} + \tilde{v}_{\mu(i_1)} + \cdots + \tilde{u}_{i_r} + \tilde{v}_{\mu(i_r)} \leq a_{i^* \mu(i_1)} + a_{i_1 \mu(i_2)} + \cdots + a_{i_r j^*}\,.$$

Together with $(\tilde{u}, \tilde{v}) \in C(w_A)$, this implies

$$\begin{aligned} &\tilde{u}_{i^*} + \tilde{v}_{\mu(i_1)} = a_{i^* \mu(i_1)}\,, \\ &\tilde{u}_{i_l} + \tilde{v}_{\mu(i_{l+1})} = a_{i_l \mu(i_{l+1})}, \text{ for all } l \in \{1, 2, \ldots, r-1\}\,, \\ &\tilde{u}_{i_r} + \tilde{v}_{j^*} = a_{i_r j^*}\,. \end{aligned}$$

• If $\tilde{v}_{\mu(i_1)} > v_{\mu(i_1)}$, and since $\tilde{u}_{i^*} > u_{i^*}$, we are done because $(\tilde{u}, \tilde{v})\, dom^{w_A}_{\{i^*, \mu(i_1)\}}(u, v)$.

• If $\tilde{v}_{\mu(i_1)} = v_{\mu(i_1)} = \overline{v}^A_{\mu(i_1)}$, then, since $(\tilde{u}, \tilde{v})$ and $(\underline{u}^A, \overline{v}^A)$ belong to $C(w_A)$,

$$\underline{u}^A_{i^*} \leq \tilde{u}_{i^*} = a_{i^*\mu(i_1)} - \tilde{v}_{\mu(i_1)} = a_{i^*\mu(i_1)} - \overline{v}^A_{\mu(i_1)} \leq \underline{u}^A_{i^*}.$$

But then, by (19), $u_{i^*} < \tilde{u}_{i^*} = \underline{u}^A_{i^*}$ contradicts $(u, v) \in R^\mu(w_A)$.

• If $\tilde{v}_{\mu(i_1)} = v_{\mu(i_1)} < \overline{v}^A_{\mu(i_1)}$, define

$$u^\varepsilon_i = \tilde{u}_i - \varepsilon \text{ and } v^\varepsilon_{\mu(i)} = \tilde{v}_{\mu(i)} + \varepsilon \text{ for all } i \in M \text{ such that } \tilde{v}_{\mu(i)} < \overline{v}^A_{\mu(i)},$$
$$u^\varepsilon_i = \tilde{u}_i \text{ and } v^\varepsilon_{\mu(i)} = \tilde{v}_{\mu(i)} \text{ for all } i \in M \text{ such that } \tilde{v}_{\mu(i)} = \overline{v}^A_{\mu(i)}.$$

Notice that, for $\varepsilon > 0$ small enough $(u^\varepsilon, v^\varepsilon) \in C(w_A)$. Indeed, if $\tilde{v}_{\mu(i)} < \overline{v}^A_{\mu(i)}$, we have $\tilde{u}_i > \underline{u}^A_i \geq 0$ and thus, for ε small enough, $u^\varepsilon_i \geq 0$. Also, $u^\varepsilon_i = \tilde{u}_i \geq 0$ for all $i \in M$ such that $\tilde{v}_{\mu(i)} = \overline{v}^A_{\mu(i)}$, $v^\varepsilon_j \geq 0$ for all $j \in M'$ and $u^\varepsilon_i + v^\varepsilon_{\mu(i)} = a_{i\mu(i)}$ for all $i \in M$. If $(i, j) \in M \times M'$, $(i, j) \notin \mu$, is such that either $\tilde{v}_{\mu(i)} < \overline{v}^A_{\mu(i)}$ and $\tilde{v}_j < \overline{v}^A_j$ or $\tilde{v}_{\mu(i)} = \overline{v}^A_{\mu(i)}$ and $\tilde{v}_j = \overline{v}^A_j$, it holds $u^\varepsilon_i + v^\varepsilon_j = \tilde{u}_i + \tilde{v}_j \geq a_{ij}$. If $\tilde{v}_{\mu(i)} = \overline{v}^A_{\mu(i)}$ and $\tilde{v}_j < \overline{v}^A_j$, then $u^\varepsilon_i + v^\varepsilon_j > \tilde{u}_i + \tilde{v}_j \geq a_{ij}$. Finally, if $\tilde{v}_{\mu(i)} < \overline{v}^A_{\mu(i)}$ and $\tilde{v}_j = \overline{v}^A_j$, we prove that $\tilde{u}_i + \tilde{v}_j > a_{ij}$ and thus, for $\varepsilon > 0$ small enough we can guarantee $u^\varepsilon_i + v^\varepsilon_j \geq a_{ij}$. The reason is that if $\tilde{u}_i + \tilde{v}_j = a_{ij}$, then

$$\underline{u}^A_i \leq \tilde{u}_i = a_{ij} - \tilde{v}_j = a_{ij} - \overline{v}^A_j \leq \underline{u}^A_i$$

where the last inequality follows from $(\underline{u}^A, \overline{v}^A) \in C(w_A)$. Thus $\tilde{u}_i = \underline{u}^A_i$, but this implies $\tilde{v}_{\mu(i)} = \overline{v}^A_{\mu(i)}$, in contradiction with the assumption. Moreover, for $\varepsilon > 0$ small enough we can take $(u^\varepsilon, v^\varepsilon) \in C(w_A)$ satisfying $v^\varepsilon_{\mu(i_1)} > \tilde{v}_{\mu(i_1)} = v_{\mu(i_1)}$ and, by (19), also $u^\varepsilon_{i^*} > u_{i^*}$, and therefore $(u^\varepsilon, v^\varepsilon)\, dom^{w_A}_{\{i^*, \mu(i_1)\}}(u, v)$.

• If $\tilde{v}_{\mu(i_1)} < v_{\mu(i_1)}$, then, since both $(\tilde{u}, \tilde{v})$ and (u, v) are in the μ-principal section, we get $\tilde{u}_{i_1} > u_{i_1}$, and we repeat the argument above with the mixed pair $\{i_1, \mu(i_2)\}$. Either we find that there exists $(u', v') \in C(w_A)$ such that $(u', v')\, dom^{w_A}_{\{i_l, \mu(i_{l+1})\}}(u, v)$ for some $l \in \{1, 2, \ldots, r-1\}$ or we reach $\tilde{u}_{i_r} > u_{i_r}$ and, since $\tilde{v}_{j^*} > v_{j^*}$ and $\tilde{u}_{i_r} + \tilde{v}_{j^*} = a_{i_r j^*}$, we obtain $(\tilde{u}, \tilde{v})\, dom^{w_A}_{\{i_r, j^*\}}(u, v)$. □

The final (and most dificult) step is to prove that all imputations not in $V^\mu(w_A)$ that are in the μ-principal section but outside the limits of the core bounds, are also dominated by elements of $V^\mu(w_A)$.

Theorem 12 *Let $(M \cup M', w_A)$ be an assignment game and $\mu \in \mathcal{M}^*_A(M, M')$ an optimal matching. Then $V^\mu(w_A)$ is a von Neumann-Morgenstern stable set.*

PROOF. We first consider the case where there are as many buyers as sellers and moreover $a_{ij} > 0$ for all $(i,j) \in \mu$.

The internal stability of the set $V^{\mu}(w_A)$ is proved in Proposition 9 following Shubik (1984). As for the external stability, Proposition 10, also based in Shubik (1984), shows that for all $(u,v) \in I(w_A) \backslash B^{\mu}(w_A)$ there exists $(u',v') \in V^{\mu}(w_A)$ such that $(u',v')\, dom^{w_A}(u,v)$.

In Proposition 11 we show how to dominate the imputations in $R^{\mu}(w_A) \setminus V^{\mu}(w_A)$, but it remains to prove that any imputation not in $V^{\mu}(w_A)$ that is in the μ-principal section but outside the limits of the core bounds, is also dominated by an element of $V^{\mu}(w_A)$. This proof is rather large and technical, and it is consigned to the Appendix.

We now consider the general case, that is when either $a_{i\mu(i)} = 0$ for some $i \in M$ or there exist unmatched agents. In this case, let it be $I' = \{i \in M \mid a_{i\mu(i)} = 0\}$ and $J' = \mu(I')$, and define $I = I' \cup \{i \in M \mid i \text{ unmatched by } \mu\}$, and also $J = J' \cup \{j \in M' \mid j \text{ unmatched by } \mu\}$. With some abuse of notation we also denote by μ the restriction of μ to $(M \setminus I) \times (M' \setminus J)$. Then, if we consider the submarket $((M \setminus I) \cup (M' \setminus J), w_{A'})$ where $A' = A_{|(M \setminus I) \times (M' \setminus J)}$, we have just proved that $V' = V^{\mu}(w_{A'}) = \bigcup_{(R,S) \in \mathcal{C}^{\mu}_{A'}} \hat{C}(w_{A'_{-R \cup S}})$ is a stable set of $((M \setminus I) \cup (M' \setminus J), w_{A'})$. Notice also that

$$B^{\mu}(w_A) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} (u_{-I}, v_{-J}) \in B^{\mu}(w_{A'}), \\ u_i = 0 \text{ for all } i \in I,\ v_j = 0 \text{ for all } j \in J \end{array} \right\}.$$

We now claim that the set

$$V = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} (u_{-I}, v_{-J}) \in V', \\ u_i = 0 \text{ for all } i \in I,\ v_j = 0 \text{ for all } j \in J \end{array} \right\}$$

is a stable set for the initial market $(M \cup M', w_A)$. To prove the internal stability of V, notice that if $(u,v), (u',v') \in V$ are such that $(u,v) dom^{w_A}_{\{i,j\}}(u',v')$, for some $(i,j) \in M \times M'$, then $i \notin I$ and $j \notin J$, and thus $(u_{-I}, v_{-J}) dom^{w_{A'}}_{\{i,j\}}(u'_{-I}, v'_{-J})$ contradicts internal stability of V'.

To prove the external stability of V, take $(u,v) \in I(w_A) \backslash V$ and consider two different cases. Take first $(u,v) \in I(w_A) \setminus B^{\mu}(w_A)$. Then it is immediate to see that if L' is a connected piecewise linear curve from $(a_{-I}, 0)$ to $(0, a_{-J})$ included in V', then $L = \{(u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \mid (u_{-I}, v_{-J}) \in L',\ u_i = 0 \text{ for all } i \in I,\ v_j = 0 \text{ for all } j \in J\}$ is a connected piecewise linear curve from $(a,0)$ to $(0,a)$ and contained in V. The existence of L' guarantees the existence of L and thus, as in Proposition 10, we obtain that there exists $(u',v') \in L$ such that $(u',v') dom^{w_A}(u,v)$.

Secondly, if $(u,v) \in B^{\mu}(w_A)\backslash V$, then $(u_{-I}, v_{-J}) \in B^{\mu}(w_{A'})\backslash V'$ and by the external stability of V' there exists $(u'_{-I}, v'_{-J}) \in V'$ such that $(u'_{-I}, v'_{-J}) dom^{w_{A'}}_{\{i,j\}} (u_{-I}, v_{-J})$. As a consequence, the payoff vector $(x,y) \in V$ defined by $(x_{-I}, y_{-J}) = (u'_{-I}, v'_{-J})$, $x_i = 0$ for all $i \in I$ and $y_j = 0$ for all $j \in J$, satisfies $(x,y) dom^{w_A}_{\{i,j\}} (u,v)$, and this finishes the proof of the stability of V.

Finally we show that not only the assignment game $(M \cup M', w_A)$ has a stable set V but also that $V = V^{\mu}(w_A) = \bigcup_{(R,S)\in \mathcal{C}^{\mu}_A} \hat{C}(w_{A_{-R\cup S}})$, as claimed by the theorem. Notice first that, by Definition 1 and the definition of the sets I and J, $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame of $(M \cup M', w_A)$. Moreover, by Remark 6, any μ-compatible subgame of $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is also a μ-compatible subgame of $(M \cup M', w_A)$. This implies $V \subseteq V^{\mu}(w_A)$. Now, if there existed $(u,v) \in V^{\mu}(w_A) \setminus V$, then the external stability of V would imply the existence of $(u',v') \in V$ such that $(u',v') dom^{w_A} (u,v)$, but this would contradict the internal stability of $V^{\mu}(w_A)$, which has been established in Proposition 9. $\square$

An important consequence of the above proof is that given an assignment market and an optimal matching μ, if there exist unassigned agents or matched pairs with null profit, then all these agents can be removed and the stable set associated to the corresponding submatrix provides an stable set of the initial game that moreover coincides with the union of the extended cores of the μ-compatible subgames of the initial market (see for instance Example 8).

As a final application, we now obtain a stable set for the assignment game proposed in Shapley and Shubik (1972), already recalled in our Example 2:

$$V^{\mu}(w_A) = C(w_A) \cup \hat{C}(w_{A_{-\{2\}}}) \cup \hat{C}(w_{A_{-\{2,3\}}}) \cup \hat{C}(w_{A_{-\{1'\}}}) \cup \hat{C}(w_{A_{-\{1'2'\}}}) \cup \hat{C}(w_{A_{-\{2,1'\}}}) .$$

Figure 2 illustrates this stable set.

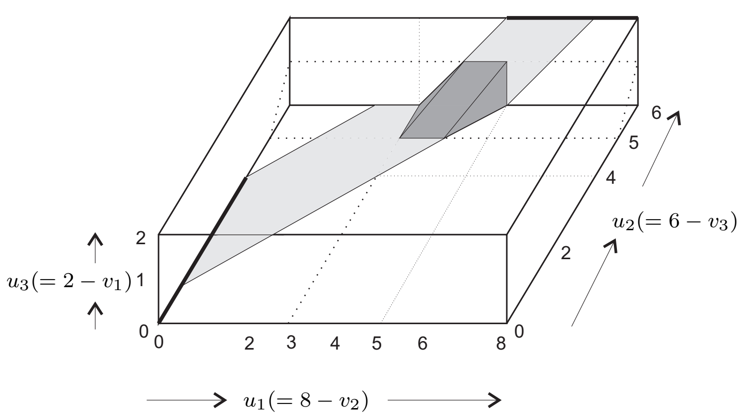

Fig. 2.

The parallelepiped $[0,8]\times[0,6]\times[0,2]$ is the projection of the principal section $B^\mu(w_A)$ to the space of the buyers' payoff. To obtain the sellers' payoffs, remember that $\mu=\{(1,2'),(2,3'),(3,1')\}$, and $u_i+v_{\mu(i)}=a_{i\mu(i)}$ for all $i\in M$ and $(u,v)\in B^\mu(w_A)$. Inside this parallelepiped we represent the core $C(w_A)$ in dark grey, with buyers-optimal core allocation $(\overline{u},\underline{v})=(5,6,1;1,3,0)$ and sellers-optimal core allocation $(\underline{u},\overline{v})=(3,5,0;2,5,1)$. The shadowed area in the face $u_2=a_{23}=6$ is the extended core of one μ-compatible subgame, $\hat{C}(w_{A_{-\{2\}}})$; and the shadowed area in the face $v_1=a_{31}=2$ (or equivalently $u_3=0$) is the extended core of another μ-compatible subgame, $\hat{C}(w_{A_{-\{1'\}}})$. Finally, the segment from (5,6,2;0,3,0) to (8,6,2;0,0,0) is $\hat{C}(w_{A_{-\{2,3\}}})$, while the segment from (0,4,0;2,8,2) to (0,0,0;2,8,6) is $\hat{C}(w_{A_{-\{1'2'\}}})$. In this example, $\hat{C}(w_{A_{-\{2,1'\}}})\subseteq\hat{C}(w_{A_{-\{1'\}}})$.

Let us now consider the usual partial order in the μ-principal section. Given $(u,v),\,(u',v')\in B^\mu(w_A)$, we say $(u,v)\leq_M(u',v')$ if and only if $u_i\leq u'_i$ for all $i\in M$ and $v_j\geq v'_j$ for all $j\in M'$, and then define

$$\begin{aligned} u_i^*&=\max\{u_i,u'_i\} \text{ and } u_{*i}=\min\{u_i,u'_i\}, \text{ for all } i\in M\,,\\ v_j^*&=\max\{v_j,v'_j\} \text{ and } v_{*j}=\min\{v_j,v'_j\}, \text{ for all } j\in M'\,. \end{aligned} \tag{20}$$

It is known from Shapley and Shubik (1972) that if (u,v) and (u',v') are core allocations, then the payoff vectors (u^*,v_*) and (u_*,v^*) also belong to the core, and this is why $C(w_A)$ is a lattice. Theorem 13 states that this is also true regarding the stable set.

Theorem 13 *Let $(M\cup M',w_A)$ be an assignment game and $\mu\in\mathcal{M}_A^*(M,M')$. If $(u,v),\,(u',v')\in V^\mu(w_A)$, then $(u^*,v_*)\in V^\mu(w_A)$ and $(u_*,v^*)\in V^\mu(w_A)$.*

PROOF. We will prove that $(u^*,v_*)\in V^\mu(w_A)$, since $(u_*,v^*)\in V^\mu(w_A)$ is proved analogously and thus left to the reader.

If $(u,v),\,(u',v')\in V^\mu(w_A)$, there exist $I_1,\,I_2\subseteq M$ and $J_1,\,J_2\in M'$ such that $((M\setminus I_1)\cup(M'\setminus J_1),w_{A_{-I_1\cup J_1}})$ and $((M\setminus I_2)\cup(M'\setminus J_2),w_{A_{-I_2\cup J_2}})$ are μ-compatible subgames of $(M\cup M',w_A)$, $(u,v)\in\hat{C}(w_{A_{-I_1\cup J_1}})$ and $(u',v')\in\hat{C}(w_{A_{-I_2\cup J_2}})$.

Let us see that (u^*,v_*) belongs to $B^\mu(w_A)$ (this follows straightforwardly from $(u,v),\,(u',v')\in B^\mu(w_A)$) and moreover satisfies one of the conditions in (15).

Let us now take $(i,j) \in \mu^{-1}(M) \times \mu(M)$. From (6), for all $i \in I_1 \cup I_2$ assigned by μ , we have $u_i^* = \max\{u_i, u_i'\} = a_{i\mu(i)}$, and for all $j \in J_1 \cap J_2$ assigned by μ , it holds $v_j = v_j' = a_{\mu^{-1}(j)j}$ and thus $v_{*j} = \min\{v_j, v_j'\} = a_{\mu^{-1}(j)j}$.

So let us assume that $i \notin I_1 \cup I_2$ and $j \notin J_1 \cap J_2$. This means that $i \notin I_1$, $i \notin I_2$ and either $j \notin J_1$ or $j \notin J_2$. In the first case $(i,j) \in (M \setminus I_1) \times (M' \setminus J_1)$, and in the second case $(i,j) \in (M \setminus I_2) \times (M' \setminus J_2)$. We will prove that in the first case $u_i + v_j \geq a_{ij}$ holds (the proof in the second case is similar and left to the reader). If $v_{*j} = v_j$, then $u_i^* + v_{*j} \geq u_i + v_j \geq a_{ij}$, where the inequality follows from $(u,v) \in \hat{C}(w_{A_{-I_1 \cup J_1}})$. If $v_{j^*} = v_j' < v_j$, we can guarantee that $j \notin J_2$ and then $u_i^* + v_{*j} \geq u_i' + v_j' \geq a_{ij}$, where the inequality follows from $(u', v') \in \hat{C}(w_{A_{-I_2 \cup J_2}})$. □

Notice that the buyers-optimal and the sellers-optimal allocations of $V^\mu(w_A)$ as a lattice are always $(a, 0)$ and $(0, a)$.

Let us finally remark that the stable set $V^\mu(w_A)$ is the only one contained in the μ-principal section, and thus, the only one that excludes third-party payments (according to μ). The reason is that, by the proof of Proposition 9, the elements in $V^\mu(w_A)$ cannot be dominated by any imputation in the μ-principal section. Therefore, $V^\mu(w_A)$ must be included in any stable set V contained in $B^\mu(w_A)$. But then, by the external stability of $V^\mu(w_A)$, the elements in $V \setminus V^\mu(w_A)$ would be dominated by elements in $V^\mu(w_A)$ in contradiction with the internal stability of V . This fact was already mentioned in Shubik (1984).

A Appendix

Lemma 14 *Let $(M \cup M', w_A)$ be an assignment game, $\mu \in \mathcal{M}_A^*(M, M')$ and $a_{ij} > 0$ for all $(i,j) \in \mu$. If $(I_1, J_1) \in \mathcal{C}_A^\mu$ and $(I_2, J_2) \in \mathcal{C}_A^\mu$, with $I_1 \supseteq I_2$ and $J_1 \supseteq J_2$, then $(I_1, J_2) \in \mathcal{C}_A^\mu$.*

PROOF. Let us assume that $((M \setminus I_1)) \cup (M' \setminus J_2)), w_{A_{-I_1 \cup J_2}})$ is not a μ-compatible subgame. From $(I_1, J_1) \in \mathcal{C}_A^\mu$ and $(I_2, J_2) \in \mathcal{C}_A^\mu$, and taking into account by assumption that $\{i \in M \mid a_{i\mu(i)} = 0\} = \emptyset$, we have $I_1 \cap \mu^{-1}(J_1) = \emptyset$ and $I_2 \cap \mu^{-1}(J_2) = \emptyset$. Now, since $I_1 \cap \mu^{-1}(J_2) \subseteq I_1 \cap \mu^{-1}(J_1)$, we obtain $I_1 \cap \mu^{-1}(J_2) = \emptyset$. Thus, $(I_1, J_2) \notin \mathcal{C}_A^\mu$ means that there exists $\mu' \in \mathcal{M}(M \setminus I_1, M' \setminus J_2)$ such that

$$\sum_{(i,j) \in \mu'} a_{ij} > \sum_{(i,j) \in \mu_{|(M \setminus I_1) \times (M' \setminus J_2)}} a_{ij} .$$

Notice that if for all $(i,j) \in \mu'$ we had that $(i,j) \in (M \setminus \mu^{-1}(J_2)) \times (M' \setminus \mu(I_1))$, then the matching obtained by the union of μ' with the sets $\{(i,\mu(i)) \mid i \in I_1\}$ and $\{(\mu^{-1}(j),j) \mid j \in J_2\}$ would contradict that μ is an optimal matching for the market (M, M', A), since $\sum_{(i,j)\in\mu'} a_{ij} + \sum_{i\in I_1} a_{i\mu(i)} + \sum_{j\in J_2} a_{\mu^{-1}(j)j} > \sum_{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}} a_{ij} + \sum_{i\in I_1} a_{i\mu(i)} + \sum_{j\in J_2} a_{\mu^{-1}(j)j} = \sum_{(i,j)\in\mu} a_{ij}$, where the last equality follows from $I_1 \cap \mu^{-1}(J_2) = \emptyset$.

Thus, there exists $(i,j) \in \mu'$ such that either $j = \mu'(i) \in \mu(I_1)$ or $i = \mu'^{-1}(j) \in \mu^{-1}(J_2)$.

Let us denote by μ_1 the restriction of μ to $(M \setminus I_1) \times (M' \setminus J_1)$, and by μ_2 the restriction of μ to $(M \setminus I_2) \times (M' \setminus J_2)$. Notice that, by the μ-compatibility of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$, μ_1 leaves unmatched the agents in $\mu^{-1}(J_1) \cap (M \setminus I_1)$ and also of $\mu(I_1) \cap (M' \setminus J_1)$. The same happens with μ_2 and the sets $\mu^{-1}(J_2) \cap (M \setminus I_2)$ and $\mu(I_2) \cap (M' \setminus J_2)$. Notice also that

$$\mu_1 \subseteq \mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \subseteq \mu_2 \subseteq \mu . \tag{A.1}$$

The two following subsets of players are defined:

$$K_1 = \{i \in M \setminus I_1 \mid \mu'(i) \in \mu(I_1)\},$$
$$K_2 = \{j \in M' \setminus J_2 \mid \mu'^{-1}(j) \in \mu^{-1}(J_2)\}.$$

and by the above argument, at least one of these sets must be non-empty.

a) If $K_1 \neq \emptyset$, to each $i \in K_1$ we associate a chain $C(i)$ of agents in the following way. Define $i_1 = i$, and take also $\mu'(i_1)$. If i_1 is not matched by μ_1 or $\mu_1(i_1)$ is not matched by μ', then $C(i) = (\mu'(i_1), i_1)$. Otherwise, let it be $i_2 \in M \setminus I_1$ the unique agent such that $\mu'(i_2) = \mu_1(i_1)$. Construct recursively the chain[7]

$$C(i) = (\mu'(i_1), i_1, \mu'(i_2), i_2, \ldots, \mu'(i_l), i_l, \mu'(i_{l+1}), i_{l+1}, \cdots, \mu'(i_n), i_n)$$

where for all $l \in \{1, \ldots, n-1\}$, i_{l+1} is the unique agent in $M \setminus I_1$ such that $\mu'(i_{l+1}) = \mu_1(i_l)$, and either i_n is unmatched by μ_1 or $\mu_1(i_n)$ is unmatched by μ'.

Several properties are satisfied by the chain $C(i)$. First, taking into account that $\mu_1(i_l) = \mu'(i_{l+1})$ for $l \in \{1, 2, \ldots, n-1\}$ and also that $i_l \in M \setminus I_1$ for $l \in \{1, 2, \ldots, n\}$, we obtain $C(i) \cap \mu(I_1) = \{\mu'(i_1)\}$.

[7] With some abuse of notation, $C(i)$ will also stand for the set of agents in the chain.

Notice secondly that if $i_l, i_t \in C(i)$, then $i_l \neq i_t$. Assume without loss of generality that $i_l = i_t$ and $1 \leq l < t$. If $l = 1$, we reach a contradiction since $\mu'(i_1) \in \mu(I_1)$ and $\mu'(i_t) \notin \mu(I_1)$. If $1 < l < t$, then $\mu'(i_l) = \mu'(i_t)$ implies $\mu_1(i_{l-1}) = \mu_1(i_{t-1})$ and thus $i_{l-1} = i_{t-1}$. The repetition of this argument leads to $i_1 = i_{t-l+1}$ and the same argument above leads to a contradiction. This property guarantees that the defined chains $C(i)$ are always finite.

Moreover, if $i, i' \in K_1$ and $i \neq i'$, then $C(i) \cap C(i') = \emptyset$. Assume otherwise that $k \in C(i) \cap C(i')$, where $C(i) = (\mu'(i_1), i_1, \mu'(i_2), i_2, \ldots, \mu'(i_l), i_l, \mu'(i_{l+1}), \cdots, \mu'(i_n), i_n)$ and $C(i') = (\mu'(i'_1), i'_1, \mu'(i'_2), i'_2, \ldots, \mu'(i'_t), i'_t, \mu'(i'_{t+1}), \cdots, \mu'(i'_s), i'_s)$, $i_1 = i$ and $i'_1 = i'$. We may assume without loss of generality that $k = i_l = i'_t$ with $l \leq t$. If $k = i_1 = i'_t$, then $\mu'(i_1) = \mu'(i'_t)$ and thus $\mu'(i'_t) \in \mu(I_1)$ implies $i'_t = i'_1$ and consequently $i = i_1 = i'_1 = i'$. Assume now that $l > 1$ is minimal such that $i_l = i'_t$ with $l \leq t$. Then, $\mu_1(i_{l-1}) = \mu'(i_l) = \mu'(i'_t) = \mu_1(i'_{t-1})$ implies $i_{l-1} = i'_{t-1}$ and contradicts the minimality of l.

Finally, if for some $i \in K_1$, $i_l \in C(i) \cap \mu^{-1}(J_2)$, then, since $\mu^{-1}(J_2) \subseteq \mu^{-1}(J_1)$ we have that i_l is not matched by μ_1 and thus $i_l = i_n$ is the last agent in the chain.

b) If $K_2 \neq \emptyset$ and $j \in K_2$ recall that, by definition of this set, $j \in M' \setminus J_2$ and $\mu'^{-1}(j) \in \mu^{-1}(J_2)$. Then, for all $j \in K_2$ we define the chain $C(j)$ by

$$C(j) = (i_1, \mu'(i_1), i_2, \mu'(i_2), \ldots, i_l, \mu'(i_l), i_{l+1}, \ldots, i_n, \mu'(i_n))$$

where $j = \mu'(i_1)$, $\mu_2(i_{l+1}) = \mu'(i_l)$ for all $l \in \{1, 2, \ldots, n-1\}$ and either $\mu'(i_n)$ is unmatched by μ_2 or $\mu_2^{-1}(\mu'(i_n))$ is unmatched by μ'.

Notice first that, since μ' is defined on $(M \setminus I_1) \times (M' \setminus J_2)$, we have $i_l \in M \setminus I_1$ and $\mu'(i_l) \in M' \setminus J_2$ for all $l \in \{1, 2, \ldots, n\}$. As a consequence, $C(j) \cap \mu^{-1}(J_2) = \{\mu'^{-1}(j)\} = \{i_1\}$. Indeed, if $i_{l+1} \in \mu^{-1}(J_2)$ for some $l \in \{1, 2, \ldots, n-1\}$, since $i_{l+1} = \mu_2^{-1}(\mu'(i_l)) = \mu^{-1}(\mu'(i_l))$ by (A.1), then $\mu'(i_l) \in J_2$, which implies a contradiction.

Secondly, if $i_l, i_t \in C(j)$, then $i_l \neq i_t$, and the proof is analogous to the one in a).

Also, if $j, j' \in K_2$, $j \neq j'$, $C(j) \cap C(j') = \emptyset$. To see that, let us denote these two chains by $C(j) = (i_1, \mu'(i_1), i_2, \mu'(i_2), \ldots, i_l, \mu'(i_l), i_{l+1}, \ldots, i_n, \mu'(i_n))$ and $C(j) = (i'_1, \mu'(i'_1), i'_2, \mu'(i'_2), \ldots, i'_l, \mu'(i'_l), i'_{l+1}, \ldots, i'_n, \mu'(i'_n))$ and assume without loss of generality that $k = i_l = i'_t$ with $1 \leq l \leq t$. If $l = 1$, then, $i'_t \in \mu^{-1}(J_2)$ implies $i'_t = i'_1$ and consequently $i_1 = i'_1$ implies $j = j'$ in contradiction with the assumption. Assume $l > 1$ is minimal such that $i_l = i'_t$ with $l \leq t$. Then, $\mu_2(i_l) = \mu_2(i'_t)$ and thus $\mu'(i_{l-1}) = \mu'(i'_{t-1})$ implies $i_{l-1} = i'_{t-1}$ and contradicts the minimality of l.

Finally, if for some $l \in \{1,2,\ldots,n\}$ we have $\mu'(i_l) \in C(j) \cap \mu(I_2)$, then $\mu'(i_l)$ is unmatched by μ_2 and thus $i_l = i_n$.

c) Moreover, if $i \in K_1$ and $j \in K_2$, then either $C(i) \cap C(j) = \emptyset$ or $C(i) \subseteq C(j)$. To prove this, let us write

$$\begin{aligned} C(i) &= (\mu'(i'_1), i'_1, \ldots, \mu'(i'_l), i'_l, \mu'(i'_{l+1}), i'_{l+1}, \ldots, \mu'(i'_k), i'_k), \\ C(j) &= (i_1, \mu'(i_1), \ldots, i_l, \mu'(i_l), i_{l+1}, \mu'(i_{l+1}), \ldots, i_n, \mu'(i_n)), \end{aligned}$$

where $i'_1 = i$ and $\mu'(i_1) = j$. If $C(i) \cap C(j) \neq \emptyset$, then $i'_r = i_s$ for some $r \in \{1,2,\ldots,k\}$ and $s \in \{1,2,\ldots,n\}$. Then $\mu'(i'_r) = \mu'(i_s) \in C(i) \cap C(j)$. If $r = k = 1$ we are done. Otherwise, since $1 \leq r \leq k$, either $r < k$ or $r > 1$.

If $r < k$, take i'_{r+1} and recall by the definition of $C(i)$ in a) that $\mu'(i'_{r+1}) = \mu_1(i'_r)$. Now, from (A.1), $\mu'(i'_{r+1}) = \mu_2(i'_r) = \mu_2(i_s)$ and, by the construction of chain $C(j)$ in b), we obtain $i'_{r+1} = i_{s-1}$ and as a consequence both i'_{r+1} and $\mu'(i'_{r+1}) = \mu'(i_{s-1})$ belong to $C(j)$. The repetition of the same argument shows that for $r \leq l \leq k$, both i'_l and $\mu'(i'_l)$ belong to $C(j)$.

If $r > 1$, again by the definition of $C(i)$ we know that $\mu_1(i'_{r-1}) = \mu'(i'_r)$, and by (A.1) $\mu_2(i'_{r-1}) = \mu'(i'_r) = \mu'(i_s)$. This, by the construction of $C(j)$ in b), implies that $i'_{r-1} = i_{s+1}$, and thus both i'_{r-1} and $\mu'(i'_{r-1})$ belong to $C(j)$. The repetition of this argument guarantees that i'_l and $\mu'(i'_l)$ belong to $C(j)$ for all $1 \leq l \leq r$. To sum up, we have proved that if $C(i) \cap C(j) \neq \emptyset$, where $i \in K_1$ and $j \in K_2$, then $C(i) \subseteq C(j)$.

Let us now define

$$\begin{aligned} &C' = \textstyle\bigcup_{k \in K_2} C(k), \\ &C = \textstyle\bigcup_{k \in K'_1} C(k) \text{ where } K'_1 = \{k \in K_1 \mid C(k) \cap C' = \emptyset\}. \end{aligned}$$

Let us now point out that the sets $\{(i,j) \in \mu' \mid i \in C\}$, $\{(i,j) \in \mu' \mid j \in C'\}$ and $\{(i,j) \in \mu' \mid i \notin C \text{ and } j \notin C'\}$ form a partition of $\mu' \subseteq (M \setminus I_1) \times (M' \setminus J_2)$. The reason is that if $j \in C'$, then $j \in C(k)$ for some $k \in K_2$. But then, by the definition of the chains, $\mu'^{-1}(j)$ also belongs to $C(k) \subseteq C'$ and consequently, by the definition of K'_1, $\mu'^{-1}(j) \notin C$.

Similarly, the sets $\{(i,j) \in \mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \mid i \in C\}$, $\{(i,j) \in \mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \mid j \in C'\}$ and $\{(i,j) \in \mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \mid i \notin C \text{ and } j \notin C'\}$ form a partition of $\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}$. If $j \in C'$, that is to say, $j \in C(k)$ for some $k \in K_2$, then whenever j is matched by $\mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \subseteq \mu_2$ it follows that either $\mu_2^{-1}(j) \in C(k) \subseteq C'$, and thus $\mu^{-1}(j) \notin C$, or $\mu_2^{-1}(j)$ is not matched by μ'. But in that last case $\mu_2^{-1}(j) \notin C$, since each buyer in a chain $C(k')$, for $k' \in K_1$, is preceded by his or her partner by μ'.

Then, since

$$\begin{aligned}\sum_{(i,j)\in\mu'} a_{ij} = \sum_{\substack{(i,j)\in\mu'\\ i\in C}} a_{ij} + \sum_{\substack{(i,j)\in\mu'\\ j\in C'}} a_{ij} + \sum_{\substack{(i,j)\in\mu'\\ i\notin C,\, j\notin C'}} a_{ij} \\ > \sum_{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}} a_{ij} = \\ = \sum_{\substack{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}\\ i\in C}} a_{ij} + \sum_{\substack{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}\\ j\in C'}} a_{ij} + \sum_{\substack{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)}\\ i\notin C,\, j\notin C'}} a_{ij}.\end{aligned}$$

one of the following cases must hold.

• There exists $k \in K_1$ with $C(k) \subseteq C$ and such that $\sum_{\substack{(i,j)\in\mu'\\ i\in C(k)}} a_{ij} > \sum_{\substack{(i,j)\in\mu\\ i\in C(k)}} a_{ij}$. Then we define

$$\tilde{\mu} = \{(i, \mu'(i)) \mid i \in C(k)\} \cup \{(i, \mu_1(i)) \mid i \in M \setminus (I_1 \cup C(k))\}$$

and we check that this is a matching in $\mathcal{M}(M \setminus I_1, M' \setminus J_1)$. First we prove that $\tilde{\mu}$ is included in $(M \setminus I_1) \times (M' \setminus J_1)$. Notice indeed that by definition of μ_1, $\mu_1(i) \in M' \setminus J_1$. Also, if $i \in C(k)$ then $\mu'(i) \in C(k)$ and thus there exists $\mu_1^{-1}(\mu'(i))$, which implies $\mu'(i) \in M' \setminus J_1$, except if $\mu'(i) = \mu'(k)$. But then $\mu'(i) \in \mu(I_1)$ and by the μ-compatibility of $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$ we have $\mu(I_1) \cap J_1 \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$ and this set is empty by assumption. It now remains to prove that $\tilde{\mu}$ is a matching. To see that, notice that if $i \in C(k)$ and $\mu'(i) = \mu_1(i')$, then $i' \in C(k)$.

Thus we get that $\sum_{(i,j)\in\tilde{\mu}} a_{ij} > \sum_{(i,j)\in\mu_1} a_{ij}$, which contradicts that $((M \setminus I_1) \cup (M' \setminus J_1), w_{A_{-I_1 \cup J_1}})$ is a μ-compatible subgame of $(M \cup M', w_A)$.

• There exists $k \in K_2$ such that $\sum_{\substack{(i,j)\in\mu'\\ j\in C(k)}} a_{ij} > \sum_{\substack{(i,j)\in\mu\\ j\in C(k)}} a_{ij}$. Then we define

$$\tilde{\mu} = \{(\mu'^{-1}(j), j) \mid j \in C(k)\} \cup \{(\mu_2^{-1}(j), j) \mid j \in M' \setminus (J_2 \cup C(k))\}$$

and check that it is a matching in $\mathcal{M}(M \setminus I_2, M' \setminus J_2)$. Notice first that $\tilde{\mu} \subseteq (M \setminus I_2) \times (M' \setminus J_2)$. Indeed, for all $j \in M' \setminus (J_2 \cup C(k))$ that is assigned by μ_2, $\mu_2^{-1}(j) \in M \setminus I_2$ by definition of μ_2 and, by definition of μ', $\mu'^{-1}(j) \in M \setminus I_1 \subseteq M \setminus I_2$ for all $j \in C(k)$. Also, each $j \in C(k) \cap M'$ is matched by μ', and thus $j \in M' \setminus J_2$. Let us now see that $\tilde{\mu}$ is really a matching. It is enough to prove that if $j \in C(k) \cap M'$ then it does not hold $\mu'^{-1}(j) = \mu_2^{-1}(j')$ for some $j' \in M \setminus (J_2 \cup C(k))$. Indeed, if $\mu'^{-1}(j) = \mu_2^{-1}(j')$ for some $j' \in M \setminus (J_2 \cup C(k))$, let us write $i = \mu'^{-1}(j)$ and notice that from $j \in C(k) \cap M'$, we deduce $i = \mu'^{-1}(j) \in C(k)$. Now, if $i = \mu'^{-1}(j)$ is matched by μ_2, then $\mu_2(i) = \mu_2(\mu'^{-1}(j)) = j'$ also belongs to $C(k)$, in contradiction with $j' \in M' \setminus (J_2 \cup C(k))$.

Once proved that $\tilde{\mu} \in \mathcal{M}(M\backslash I_2, M'\backslash J_2)$ we get that $\sum_{(i,j)\in\tilde{\mu}} a_{ij} > \sum_{(i,j)\in\mu_2} a_{ij}$, contradicts that $((M \setminus I_2) \cup (M' \setminus J_2), w_{A_{-I_2 \cup J_2}})$ is a μ-compatible subgame of $(M \cup M', w_A)$.

- Otherwise,

$$\sum_{\substack{(i,j)\in\mu' \\ i\notin C,\, j\notin C'}} a_{ij} > \sum_{\substack{(i,j)\in\mu_{|(M\setminus I_1)\times(M'\setminus J_2)} \\ i\notin C,\, j\notin C'}} a_{ij}\,. \tag{A.2}$$

In this case, we define

$$\begin{aligned}\tilde{\mu} = &\{(i,\mu'(i)) \mid i \notin I_1 \cup C \text{ and } \mu'(i) \notin J_2 \cup C'\} \\ &\cup \{(i,\mu(i)) \mid i \in I_1 \cup C \text{ or } \mu(i) \in J_2 \cup C'\}\end{aligned}$$

and will prove that $\tilde{\mu} \in \mathcal{M}(M, M')$. Let us write $A = \{(i,\mu'(i)) \mid i \notin I_1 \cup C\,,\ \mu'(i) \notin J_2 \cup C'\}$, $B = \{(i,\mu(i)) \mid i \in I_1 \cup C \text{ or } \mu(i) \in J_2 \cup C'\}$ and prove that if $(i,\mu(i)) \in B$, then neither $(i,\mu'(i)) \in A$ nor $(i',\mu(i)) \in A$ for any $i' \in M \setminus (I_1 \cup C)$. We distinguish two cases: $i \in I_1 \cup C$ or $\mu(i) \in J_2 \cup C'$.

If $(i,\mu(i)) \in B$ with $i \in I_1 \cup C$, then it is immediate that $(i,\mu'(i)) \notin A$. On the other hand, if $(i',\mu(i)) \in A$ for some $i' \in M \setminus (I_1 \cup C)$, then $\mu(i) = \mu'(i')$. Now, if $i \in C$, then $\mu(i) = \mu'(i')$ implies $i' \in C$ in contradiction with $(i',\mu(i)) \in A$. Otherwise, $i \in I_1$, but then $\mu(i) = \mu'(i') \in \mu(I_1)$ implies $i' \in K_1 \subseteq C$ and also contradicts $(i',\mu(i)) \in A$.

If $(i,\mu(i)) \in B$ with $\mu(i) \in J_2 \cup C'$, then it is immediate that $(i',\mu(i)) \notin A$ for any $i' \in M \setminus (I_1 \cup C)$ such that $\mu'(i') = \mu(i)$. On the other hand, if $(i,\mu'(i)) \in A$, from $\mu(i) \in J_2 \cup C'$ we get that either $\mu(i) \in C'$ or $\mu(i) \in J_2$. In the first case, that is $\mu(i) \in C'$, by the definition of the chains in C', $\mu(i) = \mu'(i')$ for some $i' \in M \setminus I_1$. As a consequence we have $i = \mu^{-1}(\mu'(i))$ where $\mu'(i')$ is matched by μ and thus by μ_2 and, moreover, $\mu^{-1}(\mu'(i))$ is matched by μ', since we are assuming $(i,\mu'(i)) \in A$. All this, by the definition of C' (in fact of the chains $C(k)$ for $k \in K_2$) implies that $i \in C'$ and thus $\mu'(i) \in C'$, in contradiction with $(i,\mu'(i)) \in A$. In the second case, that is, $\mu(i) \in J_2$, $i = \mu^{-1}(\mu(i)) \in \mu^{-1}(J_2)$ and, since $i = \mu'^{-1}(\mu'(i))$, we obtain $\mu'(i) \in M' \setminus J_2$ and $\mu'^{-1}(\mu'(i)) \in \mu^{-1}(J_2)$ which implies $\mu'(i) \in K_2 \subseteq C'$, in contradiction with $(i,\mu'(i)) \in A$.

Once proved that $\tilde{\mu} \in \mathcal{M}(M, M')$, we have

$$\sum_{(i,j)\in\tilde{\mu}} a_{ij} = \sum_{(i,j)\in A} a_{ij} + \sum_{(i,j)\in B} a_{ij}$$

$$> \sum_{\substack{(i,j)\in\mu \\ i\notin I_1\cup C,\, j\notin J_2\cup C'}} a_{ij} + \sum_{\substack{(i,j)\in\mu \\ i\in I_1\cup C \text{ or } j\in J_2\cup C'}} a_{ij} = \sum_{(i,j)\in\mu} a_{ij},$$

where the inequality follows from (A.2), and this contradicts $\mu \in \mathcal{M}^*_A(M, M')$. □

PROOF OF PROPOSITION 10

1) Recall we are assuming that $|M| = |M'|$ and μ does not leave agents unmatched. If $(\overline{u}^A, \underline{v}^A) = (a, 0)$ and $(\underline{u}^A, \overline{v}^A) = (0, a)$, then L is the segment between $(\overline{u}^A, \underline{v}^A)$ and $(\underline{u}^A, \overline{v}^A)$, which we denote by $L = [(\overline{u}^A, \underline{v}^A), (\underline{u}^A, \overline{v}^A)]$, and we have $L \subseteq C(w_A) \subseteq V^\mu(w_A)$.

Assume now that $(\underline{u}^A, \overline{v}^A) = (0, a)$ but $(\overline{u}^A, \underline{v}^A) \neq (a, 0)$. By part 1) of Corollary 5 we know that the subset $I_1 \subseteq M$ such that $I_1 = \{i \in M \mid \overline{u}^A_i = a_{i\mu(i)}\}$ is non-empty and, by and part 1) of Proposition 4, $((M\backslash I_1)\cup M', w_{A_{-I_1}})$ is a μ-compatible subgame of $(M\cup M', w_A)$. Let $(\overline{u}^{A-I_1}, \underline{v}^{A-I_1})$ be the buyers-optimal core allocation of this subgame and let $(\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e$ be its extension to $B^\mu(w_A)$:

$$(\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e_k = \begin{cases} \overline{u}^{A-I_1}_k & \text{if } k \in M \setminus I_1, \\ a_{k\mu(k)} & \text{if } k \in I_1, \\ \underline{v}^{A-I_1}_k & \text{if } k \in M'. \end{cases}$$

Consider now the piecewise linear curve

$$L^1 = [(\underline{u}^A, \overline{v}^A), (\overline{u}^A, \underline{v}^A)] \cup [(\overline{u}^A, \underline{v}^A), (\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e].$$

Notice that $[(\underline{u}^A, \overline{v}^A), (\overline{u}^A, \underline{v}^A)] \subseteq C(w_A)$ and, since $\overline{u}^A_i = a_{i\mu(i)}$, for all $i \in I_1$, trivially by (6) and (7) we have $(\overline{u}^A, \underline{v}^A) \in \hat{C}(w_{A_{-I_1}})$ and thus $[(\overline{u}^A, \underline{v}^A), (\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e] \subseteq \hat{C}(w_{A_{-I_1}})$.

Then,

$$L^1 \subseteq C(w_A) \cup \hat{C}(w_{A_{-I_1}}) \subseteq V^\mu(w_A).$$

and, if $(\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e = (a, 0)$, we are done.

Assume there exists a chain $I_0 = \emptyset \subset I_1 \subset I_2 \subset \cdots \subset I_r \subseteq M$ such that, for all $k \in \{1, 2, \ldots, r\}$, $((M \setminus I_k) \cup M', w_{A_{-I_k}})$ is a μ-compatible subgame of $((M \setminus I_{k-1}) \cup M', w_{A_{-I_{k-1}}})$, and thus, by Remark 6, also of $(M \cup M', w_A)$. Then

$$L^r = [(\underline{u}^A, \overline{v}^A), (\overline{u}^A, \underline{v}^A)] \cup [(\overline{u}^A, \underline{v}^A), (\overline{u}^{A-I_1}, \underline{v}^{A-I_1})^e] \cup \bigcup_{k=2}^{r} \left[(\overline{u}^{A-I_{k-1}}, \underline{v}^{A-I_{k-1}})^e, (\overline{u}^{A-I_k}, \underline{v}^{A-I_k})^e\right] \subseteq V^\mu(w_A)$$

If $(\overline{u}^{A_{-I_r}}, \underline{v}^{A_{-I_r}})^e = (a,0)$ we are done. Otherwise, let it be $I = \{i \in M \setminus I_r \mid \overline{u}^{A_{-I_r}} = a_{i\mu(i)}\}$ and notice that again by Corollary 5 and Proposition 4, I is nonempty and, if we write $I_{r+1} = I_r \cup I$, we have that $((M \setminus I_{r+1}) \cup M', w_{A_{-I_{r+1}}})$ is a μ-compatible subgame of $((M \setminus I_r) \cup M', w_{A_{-I_r}})$ and thus, by Remark 6, also of $(M \cup M', w_A)$. Then, the piecewise linear curve

$$L^{r+1} = L^r \cup \left[(\overline{u}^{A_{-I_r}}, \underline{v}^{A_{-I_r}})^e, (\overline{u}^{A_{-I_{r+1}}}, \underline{v}^{A_{-I_{r+1}}})^e\right]$$

is contained in $V^\mu(w_A)$. Since the inclusion between the sets I_k is strict, the procedure will finish in a finite number s of steps and then we get a piecewise linear curve $L^s \subseteq V^\mu(w_A)$ from $(0,a)$ to $(a,0)$.

Notice that if $(\underline{u}^A, \overline{v}^A) \neq (0,a)$ we would analogously obtain a piecewise linear curve from $(\underline{u}^A, \overline{v}^A)$ to $(0,a)$, and contained in $V^\mu(w_A)$.

2) Let it be $(u,v) \in I(w_A) \setminus B^\mu(w_A)$. This implies that there exists $(i,j) \in \mu$ such that $u_i + v_j < a_{ij}$ and, as a consequence, $0 \leq u_i < a_{ij} - v_j$. By the construction of the curve L in part 1), we can find $(u',v') \in L$ such that $u_i < u'_i < a_{ij} - v_j$. Then, not only $u'_i > u_i$ but also, since $(u',v') \in L \subseteq B^\mu(w_A)$, we have $u'_i + v'_j = a_{ij}$ and then $v'_j = a_{ij} - u'_i > v_j$. Thus, $(u',v')\, dom^{w_A}_{\{i,j\}} (u,v)$.
□

Before proving the Theorem 12 we need to prove the following statement.

Lemma 15 *Let $(M \cup M', w_A)$ be an assignment game, $\mu \in \mathcal{M}^*_A(M,M')$ and $(u,v) \in B^\mu(w_A)$.*

(1) Let $I \subset M$ and $J \subset J' \subset M'$, $(J' \setminus J) \cap \mu(I) \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$, such that $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame of $(M \cup M', w_A)$. If $v_j = \overline{v}_j^{A_{-I\cup J}} = a_{\mu^{-1}(j)j}$ for all $j \in J' \setminus J$ and there exists $j^* \in M' \setminus J'$ such that $\underline{v}_{j^*}^{A_{-I\cup J}} \leq v_{j^*} < \underline{v}_{j^*}^{A_{-I\cup J'}}$, then there exists $(u'',v'') \in \hat{C}(w_{A_{-I\cup J}})$ such that $(u'',v'') dom^{w_A} (u,v)$.

(2) Let $J \subset M'$ and $I \subset I' \subset M$, $(I' \setminus I) \cap \mu^{-1}(J) \subseteq \{i \in M \mid a_{i\mu(i)} = 0\}$, such that $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is a μ-compatible subgame of $(M \cup M', w_A)$. If $u_i = \overline{u}_i^{A_{-I\cup J}} = a_{i\mu(i)}$ for all $i \in I' \setminus I$ and there exists $i^* \in M \setminus I'$ such that $\underline{u}_{i^*}^{A_{-I\cup J}} \leq u_{i^*} < \underline{u}_{i^*}^{A_{-I'\cup J}}$, then there exists $(u'',v'') \in \hat{C}(w_{A_{-I\cup J}})$ such that $(u'',v'') dom^{w_A} (u,v)$.

PROOF. We prove part 1), since 2) is proved analogously. By Proposition 4, $((M \setminus I) \cup (M' \setminus J'), w_{A_{-I\cup J'}})$ is a μ-compatible subgame of $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$, and, since $(J' \setminus J) \cap \mu(I) \subseteq \{j \in M' \mid a_{\mu^{-1}(j)j} = 0\}$, by Remark 6, also of the initial game $(M \cup M', w_A)$.

Consider the oriented tight graph $G^* = G(\overline{u}^{A_{-I\cup J'}}, \underline{v}^{A_{-I\cup J'}})$ defined by the set of nodes $(M \setminus I) \cup (M' \setminus J')$ and the arcs $i \longrightarrow j$ whenever $(i,j) \in \mu$ and $j \longrightarrow i$ whenever $\overline{u}_i^{A_{-I\cup J'}} + \underline{v}_j^{A_{-I\cup J'}} = a_{ij}$ but $(i,j) \notin \mu$. Since $\underline{v}_{j^*}^{A_{-I\cup J'}} > 0$, following p. 213 of Roth and Sotomayor (1990) we know there exists a path $c^* = (j^*, i_1^*, j_1^*, i_2^*, \ldots, i_s^*, (j_s^*))$ in G ending either at $i_s^* \in M \setminus I$ unmatched by $\mu_{|(M\setminus I)\times(M'\setminus J')}$ or at $j_s^* \in M' \setminus J'$ with $\underline{v}_{j_s^*}^{A_{-I\cup J'}} = 0$.

If $\underline{v}_{j_s^*}^{A_{-I\cup J'}} = 0$, from (12) we have $\underline{v}_{j_s^*}^{A_{-I\cup J}} \le \underline{v}_{j_s^*}^{A_{-I\cup J'}}$, and thus we get $\underline{v}_{j_s^*}^{A_{-I\cup J}} = 0$ and consequently, since $i_s^* = \mu^{-1}(j_s^*)$, $\overline{u}_{i_s^*}^{A_{-I\cup J'}} = \overline{u}_{i_s^*}^{A_{-I\cup J}}$. But then we have

$$\underline{v}_{j_{s-1}^*}^{A_{-I\cup J'}} = a_{i_s^* j_{s-1}^*} - \overline{u}_{i_s^*}^{A_{-I\cup J'}} = a_{i_s^* j_{s-1}^*} - \overline{u}_{i_s^*}^{A_{-I\cup J}} \le \underline{v}_{j_{s-1}^*}^{A_{-I\cup J}},$$

where the last inequality follows from the fact that $(\overline{u}^{A_{-I\cup J}}, \underline{v}^{A_{-I\cup J}}) \in C(w_{A_{-I\cup J}})$. This, taking into account that from (12) we have $\underline{v}_{j_{s-1}^*}^{A_{-I\cup J}} \le \underline{v}_{j_{s-1}^*}^{A_{-I\cup J'}}$, leads to $\underline{v}_{j_{s-1}^*}^{A_{-I\cup J}} = \underline{v}_{j_{s-1}^*}^{A_{-I\cup J'}}$ and, repeatedly applying the same argument along the path c^*, we would deduce $\underline{v}_{j^*}^{A_{-I\cup J}} = \underline{v}_{j^*}^{A_{-I\cup J'}}$, in contradiction with the assumption.

Then the path c^* must end at some $i_s^* \in M \setminus I$ unmatched by $\mu_{|(M\setminus I)\times(M'\setminus J')}$. If i_s^* were also unmatched by $\mu_{|(M\setminus I)\times(M'\setminus J)}$, then $\overline{u}_{i_s^*}^{A_{-I\cup J}} = \overline{u}_{i_s^*}^{A_{-I\cup J'}} = 0$ and following the same argument as in the paragraph above we reach again the contradiction $\underline{v}_{j^*}^{A_{-I\cup J}} = \underline{v}_{j^*}^{A_{-I\cup J'}}$.

As a consequence of all that, the path c^* ends at some i_s^* unmatched by $\mu_{|(M\setminus I)\times(M'\setminus J')}$, but that is matched by $\mu_{|(M\setminus I)\times(M'\setminus J)}$ and so $\mu(i_s^*) \in J' \setminus J$ and by assumption in part 1) of the lemma, $v_{\mu(i_s^*)} = a_{i_s^* \mu(i_s^*)}$. Then, since $(u,v) \in B^\mu(w_A)$ we get

$$u_{i_s^*} = 0. \tag{A.3}$$

Let us now consider the μ-compatible subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$. From part 2) of Proposition 7 applied to the game $((M \setminus I) \cup (M' \setminus J'), w_{A_{-I\cup J'}})$, the payoff vector obtained by completing $(\overline{u}^{A_{-I\cup J'}}, \underline{v}^{A_{-I\cup J'}})$ with $a_{\mu^{-1}(j)j}$ for all $j \in J' \setminus J$, that is $(\overline{u}^{A_{-I\cup J'}}, \underline{v}^{A_{-I\cup J'}}, (a_{\mu^{-1}(j)j})_{j\in J'\setminus J})$, belongs to $C(w_{A_{-I\cup J}})$. Let us denote by (u', v') this payoff vector and construct its oriented tight graph, $G(u', v')$, now defined by the set of nodes $(M \setminus I) \cup (M' \setminus J)$ and the arcs $i \longrightarrow j$ whenever $(i,j) \in \mu$ and $j \longrightarrow i$ whenever $u'_i + v'_j = a_{ij}$ but $(i,j) \notin \mu$. Notice that all the nodes and arcs of the former graph G^* also belong to $G(u', v')$ but this new graph may have some additional nodes and arcs. In particular, c^* is a path in $G(u', v')$.

Since, by assumption, $\underline{v}_{j^*}^{A_{-I\cup J}} \le v_{j^*} < \underline{v}_{j^*}^{A_{-I\cup J'}}$ holds, we know there exists $(\tilde{u}, \tilde{v}) \in C(w_{A_{-I\cup J}})$ such that $\tilde{v}_{j^*} < \underline{v}_{j^*}^{A_{-I\cup J'}}$, let us say $\tilde{v}_{j^*} = \underline{v}_{j^*}^{A_{-I\cup J'}} - \varepsilon = v'_{j^*} - \varepsilon$ for some $\varepsilon > 0$. Take any path in $G(u', v')$ starting at j^*, let us say

$c = (j^*, i_1, j_1, i_2, \ldots, i_t, (j_t))$, where the parenthesis in the last node indicates that the path may end either at a buyer or at a seller. Since $(\tilde{u}, \tilde{v})$ is in $C(w_{A_{-I\cup J}})$ it must be that $\tilde{u}_{i_1} = u'_{i_1} + \varepsilon_1$ for some $\varepsilon_1 \geq \varepsilon$. Then, $\tilde{v}_{j_1} = v'_{j_1} - \varepsilon_1$, which implies $\tilde{u}_{i_2} = u'_{i_2} + \varepsilon_2$ for some $\varepsilon_2 \geq \varepsilon_1$. By repeatedly applying the same argument we get that, for all i_l in this path, there exists $\varepsilon_l \geq \varepsilon_{l-1} > 0$ such that $\tilde{u}_{i_l} = u'_{i_l} + \varepsilon_l$ and $\tilde{v}_{j_l} = v'_{j_l} - \varepsilon_l$. This implies that $v'_{j_l} > 0$ for all j_l in the path c, and also that, since $\tilde{u}_{i_l} > 0$, there is no i_l in c unmatched by $\mu_{|(M\setminus I)\times(M'\setminus J)}$.

Let S be the set of $i \in M \setminus I$ that can be reached from j^* in $G(u', v')$ and T be the set of $j \in M' \setminus J$ that can be reached from j^*. Notice that $j \in T$ if and only if $\mu^{-1}(j) \in S$. Also, if $j \in T$ and $i \in M \setminus I$ is such that $u'_i + v'_j = a_{ij}$, then $i \in S$. Define now

$$
\begin{aligned}
u''_i &= u'_i + \varepsilon, \text{ for all } i \in S \cup \{\mu^{-1}(j^*)\}, \; u''_i = u'_i, \text{ for all } i \in M \setminus (I \cup S \cup \{\mu^{-1}(j^*)\}), \\
v''_j &= v'_j - \varepsilon, \text{ for all } j \in T, \qquad\qquad v''_j = v'_j, \text{ for all } j \in M' \setminus (J \cup T).
\end{aligned}
$$

For $\varepsilon > 0$ small enough we claim (and the proof is left to the reader) that $(u'', v'') \in C(w_{A_{-I\cup J}})$, $v_{j^*} < v''_{j^*}$ (since $v_{j^*} < \underline{v}^{A_{-I\cup J'}}_{j^*} = v'_{j^*}$) and $v''_{j^*_l} \neq v_{j^*_l}$ for all j^*_l in the inicial path c^*.

Recall from (A.3) that $u_{i^*_s} = 0$ and notice that $i^*_s \in S$ and thus $u_{i^*_s} = 0 < u''_{i^*_s}$. If $v_{j^*_{s-1}} < v''_{j^*_{s-1}}$ we are done, since $u''_{i^*_s} + v''_{j^*_{s-1}} = a_{i^*_s j^*_{s-1}}$. Thus, the extension of (u'', v'') by paying $a_{i\mu(i)}$ to all $i \in I \cap \mu^{-1}(M')$, $a_{\mu^{-1}(j)j}$ to all $j \in J \cap \mu(M)$ and zero to all agents in $I \cup J$ unassigned by μ, belongs to $\hat{C}(w_{A_{-I\cup J}}) \subseteq V^{\mu}(w_A)$ and dominates (u, v) via coalition $\{i^*_s, j^*_{s-1}\}$. Otherwise, we have $v_{j^*_{s-1}} > v''_{j^*_{s-1}}$ and consequently $u_{i^*_{s-1}} < u''_{i^*_{s-1}}$. We then repeat the argument replacing j^*_{s-1} with j^*_{s-2} and i^*_s with i^*_{s-1}. In this way we will either get that (u'', v'') dominates (u_{-I}, v_{-J}) via coalition $\{i^*_l, j^*_{l-1}\}$ for some $l \in \{2, \ldots, s\}$ or else we reach $u_{i^*_1} < u''_{i^*_1}$ and since $v_{j^*} < v''_{j^*}$ and $u''_{i^*_1} + v''_{j^*} = u'_{i^*_1} + v'_{j^*} = a_{i^*_1 j^*}$ we obtain that the extension of (u'', v'') by paying $a_{i\mu(i)}$ to all $i \in I \cap \mu^{-1}(M')$, $a_{\mu^{-1}(j)j}$ to all $j \in J \cap \mu(M)$ and zero to all agents in $I \cup J$ unassigned by μ, belongs to $\hat{C}(w_{A_{-I\cup J}}) \subseteq V^{\mu}(w_A)$ and dominates (u, v) via coalition $\{i^*_1, j^*\}$. $\square$

Proof of Theorem 12 (Continuation)

PROOF. It remained to see that, under the assumption that $\mu(M) = M'$ and $a_{ij} > 0$ for all $(i, j) \in \mu$, any element in $B^{\mu}(w_A) \setminus (R^{\mu}(w_A) \cup V^{\mu}(w_A))$ is also dominated by some element of $V^{\mu}(w_A)$.

Take $(u,v)\in B^{\mu}(w_A)\backslash(R^{\mu}(w_A)\cup V^{\mu}(w_A))$. Let us define the following subsets of players.

Take first $I_0^0=J_0^0=\emptyset$.

Recursively, for $k\geq 1$, define $\tilde{I}_0^k\subseteq M\setminus(I_0^{k-1}\cup\mu^{-1}(J_0^{k-1}))$ and $\tilde{J}_0^k\subseteq M'\setminus(J_0^{k-1}\cup\mu(I_0^{k-1}))$ in such a way that $|\tilde{I}_0^k|=|\tilde{J}_0^k|=1$ and there exists $(x,y)\in C(w_{A_{-I_0^{k-1}\cup J_0^{k-1}}})$ such that $u_i=x_i=a_{i\mu(i)}$ for $i\in\tilde{I}_0^k$ and $v_j=y_j=a_{\mu^{-1}(j)j}$ for all $j\in\tilde{J}_0^k$. Then we write $I_0^k=\tilde{I}_0^k\cup I_0^{k-1}$ and $J_0^k=\tilde{J}_0^k\cup J_0^{k-1}$.

By Proposition 4, $((M\setminus I_0^1)\cup(M'\setminus J_0^1),w_{A_{-I_0^1\cup J_0^1}})$ is a μ-compatible subgame of $(M\cup M',w_A)$ and by an induction argument, together with Proposition 4 and Remark 6, we obtain that, for all $k\geq 0$, $((M\setminus I_0^k)\cup(M'\setminus J_0^k),w_{A_{-I_0^k\cup J_0^k}})$ is a μ-compatible subgame of $((M\setminus I_0^{k-1})\cup(M'\setminus J_0^{k-1}),w_{A_{-I_0^{k-1}\cup J_0^{k-1}}})$ and thus of $(M\cup M',w_A)$. Moreover, $(u_{-I_0^k},v_{-J_0^k})\in B^{\mu}(w_{A_{-I_0^k\cup J_0^k}})$. Notice also that, as an immediate consequence of the definition of $\tilde{I}_0^k$ and $\tilde{J}_0^k$, we have that $\overline{u}_i^{A_{-I_0^{k-1}\cup J_0^{k-1}}}=a_{i\mu(i)}$ for $i\in\tilde{I}_0^k$ and $\overline{v}_j^{A_{-I_0^{k-1}\cup J_0^{k-1}}}=a_{\mu^{-1}(j)j}$ for $j\in\tilde{J}_0^k$.

By the finiteness of M and M', there exists $d\geq 0$ such that $\tilde{I}_0^{d+1}\times\tilde{J}_0^{d+1}=\emptyset$. Then, $I_0^d\times J_0^d$ is the last subset of the sequence. To simplify notation, in the sequel we write $I_0=I_0^d$ and $J_0=J_0^d$.

Once we have exhausted the possibility of building μ-compatible subgames (related to the payoff vector (u,v)) by removing one agent on each side of the market, we continue the procedure but now removing several agents of only one side, let us say the buyers' side.

Now, let us define

$$\tilde{I}_1=\{i\in M\setminus(I_0\cup\mu^{-1}(J_0))\mid u_i=\overline{u}_i^{A_{-I_0\cup J_0}}=a_{i\mu(i)}\},\qquad I_1=\tilde{I}_1\cup I_0$$

and, for all $k>1$

$$\tilde{I}_k=\{i\in M\setminus(I_{k-1}\cup\mu^{-1}(J_0))\mid u_i=\overline{u}_i^{A_{-I_{k-1}\cup J_0}}=a_{i\mu(i)}\}$$

and, as long as $\tilde{I}_k\neq\emptyset$, $I_k=I_{k-1}\cup\tilde{I}_k$.

By definition, $I_0\subset I_1\subset\cdots\subset I_{k-1}\subset I_k$ and there exists $r+1\geq 0$ such that $\tilde{I}_{r+1}=\emptyset$, which implies that the sequence[8] ends at I_r. Notice that $M\backslash I_r\neq\emptyset$

[8] Notice that the sequences of sets, $I_0^1\times J_0^1\subset I_0^2\times J_0^2\subset\cdots\subset I_0^d\times J_0^d=I_0\times J_0$ and $I_0\subset I_1\subset\cdots\subset I_{r-1}\subset I_r$ depend on the allocation (u,v) we want to dominate.

since otherwise we would have $(u,v)=(a,0)$, and, by Proposition 10, $(a,0)\in V^{\mu}(w_A)$, in contradiction with $(u,v)\in B^{\mu}(w_A)\setminus(R^{\mu}(w_A)\cup V^{\mu}(w_A))$.

By repeatedly applying part 1) of Proposition 4 and Remark 6 we obtain that, for all $k\in\{0,1,\ldots,r\}$, $((M\setminus I_k)\cup(M'\setminus J_0),w_{A_{-I_k\cup J_0}})$ is a μ-compatible subgame of $(M\cup M',w_A)$. Moreover, $(u_{-I_r},v_{-J_0})\in B^{\mu}(w_{A_{-I_r\cup J_0}})$ and different cases are considered.

Case 1: $(u_{-I_r},v_{-J_0})\in R^{\mu}(w_{A_{-I_r\cup J_0}})$.

In this case, since $(u,v)\notin V^{\mu}(w_A)$, it must be the case that $(u_{-I_r},v_{-J_0})\notin C(w_{A_{-I_r\cup J_0}})$ and thus we can apply Proposition 11 to the game $((M\setminus I_r)\cup(M'\setminus J_0),w_{A_{-I_r\cup J_0}})$ and obtain there exists $(u',v')\in C(w_{A_{-I_r\cup J_0}})$ that dominates (u_{-I_r},v_{-J_0}). We then define (u'',v'') by $u''_i=u'_i$ for all $i\in M\setminus I_r$, $u''_i=a_{i\mu(i)}$ for all $i\in I_r$, $v''_j=v'_j$ for all $j\in M'\setminus J_0$ and $v''_j=a_{\mu^{-1}(j)j}$ for all $j\in J_0$. Then, $(u'',v'')\,dom^{w_A}(u,v)$ and, by (6) and (14), $(u'',v'')\in\hat{C}(w_{A_{-I_r\cup J_0}})\subseteq V^{\mu}(w_A)$.

Case 2: There exists $i^*\in M\setminus I_r$ such that $u_{i^*}>\overline{u}_{i^*}^{A_{-I_r\cup J_0}}$.

Since $(u_{-I_r},v_{-J_0})\in B^{\mu}(w_{A_{-I_r\cup J_0}})$ and $u_{i^*}>0$ we can guarantee that i^* is matched by $\mu_{|(M\setminus I_r)\times(M'\setminus J_0)}$. Then, the above statement is equivalent to

$$0\le v_{j_0}<\underline{v}_{j_0}^{A_{-I_r\cup J_0}},\ \text{for}\ j_0=\mu(i^*). \tag{A.4}$$

Since $((M\setminus I_r)\cup(M'\setminus J_0),w_{A_{-I_r\cup J_0}})$ and $((M\setminus I_0^k)\cup(M'\setminus J_0^k),w_{A_{-I_0^k\cup J_0^k}})$, for all $k\in\{0,1,\ldots,d\}$, are μ-compatible subgames of $(M\cup M',w_A)$, by Lemma 14 we obtain that $((M\setminus I_r)\cup(M'\setminus J_0^k),w_{A_{-I_r\cup J_0^k}})$ is also a μ-compatible subgame of $(M\cup M',w_A)$, and this for all $k\in\{0,1,\ldots,d\}$. Moreover, from (12), and taking into account that $J_0^0=\emptyset$ and $J_0^d=J_0$, it holds

$$\underline{v}_{j_0}^{A_{-I_r}}\le\underline{v}_{j_0}^{A_{-I_r\cup J_0^1}}\le\cdots\le\underline{v}_{j_0}^{A_{-I_r\cup J_0^{d-1}}}\le\underline{v}_{j_0}^{A_{-I_r\cup J_0}} \tag{A.5}$$

and thus, either $\underline{v}_{j_0}^{A_{-I_r\cup J_0^{k-1}}}\le v_{j_0}<\underline{v}_{j_0}^{A_{-I_r\cup J_0^k}}$, for some $k\in\{1,2,\ldots,d\}$ or $v_{j_0}<\underline{v}_{j_0}^{A_{-I_r}}$.

Case 2.1: There exists $k\in\{1,2,\ldots,d\}$ such that $\underline{v}_{j_0}^{A_{-I_r\cup J_0^{k-1}}}\le v_{j_0}<\underline{v}_{j_0}^{A_{-I_r\cup J_0^k}}$.

Since both $((M\setminus I_r)\cup(M'\setminus J_0^k),w_{A_{-I_r\cup J_0^k}})$ and $((M\setminus I_r)\cup(M'\setminus J_0^{k-1}),w_{A_{-I_r\cup J_0^{k-1}}})$ are μ-compatible subgames of $(M\cup M',w_A)$, we obtain that $((M\setminus I_r)\cup(M'\setminus$

$J_0^k), w_{A_{-I_r\cup J_0^k}})$ is a μ-compatible subgame of $((M\setminus I_r)\cup(M'\setminus J_0^{k-1}), w_{A_{-I_r\cup J_0^{k-1}}})$. To prove this, recall that $\tilde{J}_0^k = J_0^k \setminus J_0^{k-1}$ and $|\tilde{J}_0^k| = 1$, and write $\{j'\} = \tilde{J}_0^k$. Notice then that

$$\begin{aligned} w_A(M\cup M') &= w_A((M\setminus I_r)\cup(M'\setminus J_0^k)) + \sum_{i\in I_r} a_{i\mu(i)} + \sum_{j\in J_0^k} a_{\mu^{-1}(j)j} = \\ &= \sum_{(i,j)\in\mu\cap(M\setminus I_r)\times(M'\setminus J_0^k)} a_{ij} + \sum_{i\in I_r} a_{i\mu(i)} + \sum_{j\in J_0^k} a_{\mu^{-1}(j)j} = \\ &= \sum_{(i,j)\in\mu\cap(M\setminus I_r)\times(M'\setminus J_0^k)} a_{ij} + a_{\mu^{-1}(j')j'} + \sum_{i\in I_r} a_{i\mu(i)} + \sum_{j\in J_0^{k-1}} a_{\mu^{-1}(j)j} \leq \\ &\leq w_A((M\setminus I_r)\cup(M'\setminus J_0^{k-1})) + \sum_{i\in I_r} a_{i\mu(i)} + \sum_{j\in J_0^{k-1}} a_{\mu^{-1}(j)j} = w_A(M\cup M'). \end{aligned}$$

where the inequality holds because $\mu^{-1}(j') \notin I_r$ [9] and thus $\mu\cap((M\setminus I_r)\times(M'\setminus J_0^k))\cup\{(\mu^{-1}(j'),j')\}$ is a matching of $(M\setminus I_r)\times(M'\setminus J_0^{k-1})$.

As a consequence, $w_A((M\setminus I_r)\cup(M'\setminus J_0^k)) + a_{\mu^{-1}(j')j'} = w_A((M\setminus I_r)\cup(M'\setminus J_0^{k-1}))$, which means that $((M\setminus I_r)\cup(M'\setminus J_0^k), w_{A_{-I_r\cup J_0^k}})$ is a μ-compatible subgame of $((M\setminus I_r)\cup(M'\setminus J_0^{k-1}), w_{A_{-I_r\cup J_0^{k-1}}})$.

Then, since $|\tilde{J}_0^k| = 1$, Proposition 4 guarantees that $\overline{v}_{j'}^{A_{-I_r\cup J_0^{k-1}}} = a_{\mu^{-1}(j')j'}$, and by definition of $\tilde{J}_0^k$ we get $v_{j'} = \overline{v}_{j'}^{A_{-I_r\cup J_0^{k-1}}} = a_{\mu^{-1}(j')j'}$. We then are on the assumptions of Lemma 15 taking $I = I_r$, $J = J_0^{k-1}$ and $J' = J_0^k$, and thus obtain that there exists $(u'',v'') \in \hat{C}(w_{A_{-I_r\cup J_0^{k-1}}})$ such that $(u'',v'')\, dom^{w_A}(u,v)$.

Case 2.2: $0 \leq v_{j_0} < \underline{v}_{j_0}^{A_{-I_r}}$.

We now construct $G = G(\overline{u}^{A_{-I_r}}, \underline{v}^{A_{-I_r}})$, the oriented graph related to the core constrains of the μ-compatible subgame $((M\setminus I_r)\cup M', w_{A_{-I_r}})$ that are tight at $(\overline{u}^{A_{-I_r}}, \underline{v}^{A_{-I_r}})$: the set of vertices is $(M\setminus I_r)\cup M'$ and the arcs are defined by: $i \longrightarrow j$ if $(i,j)\in\mu$, and $j \longrightarrow i$ if $\overline{u}_i^{A_{-I_r}} + \underline{v}_j^{A_{-I_r}} = a_{ij}$ but $(i,j)\notin\mu$. Since $\underline{v}_{j_0}^{A_{-I_r}} > 0$, as proved in page 213 of Roth and Sotomayor (1990), there exists a path starting at j_0 and ending either at $i_s \in M\setminus I_r$

[9] Since $I_r = \tilde{I}_{r-1}\cup\cdots\cup\tilde{I}_l\cup\cdots\cup\tilde{I}_1\cup\tilde{I}_0^d\cup\cdots\cup\tilde{I}_0^{k'}\cup\cdots\cup I_0^0$, if $\mu^{-1}(j')\in I_r$, either $\mu^{-1}(j')\in\tilde{I}_l$ and thus by definition $j'\notin J_0$, which contradicts $j'\in\tilde{J}_0^k$, or $\mu^{-1}(j')\in\tilde{I}_0^{k'}$ for some $k'\in\{1,\ldots,d\}$. In this case, if $k'>k$, then by definition of $\tilde{I}_0^{k'}$, $j'\notin J_0^{k'-1}$ which contradicts $j'\in\tilde{J}_0^k\subseteq J_0^k\subseteq J_0^{k'-1}$. If $k'<k$, then $\mu^{-1}(j')\in\tilde{I}_0^{k'}\subseteq I_0^{k-1}$ and this contradicts $j'\in\tilde{J}_0^k$. Finally, if $k'=k$, then, by definition of $\tilde{I}_0^k$ and $\tilde{J}_0^k$, there exists $(x,y)\in C(w_{A_{-I_0^{k-1}\cup J_0^{k-1}}})$ such that $x_{\mu^{-1}(j')} = a_{\mu^{-1}(j')j'}$ and $y_{j'} = a_{\mu^{-1}(j')j'}$, which implies $a_{\mu^{-1}(j')j'} = 0$, in contradiction with the assumption.

unmatched by $\mu_{|(M\setminus I_r)\times M'}$ or at $j_s \in M'$ with $\underline{v}_{j_s}^{A-I_r}=0$. Since $\mu(M)=M'$, the first possibility cannot hold, and thus it must be the case that the path ends at $j_s \in M'$ with $\underline{v}_{j_s}^{A-I_r}=0$. Let $c=(j_0,i_1,j_1,\ldots,i_s,j_s)$ be such a path and assume without loss of generality that j_s is the first $j\in M'$ in the path such that $\underline{v}_{j}^{A-I_r}=0$.

Since $\underline{v}_{j_s}^{A-I_r}=0$, we have that

$$\overline{u}_{i_s}^{A-I_r}=a_{i_sj_s} \text{ and } u_{i_s}\leq \overline{u}_{i_s}^{A-I_r}. \tag{A.6}$$

Now, several cases are to be considered:

• If $u_{i_1}<\overline{u}_{i_1}^{A-I_r}$ we are done, since also $v_{j_0}<\underline{v}_{j_0}^{A-I_r}$ and $\underline{v}_{j_0}^{A-I_r}+\overline{u}_{i_1}^{A-I_r}=a_{i_1j_0}$ because $j_0\longrightarrow i_1$ is an arc in the path c. This means that $(\overline{u}^{A-I_r},\underline{v}^{-A_{I_r}})$ dominates (u_{-I_r},v) via coalition $\{i_1,j_0\}$. We then define (u',v') by

$$\begin{aligned} u'_i &= \overline{u}_i^{A-I_r} \text{ for all } i\in M\setminus I_r\,,\\ u'_i &= a_{i\mu(i)} \text{ for all } i\in I_r\\ v'_j &= \underline{v}_j^{A-I_r} \text{ for all } j\in M'\,. \end{aligned} \tag{A.7}$$

By (6) and (14), $(u',v')\in \hat{C}(w_{A_{-I_r}})\subseteq V^{\mu}(w_A)$ and moreover $(u',v')\, dom_{\{i_1,j_0\}}^{w_A}(u,v)$.

• If $u_{i_1}>\overline{u}_{i_1}^{A-I_r}$, then, since both (u_{-I_r},v) and $(\overline{u}^{A-I_r},\underline{v}^{A-I_r})$ belong to $B^{\mu}(w_{A_{-I_r}})$, $v_{j_1}<\underline{v}_{j_1}^{A-I_r}$ and we can repeat the above argument, replacing j_0 with j_1. If at some step $k\in\{1,\ldots,s-1\}$ we get $v_{j_k}<\underline{v}_{j_k}^{A-I_r}$ and $u_{i_{k+1}}<\overline{u}_{i_{k+1}}^{A-I_r}$, we are done, since $\underline{v}_{j_k}^{A-I_r}+\overline{u}_{i_{k+1}}^{A-I_r}=a_{i_{k+1}j_k}$ (the arc $j_k\longrightarrow i_{k+1}$ belongs to the tight graph G) and thus we have that $(\overline{u}^{A-I_r},\underline{v}^{A-I_r})$ dominates (u_{-I_r},v) via coalition $\{i_{k+1},j_k\}$. Then, the payoff vector (u',v'), as defined in (A.7), dominates (u,v) via coalition $\{i_{k+1},j_k\}$.

• Otherwise, for some $k\in\{1,\ldots,s\}$ we have $u_{i_l}>\overline{u}_{i_l}^{A-I_r}$ for all $l\in\{1,\ldots,k-1\}$ and $u_{i_k}=\overline{u}_{i_k}^{A-I_r}$. (Recall that by (A.6) we can guarantee that $u_{i_s}\leq\overline{u}_{i_s}^{A-I_r}$).

Since $u_{i_k}=\overline{u}_{i_k}^{A-I_r}\leq a_{i_kj_k}$, we first analyze the case where $u_{i_k}=\overline{u}_{i_k}^{A-I_r}=a_{i_kj_k}$. In this case, either $a_{i_kj_k}=u_{i_k}=\overline{u}_{i_k}^{A-I_r}=\overline{u}_{i_k}^{A-I_r\cup J_0}$, which implies that $i_k\in\tilde{I}_{r+1}$ [10] and contradicts that I_r is the last set in the sequence, or $a_{i_kj_k}=u_{i_k}=\overline{u}_{i_k}^{A-I_r}>\overline{u}_{i_k}^{A-I_r\cup J_0}$. But then $\underline{v}_{j_k}^{A-I_r}\leq v_{j_k}<\underline{v}_{j_k}^{A-I_r\cup J_0}$ and as a

[10] Notice that $i_k\notin\mu^{-1}(J_0)$ since otherwise $u_{i_k}=0=a_{i_kj_k}$, in contradiction with the assumption.

consequence there exists $t \in \{1, \ldots, d\}$ such that $\underline{v}_{j_k}^{A_{-I_r \cup J_0^{t-1}}} \leq v_{j_k} < \underline{v}_{j_k}^{A_{-I_r \cup J_0^t}}$. Then, similarly to Case 2.1, we can apply Lemma 15 taking $I = I_r$, $J = J_0^{t-1}$ and $J' = J_0^t$. To apply the aforementioned lemma, notice first that the same argument in footnote 9 guarantees that $(J_0^t \setminus J_0^{t-1}) \cap \mu(I_r) = \tilde{J}_0^t \cap \mu(I_r) = \emptyset$. Also, it has been proved in page 37 that $((M \setminus I_r) \cup (M' \setminus J_0^t), w_{A_{-I_r \cup J_0^t}})$ is a μ-compatible subgame of $((M \setminus I_r) \cup (M' \setminus J_0^{t-1}), w_{A_{-I_r \cup J_0^{t-1}}})$ and thus by Proposition 4 we obtain $\overline{v}_j^{A_{-I_r \cup J_0^{t-1}}} = a_{\mu^{-1}(j)j}$ for $j \in J_0^{t-1} \setminus J_0^t = \tilde{J}_0^t$ and, by definition of $\tilde{J}_0^t$, $v_j = \overline{v}_j^{A_{-I_r \cup J_0^{t-1}}} = a_{\mu^{-1}(j)j}$. Then, Lemma 15 guarantees that there exists $(u'', v'') \in \hat{C}(w_{A_{-I_r \cup J_0^{t-1}}})$ such that $(u'', v'')\, dom^{w_A}(u, v)$.

Otherwise, it holds $u_{i_k} = \overline{u}_{i_k}^{A_{-I_r}} < a_{i_k j_k}$. Let it be $I' = \{i \in M \setminus I_r \mid \overline{u}_i^{A_{-I_r}} = a_{i\mu(i)}\}$ and $I = I_r \cup I'$, and notice that, by (A.6), $i_s \in I'$ and thus I' is non-empty. Since, by Proposition 4, $((M \setminus I) \cup M', w_{A_{-I}})$ is a μ-compatible subgame of $((M \setminus I_r) \cup M', w_{A_{-I_r}})$ it follows from Remark 6 that $((M \setminus I) \cup M', w_{A_{-I}})$ is a μ-compatible subgame of the initial game $(M \cup M', w_A)$.

Let us denote by G' the restriction of $G = G(\overline{u}^{A_{-I_r}}, \underline{v}^{A_{-I_r}})$ to the player set $(M \setminus I) \cup M'$. Notice also that, by the definition of I, $\overline{u}_i^{A_{-I_r}} < a_{i\mu(i)}$ for all $i \in M \setminus I$ (and consequently $\underline{v}_{\mu(i)}^{A_{-I_r}} > 0$).

Let S be the set of buyers in $M \setminus I$ that can be reached from j_0 in G' (together with $i^* = \mu^{-1}(j_0)$), and let T be the set of sellers that can be reached in G' from j_0. Both S and T are nonempty. Notice that if $i \in S$, then $\mu(i) \in T$; and if $j \in T$ then $\mu^{-1}(j)$ exists and belongs to S. Also, if $i \notin S$ and $j \in T$ it cannot be that $\overline{u}_i^{A_{-I_r}} + \underline{v}_j^{A_{-I_r}} = a_{ij}$.

Define (u'_{-I}, v') by $u'_i = \overline{u}_i^{A_{-I_r}} + \varepsilon$ for all $i \in S$; $u'_i = \overline{u}_i^{A_{-I_r}}$ for all $i \in M \setminus (I \cup S)$; $v'_j = \underline{v}_j^{A_{-I_r}} - \varepsilon$ for all $j \in T$ and $v'_j = \underline{v}_j^{A_{-I_r}}$ for all $j \in M' \setminus T$. It is easy to check that, for $\varepsilon > 0$ small enough, $(u'_{-I}, v') \in C(w_{A_{-I}})$.[11] Moreover, all buyer i_l in the chain c remains in S, for all $l \in \{1, \ldots, k\}$, since we have $\overline{u}_{i_l}^{A_{-I_r}} < u_{i_l} \leq a_{i_l j_l}$, for all $l \in \{1, \ldots, k\}$, and $u_{i_k} = \overline{u}_{i_k}^{A_{-I_r}} < a_{i_k j_k}$. As a consequence, for $\varepsilon > 0$ small enough, $u_{i_l} > u'_{i_l}$ for all $l \in \{1, 2, \ldots, k-1\}$ and $u_{i_k} < u'_{i_k}$. From $u_{i_{k-1}} > u'_{i_{k-1}}$, and taking into account that $(u'_{-I}, v') \in C(w_{A_{-I}})$ and also $u_{i_{k-1}} + v_{j_{k-1}} = a_{i_{k-1} j_{k-1}}$ since $(u, v) \in B^{\mu}(w_A)$, we get $v_{j_{k-1}} < v'_{j_{k-1}}$. Then, from $u'_{i_k} + v'_{j_{k-1}} = \overline{u}_{i_k}^{A_{-I_r}} + \underline{v}_{j_{k-1}}^{A_{-I_r}} = a_{i_k j_{k-1}}$, we obtain that (u'_{-I}, v') dominates (u_{-I}, v) via coalition $\{i_k, j_{k-1}\}$.

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[11] Notice that if $j \in M'$ is not assigned by $\mu_{|(M \setminus I) \times M'}$, then $j \notin T$, which implies $v'_j = \underline{v}_j^{A_{-I_r}}$. Moreover, either j is not assigned by $\mu_{|(M \setminus I_r) \times M'}$ and thus $v'_j = \underline{v}_j^{A_{-I_r}} = 0$ or $\mu^{-1}(j) \in I'$ and also $v'_j = \underline{v}_j^{A_{-I_r}} = 0$.

We only have to define (u'',v'') by $u''_i = u'_i$ for all $i \in M \setminus I$, $u''_i = a_{i\mu(i)}$ for all $i \in I$ and $v''_j = v'_j$ for all $j \in M'$. By (6) and (14), we have that $(u'',v'') \in \hat{C}(w_{A_{-I}}) \subseteq V^{\mu}(w_A)$ and moreover $(u'',v'')\, dom^{w_A}_{\{i_k,j_{k-1}\}}(u,v)$.

If (u_{-I_r}, v_{-J_0}) is neither in Case 1 nor in Case 2, that is to say, $(u_{-I_r}, v_{-J_0}) \notin R^{\mu}(w_{A_{-I_r \cup J_0}})$ and $u_i \leq \overline{u}_i^{A_{-I_r \cup J_0}}$ for all $i \in M \setminus I_r$, since by (9) we know that $\underline{u}_i^{A_{-I_0 \cup J_0}} \leq \underline{u}_i^{A_{-I_1 \cup J_0}} \leq \cdots \leq \underline{u}_i^{A_{-I_r \cup J_0}}$ for all $i \in M \setminus I_r$, we deduce that (u,v) must be either in Case 3 or Case 4.

Case 3: $r \geq 1$ and there exists $i^* \in M \setminus I_r$ and $k \in \{1,2,\ldots,r\}$ such that $\underline{u}_{i^*}^{A_{-I_{k-1} \cup J_0}} \leq u_{i^*} < \underline{u}_{i^*}^{A_{-I_k \cup J_0}}$.

Lemma 15 in the Appendix guarantees that in this case there exists $(u'',v'') \in \hat{C}(w_{A_{-I_{k-1} \cup J_0}})$ such that (u'',v'') dominates (u,v). We only must take $I = I_{k-1}$, $I' = I_k$ and $J = J_0$, and recall that by definition of $\tilde{I}_k = I_k \setminus I_{k-1}$, $u_i = \overline{u}_i^{A_{-I_{k-1} \cup J_0}} = a_{i\mu(i)}$ for all $i \in I_k \setminus I_{k-1}$.

Case 4: There exists $i^* \in M \setminus I_r$ such that $0 \leq u_{i^*} < \underline{u}_{i^*}^{A_{-I_0 \cup J_0}}$.

Since $(u,v) \in B^{\mu}(w_A) \setminus (R^{\mu}(w_A) \cup V^{\mu}(w_A))$, let us define the following sequence of sets of players:

$$\tilde{J}_1 = \{j \in M' \setminus (J_0 \cup \mu(I_0)) \mid v_j = \overline{v}_j^{A_{-I_0 \cup J_0}} = a_{\mu^{-1}(j)j}\}, \qquad J_1 = \tilde{J}_1 \cup J_0$$

and, for all $k > 1$,

$$\tilde{J}_k = \{j \in M' \setminus (J_{k-1} \cup \mu(I_0)) \mid v_j = \overline{v}_j^{A_{-I_0 \cup J_{k-1}}} = a_{\mu^{-1}(j)j}\}$$

and, as long as $\tilde{J}_k \neq \emptyset$, $J_k = J_{k-1} \cup \tilde{J}_k$.

Also by definition, $J_0 \subset J_1 \subset \cdots \subset J_{k-1} \subset J_k$ and there exists $p \geq 0$ such that $\tilde{J}_{p+1} = \emptyset$, which implies that the sequence ends at J_p. Moreover, $M' \setminus J_p \neq \emptyset$, because otherwise we would have $(u,v) = (0,a)$ and, by Proposition 10, $(0,a) \in V^{\mu}(w_A)$, in contradiction with $(u,v) \in B^{\mu}(w_A) \setminus (R^{\mu}(w_A) \cup V^{\mu}(w_A))$.

By repeatedly applying part 1) of Proposition 4 and Remark 6 we obtain that, for all $k \in \{0,1,\ldots,p\}$, $((M \setminus I_0) \cup (M' \setminus J_k), w_{A_{-I_0 \cup J_k}})$ is a μ-compatible subgame of $(M \cup M', w_A)$. Also, $(u_{-I_0}, v_{-J_p}) \in B^{\mu}(w_{A_{-I_0 \cup J_p}})$ and different cases are considered.

Case 4.1 $(u_{-I_0}, v_{-J_p}) \in R^{\mu}(w_{A_{-I_0 \cup J_p}})$. Then an argument analogous to that of Case 1 shows that there exists $(u'',v'') \in \hat{C}(w_{A_{-I_0 \cup J_p}}) \subseteq V^{\mu}(w_A)$ such that $(u'',v'')\, dom^{w_A}(u,v)$.

Case 4.2 There exists $j^* \in M' \setminus J_p$ with $v_{j^*} > \overline{v}_{j^*}^{A_{-I_0 \cup J_p}}$. Then an argument analogous to that of Case 2 shows that there exists $I \subseteq M$ and $J \subseteq M'$ such that $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$ is a μ-compatible subgame of $(M \cup M', w_A)$, and there exists $(u'', v'') \in \hat{C}(w_{A_{-I \cup J}}) \subseteq V^{\mu}(w_A)$ such that $(u'', v'') \, dom^{w_A} (u, v)$.

Case 4.3 $p \geq 1$ and there exists $j^* \in M' \setminus J_p$ and $k \in \{1, 2, \ldots, p\}$ such that $\underline{v}_{j^*}^{A_{-I_0 \cup J_{k-1}}} \leq v_{j^*} < \underline{v}_{j^*}^{A_{-I_0 \cup J_k}}$. Then an argument analogous to that of Case 3 shows that there exists $(u'', v'') \in \hat{C}(w_{A_{-I_0 \cup J_{k-1}}}) \subseteq V^{\mu}(w_A)$ that dominates (u, v).

Case 4.4 There exists $j^* \in M' \setminus J_p$ such that $v_{j^*} < \underline{v}_{j^*}^{A_{-I_0 \cup J_0}}$.

Recall that, by the assumption in Case 4, there also exists $i^* \in M \setminus I_r$ such that $u_{i^*} < \underline{u}_{i^*}^{A_{-I_0 \cup J_0}}$, and as a consequence is matched by $\mu_{|(M \setminus I_0) \times (M' \setminus J_0)}$. Also from $\underline{v}_{j^*}^{A_{-I_0 \cup J_0}} > 0$ we deduce that $\mu^{-1}(j^*) \in M \setminus I_0$.

Notice first that if $p = 0$ then $u_{i^*} < \underline{u}_{i^*}^{A_{-I_0 \cup J_0}}$ implies $v_{\mu(i^*)} > \overline{v}_{\mu(i^*)}^{A_{-I_0 \cup J_0}} = \overline{v}_{\mu(i^*)}^{A_{-I_0 \cup J_p}}$ where $\mu(i^*) \in M' \setminus J_0 = M' \setminus J_p$ and we are in the Case 4.2. Similarly, if $r = 0$, then $v_{j^*} < \underline{v}_{j^*}^{A_{-I_0 \cup J_0}}$ implies $u_{\mu^{-1}(j^*)} > \overline{u}_{\mu^{-1}(j^*)}^{A_{-I_0 \cup J_0}} = \overline{u}_{\mu^{-1}(j^*)}^{A_{-I_r \cup J_0}}$ where $\mu^{-1}(j^*) \in M \setminus I_0 = M \setminus I_r$, and we are then in Case 2.

Let us then assume that $r > 0$ and $p > 0$.

Let us now consider the buyer-seller exact representative (see (3)) of $((M \setminus I_0) \cup (M' \setminus J_0), w_{A_{-I_0 \cup J_0}})$. Since $((M \setminus I_0) \cup (M' \setminus J_0), w_{(A_{-I_0 \cup J_0})^r})$ is buyer-seller exact, there exists $(x, y) \in C(w_{(A_{-I_0 \cup J_0})^r}) = C(w_{A_{-I_0 \cup J_0}})$ such that $x_{i^*} + y_{j^*} = a^r_{i^* j^*}$, and moreover

$$u_{i^*} < \underline{u}_{i^*}^{A_{-I_0 \cup J_0}} \leq x_{i^*} \text{ and } v_{j^*} < \underline{v}_{j^*}^{A_{-I_0 \cup J_0}} \leq y_{j^*}. \qquad \text{(A.8)}$$

If $a^r_{i^* j^*} = a_{i^* j^*}$ we obtain $(u'', v'') \, dom^{w_A}_{\{i^* j^*\}} (u, v)$, where $u''_i = x_i$ for all $i \in M \setminus I_0$, $u''_i = a_{i \mu(i)}$ for all $i \in I_0$, $v''_j = y_j$ for all $j \in M' \setminus J_0$ and $v''_j = a_{\mu^{-1}(j) j}$ for all $j \in J_0$. Notice that $(u'', v'') \in \hat{C}(w_{A_{-I_0 \cup J_0}}) \subseteq V^{\mu}(w_A)$.

If $a^r_{i^* j^*} \neq a_{i^* j^*}$, take $\mu' \in \mathcal{M}^*_A(M \setminus I_0, M' \setminus J_0)$ defined by $\mu' = \mu_{|(M \setminus I_0) \times (M' \setminus J_0)} \cup \mu_0$, where μ_0 is any matching in $\mathcal{M}(\mu^{-1}(J_0), \mu(I_0))$ that does not leave agents unmatched. Recall that $a_{ij} = 0$ for any $(i, j) \in \mu^{-1}(J_0) \times \mu(I_0)$ and thus all matchings are optimal for this submarket. Then, since any $(u_{-I_0}, v_{-J_0}) \in B^{\mu}(w_{A_{-I_0 \cup J_0}})$ satisfies $u_i = 0$ for all $i \in \mu^{-1}(J_0)$ and $v_j = 0$ for all $j \in \mu(I_0)$, we have $B^{\mu}(w_{A_{-I_0 \cup J_0}}) = B^{\mu'}(w_{A_{-I_0 \cup J_0}})$.

By the definition of $(A_{-I_0 \cup J_0})^r$ in (3), there exist $i_1, i_2, \ldots, i_s$ in $M \setminus I_0$ and different such that

$$x_{i^*}+y_{j^*} = a^r_{i^*j^*} = a_{i^*\mu'(i_1)}+a_{i_1\mu'(i_2)}+\cdots+a_{i_{s-1}\mu'(i_s)}+a_{i_sj^*}-a_{i_1\mu'(i_1)}-\cdots-a_{i_s\mu'(i_s)}\,.$$

Since $(x,y)\in C(w_{A_{-I_0\cup J_0}})$ we have $x_{i_l}+y_{\mu(i_l)} = a_{i_l\mu'(i_l)}$ for all $l\in\{1,2,\ldots,s\}$ and thus

$$\begin{aligned} &x_{i^*}+y_{\mu'(i_1)} = a_{i^*\mu'(i_1)},\\ &x_{i_l}+y_{\mu'(i_{l+1})} = a_{i_l\mu'(i_{l+1})},\ \text{for all } l\in\{1,2,\ldots,s-1\},\\ &x_{i_s}+y_{j^*} = a_{i_sj^*}\,. \end{aligned} \tag{A.9}$$

Step 4.4.1 If $v_{\mu'(i_1)} < y_{\mu'(i_1)}$, since $u_{i^*} < x_{i^*}$, we define $(x',y')\in \hat{C}(w_{A_{-I_0\cup J_0}})$ by $x'_i = x_i$ for all $i\in M\setminus I_0$, $x'_i = a_{i\mu(i)}$ for all $i\in I_0$, $y'_j = y_j$ for all $j\in M'\setminus J_0$, $y'_j = a_{\mu^{-1}(j)j}$ for all $j\in J_0$, and obtain that $(x',y')\, dom^{w_A}_{\{i^*,\mu'(i_1)\}}(u,v)$, and this would finish the proof.

Step 4.4.2 If $v_{\mu'(i_1)} > y_{\mu'(i_1)}$, then $u_{i_1} < x_{i_1}$ and we go to 4.4.1 replacing i^* with i_1 and $\mu'(i_1)$ with $\mu'(i_2)$. If by repeting this procedure we do not finish, it is because at some iteration we reach one of the two following steps (4.4.3 or 4.4.4).

Step 4.4.3 If $v_{\mu'(i_l)} > y_{\mu'(i_l)}$ for all $l\in\{1,2,\ldots,k-1\}$, for some $k\in\{1,\ldots,s\}$, and $v_{\mu'(i_k)} = y_{\mu'(i_k)} < a_{i_k\mu'(i_k)}$ [12], consider $J' = \{j\in M'\setminus(J_0\cup\mu(I_0)) \mid y_j = a_{\mu^{-1}(j)j}\}$, and $J = J'\cup J_0$. By Proposition 4 and Remark 6, the game $((M\setminus I_0)\cup(M'\setminus J), w_{A_{-I_0\cup J}})$ is a μ-compatible subgame of $(M\cup M', w_A)$, and $\mu'(i_k)$ belongs to $M'\setminus J$. Moreover, by (6) and (7), $(x, y_{-J'})\in C(w_{A_{-I_0\cup J}})$.

Define the oriented graph $G = G(x,y)$ with set of vertices $(M\setminus I_0)\cup(M'\setminus J_0)$ and arcs $j\longrightarrow i$ whenever $(i,j)\in\mu$ and $i\longrightarrow j$ whenever $x_i+y_j = a_{ij}$ and $(i,j)\notin\mu$ [13]. Denote by G' the restriction of G to the player set $(M\setminus I_0)\cup(M'\setminus J)$. Let S be the set of buyers $i\in M\setminus I_0$ that can be reached from i^* in G'. Let T be the set of sellers $j\in M'\setminus J$ that can be reached from i^* in G', together with $\mu(i^*)\in M'\setminus J$. Both S and T are nonempty. Moreover, if $i\in S$ and $j\notin T$, then $x_i+y_j > a_{ij}$, since $x_i+y_j = a_{ij}$ and $i\in S$ would imply $j\in T$.

Now, two cases are considered.

- If $\mu(I_0)\cap T = \emptyset$, then we define the payoff vector $(x', y'_{-J'})\in \mathbb{R}^{M\setminus I_0}\times\mathbb{R}^{M'\setminus J}$

[12] The assumptions of this step trivially imply that $x_{i_l} > 0$ for all $l\in\{1,2,\ldots,k\}$ and thus, for all $l\in\{1,2,\ldots,k\}$, $i_l\notin\mu^{-1}(J_0)$, which implies $\mu(i_l) = \mu'(i_l)$.

[13] Notice that the arcs in graph G are defined making use of μ, instead of μ'.

by

$$x'_i = x_i - \varepsilon\,,\ \text{for all } i \in S\,; x'_i = x_i\,,\ \text{for all } i \in M \setminus (S \cup I_0)\,;$$

$$y'_j = y_j + \varepsilon\,,\ \text{for all } j \in T\,; y'_j = y_j\,,\ \text{for all } j \in (M' \setminus (T \cup J))\,.$$

and see that, for $\varepsilon > 0$ small enough, $(x', y'_{-J'}) \in C(w_{A_{-I_0 \cup J}})$. Notice first that $x_i > 0$ for all $i \in S$. By (A.8), $x_{i^*} > 0$. Moreover, if for some $i \in S$ we have $x_i = 0$ then we know that $\mu(i) \in T$ and $y_{\mu(i)} = a_{i\mu(i)}$, and this means that $\mu(i) \in J'$ and cannot be a node of graph G'. This guarantees that $x_i > 0$ for all $i \in S$ and thus, for $\varepsilon > 0$ small enough, $x'_i \geq 0$ for all $i \in M \setminus I_0$ (and trivially $y'_j \geq 0$ for all $j \in M' \setminus J$). For $\varepsilon > 0$, if $(i,j) \in \mu$, either $i \in S$ and $j \in T$ or else $i \notin S$ and $j \notin T$, and in both cases $x'_i + y'_j = x_i + y_j = a_{ij}$. Let us now consider the case $(i,j) \notin \mu$. If $i \in S$ and $j \in T$ or $i \notin S$ and $j \notin T$, $x'_i + y'_j = x_i + y_j \geq a_{ij}$. If $i \notin S$ and $j \in T$, $x'_i + y'_j = x_i + y_j + \varepsilon \geq a_{ij}$. And if $i \in S$ and $j \notin T$, $x_i + y_j > a_{ij}$ and so, for $\varepsilon > 0$ small enough, $x'_i + y'_j = x_i - \varepsilon + y_j \geq a_{ij}$. Notice that if i is not matched by $\mu_{|(M\setminus I_0)\times(M'\setminus J)}$, then $i \notin S$ and thus $x'_i = x_i$. Moreover, since $i \in M \setminus I_0$ is not matched by $\mu_{|(M\setminus I_0)\times(M'\setminus J)}$, either $i \in \mu^{-1}(J_0)$ and thus $x_i = 0$, or $\mu(i) \in J'$ which implies $y_{\mu(i)} = a_{i\mu(i)}$ and also $x_i = 0$. Finally, if $j \in M' \setminus J$ is not matched by $\mu_{|(M\setminus I_0)\times(M'\setminus J)}$ it must be the case that $j \in \mu(I_0)$ and thus $y_j = 0$. But then, by the assumption $\mu(I_0) \cap T = \emptyset$, $j \notin T$ and thus $y'_j = y_j = 0$.

Notice that, by the assumptions of this step 4.4.3, $\mu(i_l) \in M' \setminus J$ for all $l \in \{1, 2, \ldots, k\}$. Moreover, by (A.9), and taking into account footnote 11, we have that $c_k = (i^*, \mu(i_1), i_1, \mu(i_2), \ldots, i_{k-1}, \mu(i_k))$ is a path in the graph G'.

For $\varepsilon > 0$ small enough, not only $(x', y'_{-J'}) \in C(w_{A_{-I_0 \cup J}})$, but also $x'_{i_{k-1}} > u_{i_{k-1}}$, $y'_{\mu(i_k)} > y_{\mu(i_k)} = v_{\mu(i_k)}$, and $x'_{i_{k-1}} + y'_{\mu(i_k)} = x_{i_{k-1}} + y_{\mu(i_k)} = a_{i_{k-1}\mu(i_k)}$. Then, the imputation (x'', y'') defined by $x''_i = x'_i$ for all $i \in M \setminus I_0$, $x''_i = a_{i\mu(i)}$ for all $i \in I_0$, $y''_j = y'_j$ for all $j \in M' \setminus J$ and $y''_j = a_{\mu^{-1}(j)j}$ for all $j \in J$, belongs to $\hat{C}(w_{A_{-I_0 \cup J}}) \subseteq V^{\mu}(w_A)$ and $(x'', y'')\, dom^{w_A}_{\{i_{k-1}, \mu(i_k)\}} (u, v)$. Notice that for $k = 1$ the domination would be through coalition $\{i^*, \mu(i_1)\}$.

• If $\mu(I_0) \cap T \neq \emptyset$, take such a $j'_m \in \mu(I_0) \cap T$.

From $j'_m \in T$, there exists a path in $G' \subseteq G$ from i^* to j'_m. Let this path be $c' = (\mu(i^*), i^*, \mu(i'_1), i'_1, \ldots, \mu(i'_l), i'_l, \ldots, \mu(i'_{m-1}), i'_{m-1}, j'_m)$. By the definition of the graph G', with set of vertices $(M \setminus I_0) \cup (M' \setminus J)$, we know that $y_{\mu(i'_l)} < a_{i'_l \mu(i'_l)}$ for $l \in \{1, 2, \ldots, m-1\}$. We are now on the assumptions of Step 4.4.5, and thus go to 4.4.5.

Step 4.4.4 If $v_{\mu'(i_l)} > y_{\mu'(i_l)}$ for all $l \in \{1, 2, \ldots, k-1\}$, for some $k \in \{1, \ldots, s\}$, and $v_{\mu'(i_k)} = y_{\mu'(i_k)} = a_{i_k \mu'(i_k)}$, steps analogous to 4.4.1 to 4.4.3 are applied to the opposite end of the sequence $i^*, \mu'(i_1), i_1, \mu'(i_2), i_2, \ldots, i_{s-1}, \mu'(i_s), i_s, j^*$ given in (A.9). That is:

• If $u_{i_s} < x_{i_s}$, since $v_{j^*} < y_{j^*}$ we define as usual $(x', y') \in \hat{C}(w_{A_{-I_0 \cup J_0}})$ by $x'_i = x_i$ for all $i \in M \setminus I_0$, $x'_i = a_{i\mu(i)}$ for all $i \in I_0$, $y'_j = y_j$ for all $j \in M' \setminus J_0$, $y'_j = a_{\mu^{-1}(j)j}$ for all $j \in J_0$, and obtain that $(x', y') dom^{w_A}_{\{i_s, j^*\}}(u, v)$.

• If $u_{i_s} > x_{i_s}$, then $v_{\mu'(i_s)} < y_{\mu'(i_s)}$ and we repeat the same argument replacing j^* with $\mu'(i_s)$ and i_s with i_{s-1}. If by repeating this procedure we do not finish, we will find ourselves in one of the two following situations.

• If $u_{i_l} > x_{i_l}$ for all $l \in \{t+1, \ldots, s\}$, for some $t \geq 1$, and $u_{i_t} = x_{i_t} < a_{i_t \mu'(i_t)}$, similarly to step 4.4.3 we consider the set $I' = \{i \in M \setminus (I_0 \cup \mu^{-1}(J_0)) \mid x_i = a_{i\mu'(i)}\}$, $I = I' \cup I_0$, and the μ-compatible subgame $((M \setminus I) \cup (M' \setminus J_0), w_{A_{-I \cup J_0}})$. Then either we prove the existence of $(x', y') \in \hat{C}(w_{A_{-I \cup J_0}}) \subseteq V^\mu(w_A)$ such that $(x', y') dom^{w_A}_{\{i_t, \mu(i_{t+1})\}}(u, v)$, or we are on the assumptions of Step 4.4.6.

• If $u_{i_l} > x_{i_l}$ for all $l \in \{t+1, \ldots, s\}$, for some $t \geq 1$, and $u_{i_t} = x_{i_t} = a_{i_t \mu'(i_t)}$, remember that we also have $v_{\mu'(i_k)} = y_{\mu'(i_k)} = a_{i_k \mu'(i_k)}$. Notice that in this case $u_{i_t} = \overline{u}^{A_{-I_0 \cup J_0}}_{i_t} = a_{i_t \mu'(i_t)}$ and $v_{\mu'(i_k)} = \overline{v}^{A_{-I_0 \cup J_0}}_{\mu'(i_k)} = a_{i_k \mu'(i_k)}$. Moreover, since $(x, y) \in C(w_{A_{-I_0 \cup J_0}})$, from $x_{i_k} = y_{\mu'(i_t)} = 0$ we have $a_{i_k \mu'(i_t)} = 0$.

If $\mu'(i_k) \notin \mu(I_0)$ and $i_t \notin \mu^{-1}(J_0)$ we can define $\tilde{I}^{d+1}_0 = \{i_t\}$ and $\tilde{J}^{d+1}_0 = \{\mu'(i_k)\}$, which contradicts that the sequence $I^1_0 \times J^1_0 \subseteq I^2_0 \times J^2_0 \subseteq \cdots \subseteq I^d_0 \times J^d_0 = I_0 \times J_0$ finishes at $I^d_0 \times J^d_0$.

Otherwise, either $\mu'(i_k) \in \mu(I_0)$ or $i_t \in \mu^{-1}(J_0)$. In the first case, taking into account (A.9), $(\mu'(i^*), i^*, \mu'(i_1), i_1, \ldots, i_{k-1}, \mu'(i_k))$ is a path in G, and the fact that $y_{\mu'(i_l)} > v_{\mu'(i_l)} \geq 0$ for $l \in \{1, 2, \ldots, k-1\}$ guarantees that $\mu'(i_l) = \mu(i_l)$ for all $l \in \{1, 2, \ldots, k-1\}$. We are then on the assumptions of Step 4.4.5 (with $j_m = \mu'(i_k)$). In the second case, that is when $i_t \in \mu^{-1}(J_0)$, the path $(\mu'^{-1}(j^*), j^*, i_s, \mu'(i_s), i_{s-1}, \ldots, \mu'(i_{t_1}), i_t)$ is in the assumptions of Step 4.4.6, since from $u_{i_l} > x_{i_l}$ for all $l \in \{t+1, \ldots, s\}$ we obtain $y_{\mu'(i_l)} > 0$ for all $l \in \{t+1, \ldots, s\}$ and thus $\mu'(i_l) = \mu(i_l)$.

Step 4.4.5 If there exists a path $(\mu(i^*), i^*, \mu(i_1), i_1, \ldots, \mu(i_{m-1}), i_{m-1}, j_m)$ in the graph $G = G(x, y)$ such that $j_m \in \mu(I_0)$ and $y_{\mu(i_l)} < a_{i_l \mu(i_l)}$ for all $l \in \{1, 2, \ldots, m-1\}$ we proceed in the following way.

Since $j_m \in \mu(I_0)$, j_m is not assigned by $\mu_{|(M \setminus I_0) \times (M' \setminus J_0)}$ and we have $y_{j_m} = \overline{v}^{A_{-I_0 \cup J_0}}_{j_m} = 0$. Now, the fact that $(\underline{u}^{A_{-I_0 \cup J_0}}, \overline{v}^{A_{-I_0 \cup J_0}}) \in C(w_{A_{-I_0 \cup J_0}})$ implies

$$\underline{u}^{A_{-I_0 \cup J_0}}_{i_{m-1}} \leq x_{i_{m-1}} = a_{i_{m-1} j_m} - y_{j_m} = a_{i_{m-1} j_m} - \overline{v}^{A_{-I_0 \cup J_0}}_{j_m} \leq \underline{u}^{A_{-I_0 \cup J_0}}_{i_{m-1}}.$$

Thus, $x_{i_{m-1}} = \underline{u}^{A_{-I_0 \cup J_0}}_{i_{m-1}}$, and as a consequence $y_{\mu(i_{m-1})} = \overline{v}^{A_{-I_0 \cup J_0}}_{\mu(i_{m-1})}$. The repetition of the same argument leads to

$$\begin{aligned} &\overline{v}_{\mu(i_l)}^{A_{-I_0\cup J_0}} = y_{\mu(i_l)} < a_{i_l\mu(i_l)} \text{ for all } 1 \le l \le m-1 \\ &\underline{u}_{i_l}^{A_{-I_0\cup J_0}} = x_{i_l} > 0 \text{ for all } 1 \le l \le m-1\,, \\ &\underline{u}_{i^*}^{A_{-I_0\cup J_0}} = x_{i^*} > 0 \text{ and } \overline{v}_{\mu(i^*)}^{A_{-I_0\cup J_0}} = y_{\mu(i^*)} < a_{i^*\mu(i^*)}\,. \end{aligned} \tag{A.10}$$

Since $p > 0$, recalling the definition in page 40 of the chain of subsets $J_0 \subset J_1 \subset \cdots \subset J_{p-1} \subset J_p \subset M'$, from $j_m \in \mu(I_0)$ we obtain that j_m is unmatched by μ in all the μ-compatible subgames $((M \setminus I_0) \cup (M' \setminus J_l), w_{A_{-I_0\cup J_l}})$, for all $l \in \{0, 1, \ldots, p\}$. As a consequence,

$$y_{j_m} = \overline{v}_{j_m}^{A_{-I_0\cup J_0}} = \overline{v}_{j_m}^{A_{-I_0\cup J_1}} = \cdots = \overline{v}_{j_m}^{A_{-I_0\cup J_p}} = 0\,. \tag{A.11}$$

Consider the μ-compatible subgame $((M \setminus I_0) \cup (M' \setminus J_1), w_{A_{-I_0\cup J_1}})$. Recall that $x_{i^*} > 0$ and notice from the assumptions of this step that $x_{i_l} > 0$ for all $l \in \{1, 2, \ldots, m-1\}$. This implies $\mu(i^*) \notin \mu(I_0)$ and $\mu(i_l) \notin \mu(I_0)$ for all $l \in \{1, 2, \ldots, m-1\}$. Also, from $x_{i^*} > 0$ we get $y_{\mu(i^*)} = \overline{v}_{\mu(i^*)}^{A_{-I_0\cup J_0}} < a_{i^*\mu(i^*)}$ which implies $\mu(i^*) \notin J_1$. Similarly, from $y_{\mu(i_l)} = \overline{v}_{\mu(i_l)}^{A_{-I_0\cup J_0}} < a_{i_l\mu(i_l)}$ for $l \in \{1, 2, \ldots, m-1\}$ we obtain $\mu(i_1), \ldots, \mu(i_{m-1}) \notin J_1$. Finally, $j_m \notin J_1$, since $j_m \in \mu(I_0)$.

By the definition of J_1, and recall that $J_1 = \tilde{J}_1 \cup J_0$, the restriction of $(\underline{u}^{A_{-I_0\cup J_0}}, \overline{v}^{A_{-I_0\cup J_0}})$ to the space of payoffs of $(M \setminus I_0) \cup (M' \setminus J_1)$, that we denote by $(\underline{u}^{A_{-I_0\cup J_0}}, \overline{v}_{-\tilde{J}_1}^{A_{-I_0\cup J_0}})$, belongs to $C(w_{A_{-I_0\cup J_1}})$. By, (12) and (11), we know that $\overline{v}_j^{A_{-I_0\cup J_0}} \le \overline{v}_j^{A_{-I_0\cup J_1}}$ for all $j \in M' \setminus J_1$ and $\underline{u}_i^{A_{-I_0\cup J_0}} \ge \underline{u}_i^{A_{-I_0\cup J_1}}$ for all $i \in M \setminus I_0$.

Let it be $K = \{i^*, i_1, i_2, \ldots, i_{m-1}\}$. If $\underline{u}_{i_{m-1}}^{A_{-I_0\cup J_0}} > \underline{u}_{i_{m-1}}^{A_{-I_0\cup J_1}}$, then, taking (A.10) and (A.11) into account, we obtain

$$a_{i_{m-1}j_m} = x_{i_{m-1}} + y_{j_m} = \underline{u}_{i_{m-1}}^{A_{-I_0\cup J_0}} + \overline{v}_{j_m}^{A_{-I_0\cup J_0}} > \underline{u}_{i_{m-1}}^{A_{-I_0\cup J_1}} + \overline{v}_{j_m}^{A_{-I_0\cup J_1}}$$

which contradicts $(\underline{u}^{A_{-I_0\cup J_1}}, \overline{v}_{-\tilde{J}_1}^{A_{-I_0\cup J_1}}) \in C(w_{A_{-I_0\cup J_1}})$. Thus, $\underline{u}_{i_{m-1}}^{A_{-I_0\cup J_0}} = \underline{u}_{i_{m-1}}^{A_{-I_0\cup J_1}}$ and as a consequence also $\overline{v}_{\mu(i_{m-1})}^{A_{-I_0\cup J_0}} = \overline{v}_{\mu(i_{m-1})}^{A_{-I_0\cup J_1}}$.

By repeatedly applying the same argument we obtain that for all $i \in K$, $\underline{u}_i^{A_{-I_0\cup J_0}} = \underline{u}_i^{A_{-I_0\cup J_1}}$ and also $\overline{v}_{\mu(i)}^{A_{-I_0\cup J_0}} = \overline{v}_{\mu(i)}^{A_{-I_0\cup J_1}}$, which, from (A.10), implies $\mu(i) \notin J_2$.

We now claim that for all $k \in \{1, 2, \ldots, p\}$ and for all $i \in K$, it holds $\underline{u}_i^{A_{-I_0\cup J_0}} = \underline{u}_i^{A_{I_0\cup J_k}}$ and $\mu(i) \notin J_k$.

To prove the claim, assume by induction hypothesis that for $1 < k \leq p$ we have $\underline{u}_i^{A-I_0\cup J_0} = \underline{u}_i^{A-I_0\cup J_{k-1}}$ where $\mu(i) \notin J_{k-1}$ for all $i \in K$. Then, for all $i \in K$, $\overline{v}_{\mu(i)}^{A-I_0\cup J_0} = \overline{v}_{\mu(i)}^{A-I_0\cup J_{k-1}}$ and from (A.10) we get $\overline{v}_{\mu(i)}^{A-I_0\cup J_0} = \overline{v}_{\mu(i)}^{A-I_0\cup J_{k-1}} < a_{i\mu(i)}$. Thus, $\mu(i) \notin J_k$ for all $i \in K$.

It is known by (11) that $\underline{u}_{i_{m-1}}^{A-I_0\cup J_0} \geq \underline{u}_{i_{m-1}}^{A-I_0\cup J_k}$, if we had $\underline{u}_{i_{m-1}}^{A-I_0\cup J_0} > \underline{u}_{i_{m-1}}^{A-I_0\cup J_k}$, then

$$a_{i_{m-1}j_m} = x_{i_{m-1}} + y_{j_m} = \underline{u}_{i_{m-1}}^{A-I_0\cup J_0} + \overline{v}_{j_m}^{A-I_0\cup J_0} > \underline{u}_{i_{m-1}}^{A-I_0\cup J_k} + \overline{v}_{j_m}^{A-I_0\cup J_k},$$

where the first equality follows from (A.9), the second one by (A.10) and the inequality follows from the assumption and (A.11). This contradicts $(\underline{u}^{A-I_0\cup J_k}, \overline{v}^{A-I_0\cup J_k}) \in C(w_{A-I_0\cup J_k})$ and consequently $\underline{u}_{i_{m-1}}^{A-I_0\cup J_0} = \underline{u}_{i_{m-1}}^{A-I_0\cup J_k}$ and $\overline{v}_{\mu(i_{m-1})}^{A-I_0\cup J_0} = \overline{v}_{\mu(i_{m-1})}^{A-I_0\cup J_k}$. By repeatedly applying the same argument we reach that for all $i \in K$, it holds $\underline{u}_i^{A-I_0\cup J_0} = \underline{u}_i^{A-I_0\cup J_k}$ and thus $\overline{v}_{\mu(i)}^{A-I_0\cup J_0} = \overline{v}_{\mu(i)}^{A-I_0\cup J_k}$.

Once proved the claim, from $u_{i^*} < \underline{u}_{i^*}^{A-I_0\cup J_0}$ (assumption of Case 4) we obtain $v_{\mu(i^*)} > \overline{v}_{\mu(i^*)}^{A-I_0\cup J_0} = \overline{v}_{\mu(i^*)}^{A-I_0\cup J_p}$, with $\mu(i^*) \notin J_p$, and we can again reduce to Case 4.2.

Step 4.4.6 Analogously to the previous step, let us consider the graph $H = H(x, y)$ with set of vertices $(M \setminus I_0) \cup (M' \setminus J_0)$ and arcs $i \longrightarrow j$ if $(i, j) \in \mu$ and $j \longrightarrow i$ whenever $x_i + y_j = a_{ij}$ but $(i, j) \notin \mu$. Then, if there exists a path

$$(\mu^{-1}(j^*), j^*, \mu^{-1}(j_1), j_1, \ldots, \mu^{-1}(j_{m-1}), j_{m-1}, i_m)$$

in the graph H such that $i_m \in \mu^{-1}(J_0)$ and $x_{\mu^{-1}(j_l)} < a_{\mu^{-1}(j_l)j_l}$ for all $l \in \{1, 2, \ldots, m-1\}$, we proceed in a way similar to Step 4.4.5 to show that $\overline{u}_{\mu^{-1}(j^*)}^{A-I_0\cup J_0} = \overline{u}_{\mu^{-1}(j^*)}^{A-I_r\cup J_0}$ and $\mu^{-1}(j^*) \notin I_r$. Then, since by the assumptions of Case 4.4 we have $u_{\mu^{-1}(j^*)} > \overline{u}_{\mu^{-1}(j^*)}^{A-I_0\cup J_0}$, we obtain $u_{\mu^{-1}(j^*)} > \overline{u}_{\mu^{-1}(j^*)}^{A-I_r\cup J_0}$ and we can thus reduce to Case 2.

This finishes the proof of Case 4.4 and of Theorem 12. □

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