

# **Speculative Dynamics in the Term Structure of Interest Rates**

**Kristoffer Nimark**

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# SPECULATIVE DYNAMICS IN THE TERM STRUCTURE OF INTEREST RATES

KRISTOFFER P. NIMARK

**ABSTRACT.** When long maturity bonds are traded frequently and traders have non-nested information sets, speculative behavior in the sense of Harrison and Kreps (1978) arises. Using a term structure model displaying such speculative behavior, this paper proposes a conceptually and observationally distinct new mechanism generating time varying predictable excess returns. It is demonstrated that (i) dispersion of expectations about future short rates is sufficient for individual traders to systematically predict excess returns and (ii) the new term structure dynamics driven by speculative trade is orthogonal to public information in real time, but (iii) can nevertheless be quantified using only publicly available yield data. The model is estimated using monthly data on US short to medium term Treasuries from 1964 to 2007 and it provides a good fit of the data. Speculative dynamics are found to be quantitatively important, potentially accounting for a substantial fraction of the variation of bond yields and appears to be more important at long maturities.

**Keywords:** Term structure of interest rates; Speculative dynamics; Excess returns; Non-nested information; Private information

## 1. INTRODUCTION

When long bonds are traded before they mature, the price an individual trader will be willing to pay for a bond depends on how much he thinks other traders will be willing to pay for it in the future. If traders have access to different information, this price may differ from what an individual trader would be willing to pay for the bond if he had to hold it until it matures and “speculative behavior” in the sense of Harrison and Kreps (1978) arises. That is, the possibility of reselling a bond changes its equilibrium price as traders exploit what they perceive to be market misperceptions about future short rates.

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*Address:* CREI, Universitat Pompeu Fabra, Ramon Trias Fargas 25-27, Barcelona 08005.

*e-mail:* knimark@crei.cat *webpage:* www.kris-nimark.net .

In this paper we present a term structure model populated with rational traders that engage in the type of speculative behavior described above. We use this model to argue that relaxing the assumption that all traders have access to the same information introduces empirically relevant new dynamics to the term structure of interest rates. More specifically, we show that (i) non-nested information sets imply that individual traders can systematically predict excess returns and (ii) individual traders can predict and take advantage of other traders' prediction errors even though no trader on average is better informed than other traders.

In the absence of arbitrage, expected returns in excess of the risk free rate must be compensation for risk. In the model, individual traders can identify bonds that conditional on their own information sets have a positive expected excess return. Traders will hold more of the bonds with a higher expected return in their portfolios and in equilibrium, the increased riskiness of a less balanced portfolio is exactly off-set by the higher expected return. The speculative behavior caused by private information thus gives rise to a novel source of time variation in risk premia that is complementary to, but conceptually distinct from existing theories. Particularly, in a rational setting, time variation in risk premia due to speculative trade must be orthogonal to all public information in real time.

Despite the fact that the speculative dynamics are orthogonal to public information, we can both quantify their importance as well as back out an estimated historical time series of their evolution in the past. This is possible since we as econometricians have access to the full sample of data and the speculative term is orthogonal only to public information available to traders in real time. That is, we can use public information available in period  $t + s : s > 0$  to back out an estimate of the speculative term in period  $t$ . The estimated model suggests that speculative dynamics are potentially quantitatively important and can explain a substantial fraction of the variation in US bond yields.

A necessary condition for traders to have any relevant private information about future bond yields is that bond prices do not perfectly reveal the state of the economy. Recent statistical evidence appear to support this view. In a few closely related papers, Cochrane and Piazzesi (2005, 2008), Joslin, Priebsch and Singleton (2010) and Duffee (2008) present evidence suggesting that the factors that can be found by inverting yields are not sufficient to optimally predict future bond returns. They find that while the usual level, slope and curvature factors explain virtually all of the cross sectional variation in yields, additional factors are needed to forecast excess returns. Ludvigson and Ng (2009) provide more evidence that current bond yields are not sufficient to optimally forecast bond returns. They show that drawing on a very large panel of macroeconomic data helps predict deviations from the expectations hypothesis, or equivalently, future excess returns, compared to using only yield data. Stated another way, these statistical models all suggest that linear combinations of current bond yields are not sufficient to predict future bond yields optimally.

In addition to the empirical evidence cited above, we also have a priori reasons to believe that bond prices should not reveal all information relevant to predicting future bond returns. Grossman and Stiglitz (1980) argued that if it is costly to gather information and prices are observed costlessly, prices cannot fully reveal all information relevant for predicting future returns. For the bond market, the most important variable to forecast is the short interest rate. In most developed countries, the short interest rate is set by a central bank that

responds to macroeconomic developments. If it is costly to gather information about the macro economy, Grossman and Stiglitz's argument implies that bond prices cannot reveal all information relevant to predict future bond returns.

If prices do not reveal all information relevant for predicting bond returns, it becomes more probable that traders have non-nested information sets, that is, traders will have access to and use different information when trading.<sup>1</sup> With the exceptions of bond prices, statements by central bank officials and some well publicized macroeconomic data releases, it is hard to think of sources of information that are public in the strong common knowledge sense of the word. In this paper we allow for traders to have private information that they can exploit when trading. This also seems to accord well with casual observation that at least one motive for trade in assets is possession of information that is not, or at least is not believed to be, already reflected in prices.

One implication of non-nested information sets is that expectations across individual traders will differ which provides us with another way of gauging the plausibility of this assumption. While bond traders' expectations are unobservable, Swanson (2006) presents evidence that professional forecasters' expectations of future interest rates are surprisingly widely dispersed. Citing numbers from the Blue Chip Survey of professional forecasters from 1992-2004, Swanson reports that the spread between the 10th and the 90th percentile of individual forecasts of the 3-month Treasury Bill rate 4 quarters ahead fluctuates between 80 and 220 basis points.

There is a growing literature analyzing asset pricing models under non-nested information sets (or one of its synonyms, see Footnote 1). Singleton (1987) is an early reference of a dynamic asset pricing model with an information structure similar to the one presented here. More recent examples include papers by Allen, Morris and Shin (2005), Kasa, Walker and Whiteman (2008), Bacchetta and van Wincoop (2006, 2007), Cespa and Vives (2009) and Makarov and Rytchkov (2009). These papers either present purely theoretical models or models calibrated to explain some feature of the data. In this paper, we estimate the model directly using Bayesian methods with uniform priors truncated only to ensure that the model is stationary and that variances are non-negative. To the best of my knowledge, this is the first paper to estimate a model with non-nested information sets. The fit of the model is surprisingly good and compares favorably to the fit of a standard affine three factor no-arbitrage model. The model is also used to quantify the cross-sectional dispersion of expectations implied by the posterior parameter estimates. A small implied dispersion across traders of short rate expectations is found to be sufficient to generate quantitatively important speculative dynamics in bond yields.

The estimated model displays similar dynamics to those documented by Duffee (2008) and Cochrane and Piazzesi (2005, 2008). Factors that play practically no role in explaining the cross section of bond yields have predictive power for future yields. This is arguably an intrinsic feature of models with imperfectly informed traders. If the true state of the

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<sup>1</sup>What I in this paper call *non-nested information sets* is also known as *disparately informed traders* (Singleton 1987), *private information* (Sargent 1991), *heterogenous information* (Bacchetta and van Wincoop 2006), *dispersed information* (Angeletos and Pavan 2009) and *imperfect common knowledge* (Woodford 2002, Adam 2006 and Nimark 2008). The term *non-nested information* connects naturally to the language of orthogonal projections used in this paper.

economy could be summarized by three factors that are an exact linear function of yields, no other factor could possibly add predictive power. We demonstrate that the model presented here can account for the evidence in Duffee (2008) by computing Duffee's impulse responses estimated on artificial data generated from our model.

In this paper, dispersion of expectations about returns are caused by rational traders observing different signals. Alternative approaches to model disagreement, or differences in beliefs, falls broadly into two different categories. One category consists of papers that model agents as boundedly rational, e.g. Scheinkman and Xiong (2003) who assume that two different groups of traders have different degrees of overconfidence in the precision of a commonly observed signal. Chen, Joslin and Tran (2010) propose a flexible affine term structure framework for modeling differences in beliefs in which at least one group of traders do not form model consistent expectations. Another example of the boundedly rational approach is Xiong and Yan (2009) who like the present paper analyze a term structure model. In their paper, two groups of risk neutral traders misinterpret an uninformative signal as being helpful in predicting future inflation. Since the two groups of agents have different subjective beliefs about the correlation between the uninformative signal and the inflation process, they will have different subjective beliefs about future inflation and the real return on bonds. While these model provide interesting theoretical insights, some of their maintained assumptions have implications that make them less attractive for empirical work. For instance, in the model of Xiong and Yan (2009), conditioning only on bond prices is sufficient to partly predict the forecast errors of the traders in their model. That is, using such a model for empirical work would implicitly imply that we believe that we as econometricians could make bigger profits than the traders inside the model. In contrast, in the empirical exercise of the present paper, the information set we exploit as econometricians is a strict subset of the information set of the traders inside the model. Arguably, that traders inside the model use the information contained in prices efficiently is an attractive feature of a fully rational approach.

That traders rationally use the information in endogenous prices also introduces additional restrictions on the dynamics of the model that are absent in the literature that models differences in expectations as driven by exogenously specified differences in beliefs. In the model presented below, imposing model consistent expectations restricts the joint dynamics of bond prices and the cross-sectional dispersion of bond price expectations. In equilibrium, bond prices cannot reveal too much information since too informative bond prices would imply a counter factually degenerate cross-sectional distribution of expectations. Put differently, the fact that agents learn from endogenous prices imposes more restrictions than models in which bond prices by assumption do not convey any information useful to individual traders.

In a second class of difference in beliefs models, traders learn rationally from commonly observed signals, but have different expectations due to different prior beliefs. Some examples of this approach include Morris (1996), Basak (2005), Buraschi and Jiltsov (2006) and Gallmeyer and Hollifield (2008). As pointed out by Morris (1995), assuming different information sets is no less ad hoc than assuming different priors and rationality by itself does not imply that agents must have the same priors. Yet, there might be reasons to prefer modeling differences in expectations as arising from differences in information rather than from different priors. Unless additional strong assumptions are made, rational learning from

common signals implies that beliefs will converge over time. Assuming that differences in beliefs arise solely from different initial priors is thus not suitable for modeling phenomena that do not subside over time.

Difference in beliefs models usually assume that the differences in beliefs are common knowledge, that is, individual investors know not only their own beliefs but also the beliefs held by all other traders. This is practical from a modeling perspective since it implies that traders do not need to extract information about the beliefs of others from endogenous variables, and it avoids the difficulties arising from modeling traders as forecasting the forecasts of others. But it also has the less appealing implication that predicting other traders' actions is no more difficult than predicting one's own. Also, while Morris (1995) argues that rationality by itself does not rule out different priors, it is less clear that *commonly known* but different priors is consistent with rationality. If the priors are informative about the variables of interest, all rational traders should condition their predictions also on other traders' priors. But then the pool of different priors is like a pool of public signals, and cannot be the basis for disagreement. If the prior beliefs are not informative, it is not clear why a rational trader would attach any weight to them.

There is also empirical evidence supporting modeling dispersion as arising from imperfect information rather than boundedly rational differences in beliefs. Coibion and Gorodnichenko (2011) find that using data from the Survey of Professional Forecasters, it is possible to reject the hypothesis that survey respondents are rational and have full information. However, it is not possible to reject the hypothesis that the forecasters are rational but imperfectly informed.

Based on these considerations, the next section presents a simple bond pricing model in which rational traders have non-nested information sets. Section 3 uses the properties of orthogonal projections to derive the main theoretical results and Section 4 contains the estimated model and discusses the quantitative importance of speculative dynamics. Section 5 relates the model to a popular class of alternative models. Section 6 concludes and the Appendix contains details of how the model was solved and lists some of the properties of orthogonal projections used in Section 3.

## 2. A BOND PRICING MODEL

This section presents a simple bond pricing model. Traders are risk averse, rational and *ex ante* identical but may observe different signals relevant for predicting future bond prices. They choose a portfolio of risky bonds in order to maximize next period wealth. Traders that have observed signals that make them more optimistic about the return of a given bond will hold relatively more of that bond in their portfolio and in equilibrium, the increased riskiness of a less balanced portfolio is exactly offset by a higher expected return. Equilibrium bond prices are a function of the average expectations of the price of the same bond in the next period, discounted by the risk free short rate and supply shocks that prevent equilibrium prices from revealing the average expectation of future bond prices. The simplicity of the model helps to highlight the consequences for term structure dynamics of relaxing the assumption that traders all have access to the same information. In later sections we will use

the model to show that private information may be a source of excess returns and that the speculative dynamics it gives rise to must be orthogonal to public information in real time.

**2.1. Demand for long maturity bonds.** As in Allen, Morris and Shin (2006) there are overlapping generations of agents who each live for two periods. Time is discrete and indexed by  $t$ . Each generation consists of a continuum of households with unit mass. Each household is endowed with one unit of wealth that it invests when young. When old, households unwind their asset positions and consume. Unlike in the model of Allen *et al*, the owners of wealth, i.e. the households, do not trade assets themselves. Instead, a continuum of traders, indexed by  $j \in (0, 1)$ , trade on behalf of the households, with households diversifying their available funds across the continuum of traders. While not modeled explicitly here, this set up can be motivated as a perfectly competitive limit case of the monopolistically competitive mutual funds model of Garcia and Vanden (2009) that allow uninformed households to benefit from mutual funds private information, while diversifying away idiosyncratic risk associated with individual funds. More importantly, the assumption that the ownership of the assets is separated from the privately informed agents that are doing the trading keeps the model tractable by shutting down information induced wealth dynamics.<sup>2</sup>

Trader  $j$  invests one unit of wealth in period  $t$  on behalf of all households born in period  $t$ . In period  $t + 1$  trader  $j$  unwinds the position of the now old generation of households who then use the proceeds to consume. Traders are infinitely lived and perform the same service for the next generation of households.

There are two types of assets. A risk free one period bond with (log) return  $r_t$  and risky zero-coupon bonds of various maturities. Trader  $j$  chooses a vector of portfolio weights  $\mathbf{a}_t(j)$  in order to maximize the discounted expected log of wealth under management  $W_{t+1}(j)$  in period  $t + 1$ . That is, trader  $j$  solves the problem

$$\max_{\mathbf{a}_t(j)} E [\log W_{t+1}(j) \mid \Omega_t(j)] \quad (2.1)$$

subject to

$$W_{t+1}(j) = 1 + r_t^p(j) \quad (2.2)$$

where  $\Omega_t(j)$  denotes trader  $j$ 's information set and  $r_t^p(j)$  is the log return of the portfolio chosen by trader  $j$  in period  $t$ . In the model presented below, equilibrium log returns of individual bonds are normally distributed. However, the log return on a portfolio of assets with individual log normal returns is not normally distributed. Following Campbell and Viceira (2002a, 2002b) we therefore use a second order Taylor expansion to approximate the log excess portfolio return as

$$r_t^p(j) - r_t = \mathbf{a}'_t(j) (\mathbf{b}_{t+1}^- - \mathbf{b}_t - \mathbf{r}_t) + \frac{1}{2} \mathbf{a}'_t(j) \text{diag} [\Sigma_{b,t}(j)] - \frac{1}{2} \mathbf{a}'_t(j) \Sigma_{b,t}(j) \mathbf{a}_t(j) \quad (2.3)$$

where  $\mathbf{b}_t$  is a vector of (log) bond prices for bonds of different maturities. The vector  $\mathbf{b}_{t+1}^-$  contains the prices of the same bonds in the next period when they have one period less to maturity and  $\mathbf{r}_t$  is a vector of the risk free rate  $r_t$ . The difference  $\mathbf{b}_{t+1}^- - \mathbf{b}_t - \mathbf{r}_t$  is thus a vector of excess returns on bonds of different maturities. The matrix  $\Sigma_{b,t}(j)$  is the covariance

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<sup>2</sup>Readers interested in the interaction between wealth dynamics and heterogenous beliefs can consult Xiong and Yan (2009) who analyze this issue in a model with boundedly rational traders.

of log bond returns conditional on trader  $j$ 's information set. In the next section, we give general conditions for traders' information sets that are sufficient to generate speculative behavior. In the empirical part of the paper we will be explicit about the exact nature of traders's information sets and specify processes for the short interest rate  $r_t$  and traders' information sets  $\Omega_t(j)$  such that in equilibrium, conditional returns are normally distributed, time invariant and with a common covariance across all traders. We therefore suppress the subscripts and trader indices on the conditional return covariance matrix and write  $\Sigma_b$  instead of  $\Sigma_{b,t}(j)$  for all  $t$  and  $j$ . We will also maintain the assumption throughout the paper that current bond prices  $\mathbf{b}_t$  and the current short rate  $r_t$  are observed perfectly by all traders so that

$$\{\mathbf{b}_t, r_t\} \in \Omega_t(j) \quad \forall j \quad (2.4)$$

Maximizing the (second order Taylor approximation) of the expected return (2.3) on wealth under management by trader  $j$  with respect to  $\mathbf{a}_t(j)$  then gives the optimal portfolio weights

$$\mathbf{a}_t(j) = \Sigma_b^{-1} (E [\mathbf{b}_{t+1}^- | \Omega_t(j)] - \mathbf{b}_t - \mathbf{r}_t) + \frac{1}{2} \Sigma_b^{-1} \text{diag} [\Sigma_b] \quad (2.5)$$

Since each trader  $j$  has one unit of wealth to invest, taking average across traders of the portfolio weights (2.5) yields the aggregate demand for bonds.

**2.2. Bond supply.** The vector of bond supply  $\mathbf{s}_t$  is stochastic

$$\mathbf{s}_t = \Sigma_b^{-1} \mathbf{v}_t : \mathbf{v}_t \sim N(0, \Sigma_v) \quad (2.6)$$

where to simplify notation, the vector of supply shocks  $\mathbf{v}_t$  are normalized by the inverse of the conditional variance of bond prices  $\Sigma_b^{-1}$ . The supply shocks  $\mathbf{v}_t$  play a similar role here as the noise traders in Admati (1985). That is, they will prevent equilibrium prices from fully revealing the information held by other traders. While there may be some uncertainty about the total number of bonds outstanding, an economically more appealing interpretation of the supply shocks is in terms of effective supply, as argued by Easley and O'Hara (2004). They define the "float" of an asset as the actual number of assets available for trade in a given period.

**2.3. Equilibrium bond prices.** Equating aggregate demand  $\int \mathbf{a}_t(j) dj$  and supply  $\mathbf{s}_t$

$$\Sigma_b^{-1} \mathbf{v}_t = \Sigma_b^{-1} \left( \int E [\mathbf{b}_{t+1}^- | \Omega_t(j)] dj - \mathbf{b}_t - \mathbf{r}_t \right) + \frac{1}{2} \Sigma_b^{-1} \text{diag} [\Sigma_b] \quad (2.7)$$

and solving for the vector of current log bond prices  $\mathbf{b}_t$  gives

$$\mathbf{b}_t = \frac{1}{2} \text{diag} [\Sigma_b] - \mathbf{r}_t + \int E [\mathbf{b}_{t+1}^- | \Omega_t(j)] dj - \mathbf{v}_t \quad (2.8)$$

A generic element of  $\mathbf{b}_t$  is thus the log price  $b_t^n$  of an  $n$  periods to maturity zero coupon bond given by

$$b_t^n = \alpha^n - r_t + \int E [b_{t+1}^{n-1} | \Omega_t(j)] dj - v_t^n \quad (2.9)$$

where  $\alpha^n$  and  $v_t^n$  are the relevant elements of  $\frac{1}{2} \text{diag} [\Sigma_b]$  and  $\mathbf{v}_t$  respectively. The price of an  $n$  periods to maturity bond in period  $t$  thus depends on the average expectation in period  $t$  of the price of a  $n - 1$  period bond in period  $t + 1$ . The more a trader expects to be able



to sell a bond for in the future, the more is he willing to pay for it today. However, risk aversion prevents the most optimistic trader from demanding all of the available supply.

**2.4. The term structure of interest rates.** The bond price formula (2.9) can be used to price any maturity bond. The procedure is similar to deriving bond prices under a no arbitrage assumption, though we need to be more careful in specifying the timing of the information sets that the expectations that govern prices are conditioned on. As usual, we can start from

$$b_t^1 = -r_t \quad (2.10)$$

and apply (2.9) recursively. The log price of a two period bond then is

$$b_t^2 = \alpha^2 - r_t - \int E[r_{t+1} | \Omega_t(j)] dj + v_t^2 \quad (2.11)$$

The price of a three period bond according to (2.9) is given by the average expectation of the price of a two period bond in  $t + 1$ , discounted by the short rate  $r_t$ . Leading (2.11) by one period and substituting into (2.9) with  $n = 3$  gives

$$\begin{aligned} b_t^3 = & \alpha^2 + \alpha^3 - r_t - \int E[r_{t+1} | \Omega_t(j)] dj \\ & - \int E \left[ \int E[r_{t+2} | \Omega_{t+1}(j')] dj' | \Omega_t(j) \right] dj \\ & + v_t^3 \end{aligned} \quad (2.12)$$

Applying the same procedure recursively to derive the price of an  $n$  periods to maturity bond gives

$$\begin{aligned} b_t^n = & \sum_{i=2}^n \alpha^i - r_t - \int E[r_{t+1} | \Omega_t(j)] - \\ & \int E \left[ \int E[r_{t+2} | \Omega_{t+1}(j')] dj' | \Omega_t(j) \right] dj + \dots \\ & \dots + \int E \left[ \int E \left[ \dots \int E[r_{t+n-1} | \Omega_{t+n-2}(j'')] dj'' \dots | \Omega_{t+1}(j') \right] dj' | \Omega_t(j) \right] dj + v_t^n \end{aligned} \quad (2.13)$$

The yield of a bond with  $n$  periods to maturity is (as usual) given by dividing the log bond price by  $n$

$$y_t^n = -n^{-1} b_t^n. \quad (2.14)$$

**2.5. Deviations from means.** The unconditional mean of bond prices are assumed to be known to the traders. Since we are primarily interested in the effects of information on the dynamics of bond yields we will conduct most of the analysis in terms of deviations of the log price from its (known) mean. We therefore define the log deviation  $\tilde{b}_t^n$  of the bond price as

$$\tilde{b}_t^n \equiv b_t^n - \mu_b^n \quad (2.15)$$

where  $\mu_b^n \equiv E[b_t^n]$ . Similarly, define the deviation of the short interest rate  $r_t$  from its mean  $\mu_r$

$$\tilde{r}_t \equiv r_t - \mu_r \quad (2.16)$$

where  $\mu_r \equiv E[r_t]$ . Subtracting  $\mu_b^n$  and  $\mu_r$  from (2.13) and substituting into the expression (2.14) of bond yields give the deviation of an  $n$  period bond yield from its mean

$$\tilde{y}_t^n = -n^{-1}\tilde{b}_t^n. \quad (2.17)$$

As has been noted before (e.g. Allen, Morris and Shin (2006)), the fact that (2.13) contains average expectations of average expectations (and so on) prevents us from applying the law of iterated expectations to solve for bond prices if traders have non-nested information sets. Before analyzing the consequences of this, we first establish that in the absence of supply shocks and with only common information, the model does indeed imply that the expectation hypothesis hold. Below, the properties of orthogonal projections will be used extensively and we therefore first define orthogonal projections and the relevant inner product space.

**2.6. Projections and a common information benchmark.** In the next section, the properties of orthogonal projections will be used to analyze the implications of non-nested information sets. However, we first apply some of these tools to the familiar case where all traders share the same information set. This section can thus be thought of as a benchmark, confirming that in the absence of private information the model implies that the expectation hypothesis hold. (For readers unfamiliar with orthogonal projections, the Appendix lists some of the properties that will be particularly useful below. For more details see Brockwell and Davis (2006).)

In the model presented below, all bond yields, the factors that drive them and the signals that traders observe will be elements of the inner product space  $L^2$ , which we now define.

**Definition 1.** (The inner-product space  $L^2$ .) The inner product space  $L^2$  is the collection  $\mathcal{C}$  of all random variables  $X$  with finite variance

$$EX^2 < \infty \quad (2.18)$$

and with inner-product

$$\langle X, Y \rangle \equiv E(XY) : X, Y \in L^2 \quad (2.19)$$

**Definition 2.** Let  $\Omega$  be a subspace of  $L^2$ . An orthogonal projection of  $X$  onto  $\Omega$ , denoted  $\mathcal{P}_\Omega X$ , is the unique element in  $L^2$  satisfying

$$\langle X - \mathcal{P}_\Omega X, \omega \rangle = 0 \quad (2.20)$$

for any  $\omega \in \Omega$ .

In a linear model with Gaussian shocks, conditional expectations are equivalent to orthogonal projections. We can thus use the equality

$$E(X | \Omega) = \mathcal{P}_\Omega X \quad (2.21)$$

to replace the conditional expectations in the bond pricing equation (2.13) with projections and rephrase the expectation hypothesis in the following way.

**Definition 3.** (*The Expectations Hypothesis.*) *The expectations hypothesis of the term structure of interest rates is said to hold with respect to  $\Omega_t$  if the implied forward rate*

$$f_t^n \equiv \tilde{b}_t^n - \tilde{b}_t^{n+1} \quad (2.22)$$

*equals the projection of the short rate in period  $t+n$  onto  $\Omega_t$*

$$f_t^n = \mathcal{P}_{\Omega_t} \tilde{r}_{t+n} \quad \forall t, n \quad (2.23)$$

This is a standard definition of the expectation hypothesis (e.g. Backus, Foresi, Mozumdar and Wu, 2001), apart from the explicit reference to the information set expectations are conditioned on.

We can use the definition (2.23) to demonstrate that in the absence of supply shocks, the bond pricing equation (2.13) implies that the expectations hypothesis holds if all agents share the same information set. Consider the 2 period ahead forward rate which by (2.13) and (2.22) is given by

$$\begin{aligned} f_t^2 &= \tilde{b}_t^3 - \tilde{b}_t^2 \\ &= \int E \left[ \int E [\tilde{r}_{t+2} \mid \Omega_{t+1}(j')] dj' \mid \Omega_t(j) \right] dj \end{aligned} \quad (2.24)$$

For now, let  $\Omega_t(j) = \Omega_t$  for all  $j$ , that is, let the information set  $\Omega_t$  be common across all traders. If traders do not forget, the sequence of information sets  $\{\Omega_t\}_{t=0}^{\infty}$  is a filtration so that  $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \dots \subseteq \Omega_t$ . It is a property of projections that repeated projections onto nested information sets reduces to the projection onto the smallest information set, i.e.  $\mathcal{P}_{\Omega_t} \mathcal{P}_{\Omega_{t+s}} X = \mathcal{P}_{\Omega_t} X$  for  $s \geq 0$ . This is simply the law of iterated expectations and implies that traders cannot predict in period  $t$  how they will revise their expectation in period  $t+1$  of the period  $t+2$  short rate. Stated differently, the sequence  $\{E[\tilde{r}_{t+n} \mid \Omega_{t+s}]\}_{s=0}^n$  is a martingale. Applying this result to the bond pricing equation (2.24) gives

$$f_t^2 = \mathcal{P}_{\Omega_t} (\mathcal{P}_{\Omega_{t+1}} \tilde{r}_{t+2}) \quad (2.25)$$

$$= \mathcal{P}_{\Omega_t} \tilde{r}_{t+2} \quad (2.26)$$

The expectation hypothesis then holds for a three period bond with respect to the common information set  $\Omega_t$ . A similar argument can be generalized to an  $n$ -period bond.

In the next section we analyze how the dynamics of the term structure changes when traders have non-nested information sets. The direct link between forward rates and expectations about future short rates make it more convenient to frame the analysis in terms of forward rates rather than bond yields. Of course, bond yields can always be backed out from implied forward rates by the identity (2.22).

### 3. NON-NESTED INFORMATION SETS AND THE TERM STRUCTURE OF INTEREST RATES

In the previous section, the fact that rational traders cannot predict the direction in which they will revise their own expectations in the future allowed us to solve for bond prices as a function of the common period  $t$  expectation about future short rates. With non-nested information sets, predictions about the expectations of others are distinct from one's own expectations. If a trader's current prediction about future short rates differ from his expectation about other traders' predictions, it is rational for him to believe that other traders in

the future will revise their predictions in the direction towards what he considers the best prediction, as more information becomes available. This section draws out the consequences of this fact for the term structure and contains the main theoretical contributions of the paper.

First, it is demonstrated that dispersed expectations about future short rates is sufficient for traders to be able to predict excess returns, even though risk premia are constant in the model. Secondly, we show that with non-nested information sets, individual traders can predict the average prediction error made by others which introduces speculative behavior in the sense of Harrison and Kreps (1978). Third, it is shown that the speculative dynamics introduced by non-nested information sets are orthogonal to public information which has interesting implications for how the speculative dynamics can (and cannot) be quantified using public bond price data. We start by defining what it means for information sets to be non-nested.

**Definition 4.** *The subspace  $\Omega_t(j)$  is the space spanned by the history of variables observed by trader  $j$  at period  $t$ . Projections onto  $\Omega_t(j)$  are denoted  $\mathcal{P}_{t,j}$ .*

**Definition 5.** *Information filtrations for trader  $j, i \in (0, 1)$  are said to be non-nested if and only if*

$$\mathcal{P}_{t,j}\tilde{r}_{t+n} \neq \mathcal{P}_{t,i}\tilde{r}_{t+n} : j \neq i \quad (3.1)$$

for at least some  $t = 0, 1, 2, \dots$  and some  $n = 1, 2, \dots$

Information sets are thus considered non-nested if two traders in at least some period  $t$  disagree about the best prediction of the short rate in period  $t + n$ . Defining non-nested information sets through the implications for projections of short rates onto individual trader's information sets is somewhat tailored to the needs of this paper. For instance, a more general definition may include information about second moments of the distribution of  $r_t$  since these also matter for bond prices. However, all higher moments are assumed to be constant and common knowledge among traders.

**3.1. Predictable excess returns.** We start by proving that non-nested information sets are sufficient for individual traders to be able to predict excess returns. We do this in two steps. First, we show that if individual traders' projections of future short rates are dispersed, then the  $n$  period forward rate  $f_t^n$  cannot coincide generally with traders' expectations about the corresponding future short rates.

**Lemma 1.** *The forward rate  $f_t^n$  is agent  $j$ 's optimal prediction of the short rate  $n$  periods ahead, if and only if it coincides with the orthogonal projection of  $\tilde{r}_{t+n}$  onto trader  $j$ 's information set  $\Omega_t(j)$  so that*

$$\mathcal{P}_{t,j}\tilde{r}_{t+n} = f_t^n \quad (3.2)$$

holds. The equality (3.2) can only hold generally, i.e. for all traders at all times, when traders' information sets coincide.

*Proof.* The first half of the proposition holds by the optimality and uniqueness of orthogonal projections. The second half states that  $\mathcal{P}_{t,j}\tilde{r}_{t+n} = f_t^n$  can hold generally only when information sets are nested. To see why this is true, note that if

$$\mathcal{P}_{t,j}\tilde{r}_{t+n} = f_t^n \quad \forall j, t, n$$

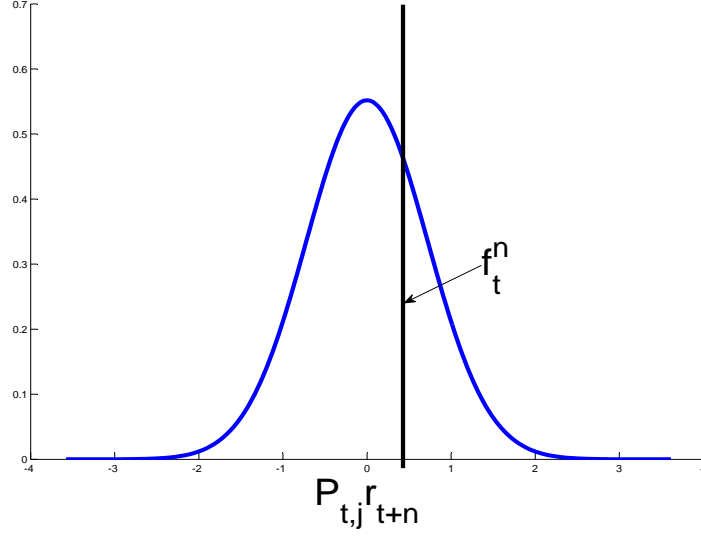


FIGURE 1. Dispersion of expectations of the short rate in period  $t+n$  and the implied forward rate.

then the ex ante symmetry of traders implies that

$$\mathcal{P}_{t,j} \tilde{r}_{t+n} = \mathcal{P}_{t,i} \tilde{r}_{t+n} \quad \forall j, i, t, n \quad (3.3)$$

or that the forward rate  $f_t^n$  is the best predictor of  $\tilde{r}_{t+n}$  for trader  $j$  at all times  $t$  and at all horizons  $n$  only when it is also the best prediction for all other traders, i.e. when information sets are nested.  $\square$

In words, Lemma 1 simply states that if the distribution across traders of expected future short rates is non-degenerate, all points on the support of the distribution cannot coincide with the forward rate, which is a single number. This is illustrated in Figure 1 and may seem like an obvious statement, but the uniqueness of orthogonal projections makes it nevertheless interesting. To see why, note that by the uniqueness of orthogonal projections,  $\mathcal{P}_{t,j} \tilde{r}_{t+n} \neq f_t^n$  implies that  $\mathcal{P}_{t,j} (\tilde{r}_{t+n} - f_t^n) \neq 0$ , i.e. trader  $j$  can systematically predict the forecast error a person would make who used the forward rate as a forecast of the short rate. The next proposition shows that this in turn implies that traders can systematically predict excess returns.

**Definition 6.** (*Excess return*) *The excess return on an  $n$  period bond is the difference in return between holding an  $n$  period bond until maturity and the return on holding a sequence of one period bonds over  $n$  periods, i.e.*

$$-\tilde{b}_t^n - (\tilde{r}_t + \tilde{r}_{t+1} + \dots + \tilde{r}_{t+n-1}) \quad (3.4)$$

An alternative (and more common) definition of excess returns would be to define it as the difference in return of holding an  $n$  period bond for one period minus the one period

yield

$$\tilde{b}_{t+1}^{n-1} - \tilde{b}_t^n - \tilde{r}_t \quad (3.5)$$

This definition is equivalent to (3.4) in the sense that if the expectations hypothesis hold, expected excess returns will be zero according to both definitions (see Singleton 2006). The definition (3.4) is somewhat more convenient to work with in the present setting, but either definition could be used without changing the substance of the results presented below. We now use the result from Lemma 1 to show that non-nested information sets imply that individual traders can predict excess returns.

**Proposition 1.** *If traders' information filtrations are non-nested, excess returns are predictable by trader  $j$  for at least some maturity  $n^*$ .*

*Proof.* Excess returns on an  $n^*$ -period bond are predictable with respect to trader  $j$ 's information set if

$$\mathcal{P}_{t,j} \left( -\tilde{b}_t^{n^*} - (\tilde{r}_t + \tilde{r}_{t+1} + \dots + \tilde{r}_{t+n^*-1}) \right) \neq 0 \quad (3.6)$$

for some  $t$ . From the identity

$$-\tilde{b}_t^{n^*} \equiv \tilde{r}_t + f_t^1 + \dots + f_t^{n^*-1} \quad (3.7)$$

excess returns on an  $n^*$  period bond are thus predictable if

$$\sum_{s=1}^{n^*-1} \mathcal{P}_{t,j} \tilde{r}_{t+s} \neq \sum_{s=1}^{n^*-1} f_t^s \quad (3.8)$$

We know from Lemma 1 that  $\mathcal{P}_{t,j} \tilde{r}_{t+n} \neq f_t^n$  holds for at least some,  $t$ ,  $n$  and  $j$  if information sets are non-nested. To prove the proposition we need to show that this in turn implies that the inequality (3.8) must be true for at least some maturity  $n^*$ . The proposition holds trivially for  $n = 2$  since the inequality (3.8) then is true by assumption. For  $n > 2$ , consider the case when (3.8) is an equality at maturity  $n + 1$  so that

$$\sum_{s=1}^n \mathcal{P}_{t,j} \tilde{r}_{t+s} = \sum_{s=1}^n f_t^s \quad (3.9)$$

which can equivalently be written as

$$\sum_{s=1}^{n-1} \mathcal{P}_{t,j} \tilde{r}_{t+s} + \mathcal{P}_{t,j} \tilde{r}_{t+n} = \sum_{s=1}^{n-1} f_t^s + f_t^n \quad (3.10)$$

Since by assumption,  $\mathcal{P}_{t,j} \tilde{r}_{t+n} \neq f_t^n$  we must have that

$$\sum_{s=1}^{n-1} \mathcal{P}_{t,j} \tilde{r}_{t+s} \neq \sum_{s=1}^{n-1} f_t^s \quad (3.11)$$

which implies that excess returns on an  $n^* = n$  period bond is predictable. To complete the proof, it is sufficient to note that the proposition is trivially true for  $n^* = n + 1$  if (3.9) fails to hold. That is, if  $\mathcal{P}_{t,j} \tilde{r}_{t+n} \neq f_t^n$  excess returns cannot simultaneously be unpredictable for bonds of both maturity  $n$  and  $n + 1$ .  $\square$

In the absence of arbitrage, predictable excess returns must be earned as compensation for risk and this is true also in the present model. Risk aversion prevents a trader from wanting to hold an infinite amount of the bond with the highest expected return. and in equilibrium, the expected excess return as perceived by an individual trader is just enough to compensate for the increased riskiness of holding a less balanced portfolio. Private information thus introduces time variation in risk premia. The magnitude of the predictable component of excess returns caused by private information will be discussed in the context of conditional Sharpe Ratios in Section 4 below.

**3.2. Non-nested information sets and speculative trade.** Proposition 2 demonstrated that a non-degenerate distribution of short rate expectations is sufficient for excess returns to be predictable by individual traders. The next propositions helps us understand more about the dynamics introduced to the term structure by non-nested information sets and how these dynamics relate to public and private information. First, we demonstrate that non-nested information sets imply that individual traders can predict the average prediction errors made by other traders.

**Proposition 2.** *Non-nested information sets imply that an individual trader  $j$  can systematically predict the average period  $t + s$  projection error of the short rate in period  $t + n$ , that is*

$$\mathcal{P}_{t,j} \left( \tilde{r}_{t+n} - \int \mathcal{P}_{t+s,j'} \tilde{r}_{t+n} dj' \right) \neq 0 \quad (3.12)$$

if  $\Omega_t(j) \not\subseteq \Omega_{t+s}(i)$  and  $\Omega_t(i) \not\subseteq \Omega_{t+s}(j)$  for  $s = 0, 1, 2 \dots n - 1$  and all  $j \neq i \in (0, 1)$ .

*Proof.* The expression (3.12) can be rearranged to

$$\mathcal{P}_{t,j} \tilde{r}_{t+n} = \mathcal{P}_{t,j} \int \mathcal{P}_{t+s,j'} \tilde{r}_{t+n} dj'. \quad (3.13)$$

Since traders do not receive signals that are informative about the idiosyncratic noise in other traders' signals, we have that

$$\mathcal{P}_{t,j} \int \mathcal{P}_{t+s,j'} \tilde{r}_{t+n} dj' = \mathcal{P}_{t,j} \mathcal{P}_{t+s,i} \tilde{r}_{t+n} : i \neq j. \quad (3.14)$$

That is, an individual trader  $j$ 's expectation about average expectations coincide with his expectation of trader  $i$ 's expectation for any  $i \neq j$ . By property (4) of projections we know that

$$\mathcal{P}_{t,j} \tilde{r}_{t+n} = \mathcal{P}_{t,j} \mathcal{P}_{t+s,i} \tilde{r}_{t+n} \quad (3.15)$$

if and only if  $\Omega_t(j) \subseteq \Omega_{t+s}(i)$  which contradicts the definition of non-nested information sets and completes the proof.  $\square$

Proposition 2 showed that individual traders can systematically predict the average prediction error made by other traders. To see how this induces speculative behavior, consider an example with  $n = 2$  and  $s = 0$  in (3.12) and

$$\mathcal{P}_{t,j} \left( \tilde{r}_{t+2} - \int \mathcal{P}_{t,j'} \tilde{r}_{t+2} dj' \right) = \Delta \quad (3.16)$$

where  $\Delta > 0$  so that trader  $j$ 's expectation about  $\tilde{r}_{t+2}$  is higher than what he thinks the current average expectation of  $\tilde{r}_{t+2}$  is. Since trader  $j$  knows that other traders will receive new information in period  $t + 1$ , he thinks that other traders on average will revise their expectations in period  $t + 1$  towards what trader  $j$  considers the best prediction of  $\tilde{r}_{t+2}$  so that

$$\mathcal{P}_{t,j} \left( \tilde{r}_{t+2} - \int \mathcal{P}_{t+1,j'} \tilde{r}_{t+2} dj' \right) < \Delta \quad (3.17)$$

A positive  $\Delta$  thus implies that trader  $j$  believes that other traders will revise their expectations of  $\tilde{r}_{t+2}$  upward in  $t + 1$

$$\mathcal{P}_{t,j} \int \mathcal{P}_{t+1,j'} \tilde{r}_{t+2} dj' > \mathcal{P}_{t,j} \int \mathcal{P}_{t,j'} \tilde{r}_{t+2} dj' \quad (3.18)$$

and trader  $j$ 's expectation of the one period return on a 3 period bond ( $\tilde{b}_{t+1}^2 - \tilde{b}_t^3$ ) is lower than his expectation about other traders expectations about the one period return, since by (2.11) the price of a 2 period bond in  $t + 1$  depends negatively on the average expectation in  $t + 1$  of  $\tilde{r}_{t+2}$ .<sup>3</sup> Ceteris paribus, for a positive  $\Delta$  trader  $j$  expects to hold less than the average trader of the 3 period bond, as he thinks others on average expects it to have a higher return. A symmetric argument holds if  $\Delta$  is negative.

**3.3. Forward rates and average higher order prediction errors.** So far, we have discussed the implications of non-nested information sets from the perspective of the individual trader. We now turn to the implications of non-nested information sets for aggregate bond prices, or more specifically, for implied forward rates. In the next proposition, we show that the speculative dynamics introduced by non-nested information sets can be expressed as average predictions about higher order prediction errors of future short rates. That is, predictions about the difference between future short rates and other traders' predictions about future short rates.

**Proposition 3.** *The forward rate  $f_t^n$  can be decomposed into the the average first order projection of  $\tilde{r}_{t+n}$ , a sum of higher order projection errors and the exogenous supply shocks  $v_t^n$  and  $v_t^{n+1}$ .*

*Proof.* For convenience, first define the notation

$$\prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \equiv \int \mathcal{P}_{t,j} \int \mathcal{P}_{t+1,j'} \dots \int \mathcal{P}_{t+n-1,j''} \tilde{r}_{t+n} dj'' \dots dj' dj. \quad (3.19)$$

and apply it to the definition of the  $n$  period forward rate (2.22) and bond prices (2.13) to get

$$f_t^n = \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} + (v_t^n - v_t^{n+1}). \quad (3.20)$$

<sup>3</sup>The argument will hold as long as there is not a perfect negative correlation between revisions to  $\int \mathcal{P}_{t+s,j'} r_{t+1}$  and  $\int \mathcal{P}_{t+s,j'} r_{t+2}$  for  $s = 0, 1$ .



Add and subtract  $\int \mathcal{P}_{t,j} \tilde{r}_{t+n}$  from the r.h.s. of (3.20) to get

$$\begin{aligned}
 f_t^n &= \int \mathcal{P}_{t,j} \tilde{r}_{t+n} \\
 &\quad - \underbrace{\int \mathcal{P}_{t,j} \left( \tilde{r}_{t+n} - \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \right)}_{n-1 \text{ order prediction error}} \\
 &\quad + (v_t^n - v_t^{n+1})
 \end{aligned} \tag{3.21}$$

□

The term on the second line of (3.21) is the average prediction of the  $n-1$  order prediction error, i.e. the average prediction of the difference between the actual short rate in period  $t+n$  and the  $n-1$  order expectation of the short rate in period  $t+n$ .<sup>4</sup> The expression (3.21) thus demonstrates that forward rates do not necessarily reflect average expectations about future short rates.

In a model with perfect or common information, the higher order prediction errors on the second line of (3.21) would of course be zero and the  $n$  period forward rate would be a function of only the period  $t$  average expectation of the the short rate in period  $t+n$  (and the exogenous supply shocks). This would also be true in a model where bonds are only traded when they are issued and then held until maturity. In such a setting, the expectation of other traders' expectations would not matter for the equilibrium price, since the price of a zero coupon bond at maturity is simply its face value, which is known to all traders. The price of the bond at the date of issue would then simply be such that the return on an  $n$  period bond equals the expected return on the alternative investment. (By imposing this condition for all maturities  $n$ , it can be shown that this alternative return is the average expectation of the cumulative return of holding a series of one period bonds for  $n$  periods.) The new dynamics introduced to the term structure by non-nested information sets contained in the higher order prediction error term is thus dependent on the fact that long bonds are traded frequently. This is also the sense in which the “speculative behavior” in this model conforms to the definition of Harrison and Kreps (1978).

There are also some differences between the model presented here and the set up of Harrison and Kreps (1978) that are worth noting. The most important of these is perhaps that Harrison and Kreps rule out short sales, with the implication that the price of the asset in their model is bounded below by what any single trader would be willing to pay for it, were he to hold on to the asset forever. In our model, there are no short sales constraints and the price of a bond can be either above or below what the equilibrium price would be if the bond could not be traded before maturity.

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<sup>4</sup>In a different context, Bacchetta and Wincoop (2006) shows that a similar term (which they label the “higher order wedge”) can be expressed as an average expectation error of the innovations to the fundamental process in their model.

**3.4. Speculative dynamics and public information.** It is straightforward to show that the speculative dynamics due to higher order prediction errors are orthogonal to public information. Before proving this statement, we first define two relevant information sets.

**Definition 7.** *The subspace  $\Omega_t^p$  is the space spanned by the history of publicly observable variables in period  $t$  so that  $\Omega_t^p \subseteq \Omega_t(j)$  for all  $j$ . Projections onto  $\Omega_t^p$  are denoted  $\mathcal{P}_t^p$ .*

**Definition 8.** *The subspace  $\Omega_t^{\perp p}(j)$  is the orthogonal complement of  $\Omega_t^p$  in  $\Omega_t(j)$ . Projections onto  $\Omega_t^{\perp p}(j)$  are denoted  $\mathcal{P}_{t,j}^{\perp p}$ .*

**Proposition 4.** *The forward rate  $f_t^n$  can be decomposed into the projection of  $\tilde{r}_{t+n}$  onto the public information set  $\Omega_t^p$ , the supply shocks and terms that are orthogonal to public information.*

*Proof.* Use that any projection onto  $\Omega_t(j)$  can be decomposed into a sum of the projection onto  $\Omega_t^p$  and a projection onto the orthogonal complement  $\Omega_t^{\perp p}(j)$  to rewrite the expression of the forward rate (3.21) as

$$\begin{aligned} f_t^n &= \mathcal{P}_t^p \tilde{r}_{t+n} + \int \mathcal{P}_{t,j}^{\perp p} \tilde{r}_{t+n} \\ &\quad - \int \mathcal{P}_{t,j}^p \left( \tilde{r}_{t+n} - \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \right) \\ &\quad - \int \mathcal{P}_{t,j}^{\perp p} \left( \tilde{r}_{t+n} - \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \right) \\ &\quad + (v_t^n - v_t^{n+1}) \end{aligned} \tag{3.22}$$

Since  $\Omega_t^p \subseteq \Omega_{t+s}(j)$  for all  $j$  and  $s = 0, 1, \dots, m-1$  and by Property 4 of orthogonal projections (see Appendix A) we have that

$$\mathcal{P}_{t,j}^p \tilde{r}_{t+n} = \mathcal{P}_{t,j}^p \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \tag{3.23}$$

for all  $j$  and  $s = 0, 1, \dots, m-1$ . The term on the second line of (3.22) is thus identically zero and the  $n$  period forward rate can thus be expressed as

$$\begin{aligned} f_t^n &= \mathcal{P}_t^p \tilde{r}_{t+n} + \int \mathcal{P}_{t,j}^{\perp p} \tilde{r}_{t+n} \\ &\quad - \int \mathcal{P}_{t,j}^{\perp p} \left( \tilde{r}_{t+n} - \prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \right) \\ &\quad + (v_t^n - v_t^{n+1}) \end{aligned} \tag{3.24}$$

which concludes the proof.  $\square$

Proposition 4 demonstrates that the new term structure dynamics introduced by non-nested information sets are orthogonal to public information which by definition is common knowledge. This is intuitive, since all traders know that all traders know, and so on, that

all traders know that all traders observe a public signal and that the public signal therefore cannot be used to predict the errors that other traders will make. The component of other traders' projection errors that are predictable by an individual trader  $j$  must therefore be orthogonal to public information.

Allen, Morris and Shin (2006) argues that with privately informed traders, asset prices may display "drift", i.e. slow adjustment to shocks and several small price changes in the same direction. While this may be true if one condition on the value of the fundamental, the result above suggests that there should be no discernable drift caused by private information that can be identified simply by observing prices or other information that is publicly available in real time. The empirical evidence summarized by Barberis, Shleifer and Vishny (1998) that there is a predictable "momentum" component in some asset prices must therefore be caused by something else than information based speculation among rational traders.

This ends the theoretical part of the paper. Before turning to the data, we can summarize our findings so far. With non-nested information sets, individual traders can identify and take advantage of predictable excess returns that would be absent in a model with only common information. We also demonstrated that the new dynamics introduced by speculative behavior is orthogonal to public information. This has an interesting empirical implication: Speculative dynamics cannot be detected using public data in real time. However, as econometricians we can use public information from periods  $t + s : s > 0$  to extract an estimate of the term due to the speculative dynamics in period  $t$ . To do so, we need to specify a process for the short rate and traders' information sets.

#### 4. THE ESTIMATED MODEL

In the previous section it was demonstrated that non-nested information sets introduce new dynamics to the term structure of interest rates. Here, we address the question whether these dynamics can be quantitatively important. Above, bond prices were derived as functions of higher order expectations of future short rates. In order to have an operational model that can be estimated, we need to specify two more objects: A process for the short rate and the information sets of the traders.

4.1. **The short rate.** The short interest rate  $\tilde{r}_t$  is the sum of three exogenous factors

$$\tilde{r}_t = x_t^1 + x_t^2 + x_t^3 \quad (4.1)$$

where the factors  $\mathbf{x}_t \equiv [x_t^1 \ x_t^2 \ x_t^3]'$  follow the vector autoregressive process

$$\mathbf{x}_t = A\mathbf{x}_{t-1} + C\mathbf{u}_t : \mathbf{u}_t \sim N(0, I_3) \quad (4.2)$$

The diagonal structure of  $A$  and the lower triangular structure of  $C$

$$A = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}, C = \begin{bmatrix} c_1 & 0 & 0 \\ c_{21} & c_2 & 0 \\ c_{31} & c_{32} & c_3 \end{bmatrix}$$

are normalizations that do not restrict the dynamics of  $\tilde{r}_t$ . The three factor structure is motivated by two considerations. It gives a sufficiently high dimensional latent state to make the filtering problem of traders interesting, while keeping the model computationally tractable. In addition, using three factors implies that if traders were perfectly informed,

the model would simply be a three factor affine no-arbitrage model with a constant price of risk.

**4.2. Traders' information sets.** All traders observe a vector of public signals containing the current short rate  $\tilde{r}_t$  and selected bond yields collected in the vector  $\mathbf{y}_t$ . Non-nested information sets are introduced through individual signals about the first two factors  $x_t^1$  and  $x_t^2$ . Each signal is the sum of the true factor and an idiosyncratic noise component and the noise is uncorrelated across signals and time. The vector of private signals  $\mathbf{z}_t(j)$  observed by trader  $j$  is thus given by

$$\mathbf{z}_t(j) = \begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} + Q\zeta_t(j) : \zeta_t \sim N(0, I_2) \quad (4.3)$$

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad (4.4)$$

Since the short rate is observed directly and is the sum of the three factors, it is without loss of generality that traders observe private signals only about the first two factors. The vector

$$S_t(j) = [ \mathbf{z}'_t(j) \quad \tilde{r}_t \quad \mathbf{y}'_t ]' \quad (4.5)$$

contains all the signals that trader  $j$  observes in period  $t$ . Trader  $j$ 's information set in period  $t$  is thus given by

$$\Omega_t(j) = \{S_t(j), \Omega_{t-1}(j)\} \quad (4.6)$$

implying that traders condition their expectations on the entire history of observed signals.

**4.3. The solved model.** When traders have non-nested information sets it becomes optimal to form expectations about other traders' expectations and natural representations of the state in this class of models tend to be infinite.<sup>5</sup> The model is solved using the method proposed in Nimark (2010) which delivers a law of motion for the (finite dimensional) state  $X_t$  of the form

$$X_t = MX_{t-1} + Ne_t \quad (4.7)$$

The state vector  $X_t$  contains stacked higher order expectations of the factors

$$X_t \equiv \left[ \mathbf{x}_t^{(0)'} \quad \mathbf{x}_t^{(1)'} \quad \dots \quad \mathbf{x}_t^{(\bar{k})'} \right]' \quad (4.8)$$

where the higher order expectations of the vector of factors  $\mathbf{x}_t$  are defined recursively as

$$\mathbf{x}_t^{(k)} \equiv \int E \left[ \mathbf{x}_t^{(k-1)} \mid \Omega_t(j) \right] dj$$

starting from  $\mathbf{x}_t^{(0)} = \mathbf{x}_t$ . The integer  $\bar{k}$  is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation in the limit as  $\bar{k} \rightarrow \infty$ .

Common knowledge of the model is used to pin down the law of motion for  $X_t$ , that is, to find  $M$  and  $N$  in (4.7). As usual in rational expectations models, first order expectations  $\mathbf{x}_t^{(1)}$  are optimal, i.e. model consistent estimates of the actual factors  $\mathbf{x}_t$ . The knowledge that other traders have model consistent estimates allow traders to treat average first order

<sup>5</sup>See Townsend (1983), Sargent (1991) and Makarov and Rytchkov (2009).

expectations as a stochastic process with known properties when they form second order expectations. Common knowledge of the model thus implies that second order expectations  $\mathbf{x}_t^{(2)}$  are optimal estimates of  $\mathbf{x}_t^{(1)}$  given the law of motion for  $\mathbf{x}_t^{(1)}$ . Imposing this structure on all orders of expectations allows us to find the law of motion for the complete hierarchy of expectations as functions of the structural parameters of the model, i.e. the parameters governing the short rate (4.1), traders' information sets (4.6) and the parameter  $\sigma_v$  governing the standard deviation of the bond supply shocks  $v_t$ . The vector  $e_t$  contains the aggregate shocks in the economy, i.e. the factor innovations  $\mathbf{u}_t$  and the bond supply shocks  $\mathbf{v}_t$ .

For a given law of motion (4.7), bond prices can be derived using the average expectation operator  $H : \mathbb{R}^{\bar{k}} \rightarrow \mathbb{R}^{\bar{k}}$  that annihilates the lowest order expectation of a hierarchy so that

$$\begin{bmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k}+1)} \end{bmatrix} = H \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(1)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k})} \end{bmatrix} \quad (4.9)$$

and where  $\mathbf{x}_t^{(k)} = 0 : k > \bar{k}$ . Combing the operator  $H$  that moves expectations one step up in orders of expectations and the matrix  $M$  from the law of motion (4.7) that moves expectations one step forward in time allows us to compute the higher order expectation in the bond pricing equation (2.13) as

$$\prod_{s=0}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} = [ \mathbf{1}_{1 \times 3} \quad \mathbf{0} ] (MH)^{s-1} X_t \quad (4.10)$$

so that bond prices are given by

$$\tilde{b}_t^n = - \sum_{s=0}^{n-1} [ \mathbf{1}_{1 \times 3} \quad \mathbf{0} ] (MH)^{s-1} X_t + v_t^n \quad (4.11)$$

The matrix  $M$  governs the actual dynamics of  $\tilde{r}_t$  while bonds are priced *as if*  $X_t$  was observed by all agents and followed a process governed by  $MH$ . The matrices  $M$  and  $MH$  are thus analogous to the ‘‘physical’’ and ‘‘risk neutral’’ dynamics in a standard no-arbitrage framework, though the interpretation is different.

Since log bond prices are a linear function of the state  $X_t$  plus the supply shocks  $\mathbf{v}_t$  we can use the definition of bond yields  $y_t^n = -n^{-1}\tilde{b}_t^n$  to also write the vector of bond yields  $\mathbf{y}_t$  as linear function of the state and the supply shocks

$$\mathbf{y}_t = BX_t + \hat{\mathbf{v}}_t \quad (4.12)$$

where the element of the vector  $\hat{\mathbf{v}}_t$  associated with the  $n$  period yield is given by  $n^{-1}v_t^n$ . The state equation (4.7) and the yield equation (4.12) constitutes a state space system that can be used to let the observed history of bond yields inform us about the most likely values of the parameters of the model. More details on how the model was solved can be found in the Appendix.

**4.4. Posterior Estimates.** The solved model, i.e. the state space system (4.7) and (4.12), is in a form that can be estimated directly by likelihood based methods. The supply shocks  $v_t^n$  are assumed to be independent across maturities. However, since traders use information in bond yields to form estimates of the state, a supply shock to a given maturity  $n$  in period  $t$  will have a persistent effect on the entire term structure for several periods. The supply shocks here are thus consistent with the evidence in Hamilton and Wu (2011). The standard deviation of an  $n$  period bond supply shock  $v_t^n$  is specified as  $n\sigma_v$ . Restricting the standard deviations of the supply shocks to be a linear function of maturity reduces the total number of parameters to be estimated. In preliminary estimations this reduction in the number of parameters came at virtually no cost in terms of fit. The presence of the supply shocks  $\mathbf{v}_t$  thus add  $\sigma_v$  as a single new parameter to estimate along with the the parameters of the short rate process (4.2) and the private signals of traders (4.3).

**Table 1**  
*Posterior Parameter Estimates 1964:1-2007:12*

$\theta$	Mode $\widehat{\theta}$	Prior dist.	Posterior 2.5%-97.5%
Short rate process			
$\rho_1$	0.990	$U(0, 0.99)$	0.989 - 0.990
$\rho_2$	0.989	$U(0, 0.99)$	0.987 - 0.990
$\rho_3$	0.696	$U(0, 0.99)$	0.655 - 0.750
$c_1$	11.2	$U(0, 50)$	7.42 - 26.8
$c_2$	0.62	$U(0, 50)$	0.59 - 0.65
$c_3$	0.48	$U(0, 50)$	0.45 - 0.51
$c_{21}$	-11.9	$U(-50, 50)$	(-27.4) - (-8.11)
$c_{31}$	-0.49	$U(-50, 50)$	(-0.54) - (-0.45)
$c_{32}$	0.38	$U(-50, 50)$	0.34 - 0.43
Noise in private signals			
$q_1$	0.35	$U(0, 50)$	0.24 - 1.14
$q_2$	1.01	$U(0, 50)$	0.61 - 1.34
Bond supply shocks			
$\sigma_v$	0.21	$U(0, 50)$	0.20 - 0.21
Log likelihood at $\widehat{\theta}$ : -248.18			

The vector of parameters to be estimated is denoted  $\theta \equiv \{A, C, Q, \sigma_v\}$  and consists of a total of 12 parameters. I use monthly data of the Federal Funds rate and the 3, 12, 24, 36, 48 and 60 month annualized interest rates on Treasuries taken from the CRSP data base. These are the same maturities that the traders inside the model are assumed to observe. The sample period is from January 1964 to December 2007 (528 monthly observations) and chosen to coincide with the sample period used by Cochrane and Piazzesi (2008) and Duffee (2008). The time series are demeaned. The posterior parameter distributions was simulated by 500 000 draws from an Adaptive Metropolis algorithm (see Haario, Saksman and Tamminen (2001)), initialized from a parameter vector found by maximizing the posterior using the simulated annealing maximizer of Goffe (1996). There are some large discrete mean adjustments in the first 200 000 draws of the Markov chain, but visual inspection suggests

that both first and second moments have converged after 500 000 draws.<sup>6</sup> The results based on the last 250 000 draws are reported in Table 1. The posterior mode  $\hat{\theta}$  is the parameter vector of the simulated posterior Markov chain that achieves the highest posterior likelihood.

By themselves, the posterior estimates are not very interesting, but we can note that all parameters appear to be well-identified. The first two factors are very persistent and traders appear to have more precise private information about the first factor than about the second, that is  $q_1 < q_2$ . The standard deviation  $\sigma_v$  of the supply shocks are similar to the estimated standard deviation of measurement errors in latent factor models (e.g. Duffee 2008).

The fit of the model is good and compares favorably to a standard three factor affine no-arbitrage model in terms of likelihood at the posterior mode. It is thus possible to construct fully dynamic models with privately informed traders that are rich enough to be taken seriously empirically. While this is reassuring, it is by itself not a strong endorsement of the mechanism proposed in the model. In later sections other ways of validating the reasonableness of the model will be discussed.

**4.5. Historical speculation.** The previous section showed that the speculative term introduced to the term structure by non-nested information sets could be expressed as a higher order prediction error that is orthogonal to public information. Nevertheless, we can quantify this term using public price data since the period  $t$  higher order prediction error is only orthogonal to public information known up to time  $t$ . As econometricians, we can use the full sample and exploit information for  $t + s : s > 0$  to back out information about the higher order prediction error in period  $t$ . The Kalman simulation smoother can be used to draw from  $p(X^T | \mathbf{y}^T, \theta)$  for a given parameter vector  $\theta$  (e.g. Durbin and Koopman 2002). The simulation smoother together with the posterior distribution of  $\theta$  can thus be used to construct the posterior distribution of the state  $p(X^T | \mathbf{y}^T)$ . Once we have a posterior distribution of the state, it is straightforward to compute the distribution of the speculative term (3.21) in the implied  $n$  period forward rate by using that

$$\begin{aligned} & - \int \mathcal{P}_{t,j} \left( \tilde{r}_{t+n} - \prod_{s=1}^{n-1} \int \mathcal{P}_{t+s,j} \tilde{r}_{t+n} \right) \\ & = \begin{bmatrix} \mathbf{1}_{1 \times 3} & \mathbf{0} \end{bmatrix} [(MH)^n - M^n H] X_t \end{aligned} \quad (4.13)$$

where the equality follows from (4.10) and the fact that

$$\int \mathcal{P}_{t,j} \tilde{r}_{t+n} = \begin{bmatrix} \mathbf{1}_{1 \times 3} & \mathbf{0} \end{bmatrix} M^n H X_t \quad (4.14)$$

$$= \mathbf{1}_{1 \times 3} \times A^n \int \mathcal{P}_{t,j} \mathbf{x}_t^{(0)} dj \quad (4.15)$$

Figure 2 displays the median and the 95% (point-wise) posterior probability intervals for the speculative term in the one ( $n = 12$ ) and five ( $n = 60$ ) year forward rates.

The speculative term in the two maturities are strongly correlated and potentially quantitatively important at both short and long maturities. At their peaks, around 1981, the

<sup>6</sup>More details, including the simulated posterior Markov chain and recursive plots of parameter standard deviations, are available to download from the author's web page.

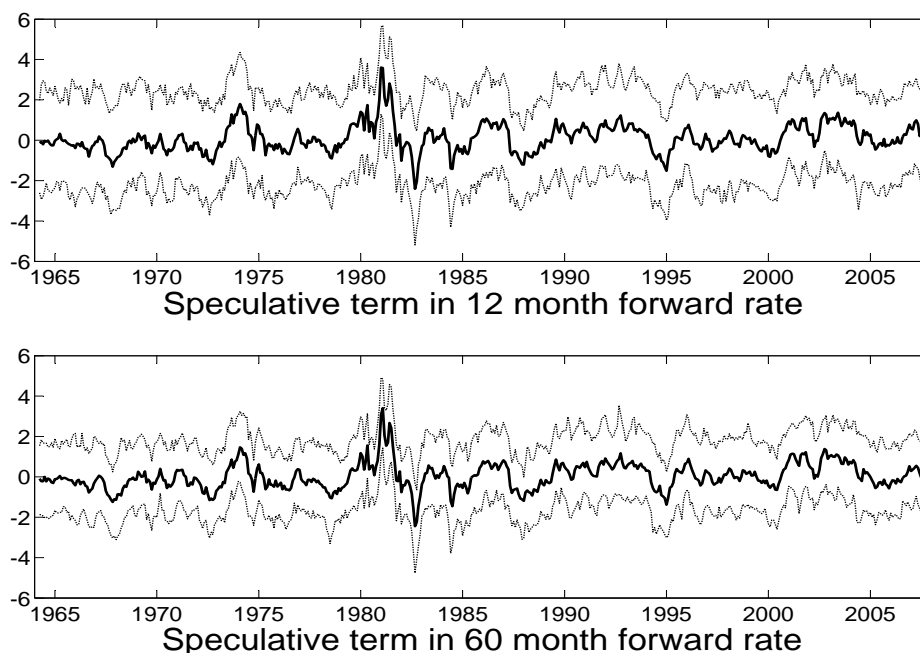


FIGURE 2. Estimated speculative term (percentage points) in 12 and 60 month implied forward rate 1964:1 - 2007:12. Median (solid) and 95% probability interval (dotted).

median estimate is 3.5% and 2.5% for respectively, the 1 and 5 year forward rates. The 95% probability interval reaches even higher, and is above 5% for the the speculative term in the 1 year forward rate. One way to interpret these numbers is the following. If a “naive” observer in 1981 took the implied forward rate as an indicator of market expectations about the short rate in 1982, the magnitude of the mistake he would make by not controlling for the speculative term would be 3.5%.

Singleton (2006) points out that violations of the expectation hypothesis in US data are most pronounced when the period 1979-1983 is included in sample, and the episode in the early 1980’s is also the most eye-catching (and volatile) in Figure 2. The period 1979 - 1983 coincides with the so-called Volcker disinflation when the then Federal Reserve chairman Paul Volcker raised interest rates sharply to bring inflation under control, causing a recession (see for instance the account in Goodfriend and King 2005). Once inflation credibility had been established, short interest rates began to fall, though long rates stayed high for some time. Viewed through the lens of the model, this was an episode when first order expectations of future short rates were significantly lower than higher order expectations. That is, individual traders may have found it credible that Volcker would be able to keep future short rates low before they believed that other traders had been convinced as well.



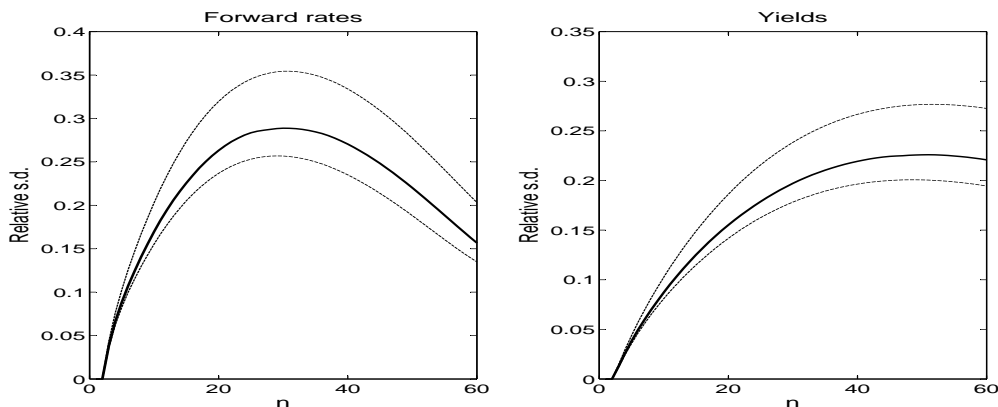


FIGURE 3. Estimated relative standard deviation of speculative term in forward rates (left panel) and yields (right panel). Median (solid) and 95% probability interval (dotted).

Alternative explanations of this episode emphasize that the early 1980s was a period when traders demanded either more compensation to hold a given amount of risk because of the recession, or when the amount of risk was perceived to be higher than usual because of more volatile interest rates. Since the model presented here abstracts from these explanations it is potentially misspecified and there is thus a risk that the importance of speculative dynamics is overstated. Of course, models that abstract from speculative dynamics may overstate the importance of other sources of variation in risk premia and it would be interesting to have a model that nests these various mechanisms in order to quantify their relative contribution to this episode, though this is beyond the scope of the present paper. While the different types of models attach importance to the same episode in the early 1980s, they are not observationally equivalent. The fact that speculative dynamics must be orthogonal to public information in real time makes it econometrically distinct from other sources of time variation in risk premia and should at least in principle allow us to identify it as separate from other sources of variation in risk premia.

**4.6. Quantifying the importance of speculative dynamics.** We can also use the estimated model to quantify the relative importance of speculation at different maturities. The left panel of Figure 3 displays the median and the 95 per cent probability interval of the relative standard deviation of the speculative term and forward rates. At the one year horizon, the relative standard deviation of the speculative term is around 15% at the median and rises to approximately 30% at the peak at around a maturity of three years ( $n = 36$ ). It is also evident from the figure that the speculative term is not only quantitatively important, but it is also statistically significant, with the 95% probability interval clearly bounded away from zero. (The statistical significance of the speculative term is somewhat obscured by the point-wise nature of the probability intervals in Figure 2.)

The right panel of Figure 3 displays the relative standard deviation of the speculative term in yields, computed as the recursive average of the speculative term in forward rates. We

can see that the speculative term is quantitatively important also for yields, and more so at long horizons, peaking at the four year maturity where it reaches 22.5 % at the median. The fact that speculative dynamics appear to be more important for longer maturities may also help explain the evidence in Gürkaynak, Sack and Swanson (2005) who argue that current macro models of the term structure have trouble explaining the “excess” variability of long bond yields. Embedding a non-nested information structure in a macro model may improve these models’ ability to match the variance of long term yields.

**4.7. The estimated dispersion of expectations.** The dynamics of the model depend importantly on that traders’ information sets are non-nested. As noted in the introduction, one implication of non-nested information sets is that expectations will be dispersed. This fact can be used as an independent check to gauge whether the estimated model requires a reasonable degree of expectations dispersion to fit the data. Since no information about expectation dispersion is used in the estimation process, this can be thought of as an informal test of over-identifying restrictions.

The standard deviation of individual traders expectations of the short rate  $n$  months ahead is given by

$$\sqrt{E \left( \mathcal{P}_{t,j} \tilde{r}_{t+n} - \int \mathcal{P}_{t,j'} \tilde{r}_{t+n} \right)^2} \quad (4.16)$$

The 95 per cent probability intervals for  $n = 12, 36$  and  $60$  of the posterior distribution of dispersion of first order expectations are reported in Table 2. The first row ( $n = 12$ ) is directly comparable to the survey evidence reported by Swanson (2006). The dispersion implied by the estimated model is significantly smaller than that of the Blue Chip survey. The spread between the 10th and the 90th percentile of a Gaussian distribution is approximately 2.6 standard deviations. The estimated spread is thus approximately  $0.12 \times 2.6 = 0.31$  or around 30 basis points at the median and around 40 basis points at the 97.5 percentile. The estimated dispersion is thus smaller than even the lower end (80 basis point) of the spread reported by Swanson. The model can thus not be considered to rely on an implausibly large dispersion of expectations in order to generate a quantitatively significant speculative term. This should increase our confidence in the model.

**Table 2**  
Estimated dispersion of short rate expectations

$n$	$\sqrt{E \left( \mathcal{P}_{t,j} \tilde{r}_{t+n} - \int \mathcal{P}_{t,j'} \tilde{r}_{t+n} \right)^2}$		
	2.5%	Median	97.5%
<b>12</b>	0.08	0.12	0.15
<b>36</b>	0.06	0.09	0.11
<b>60</b>	0.05	0.07	0.08

All numbers in percentage points (1 basis point = 0.01).

Table 2 also reports the dispersion of expectations at the three and five year forecast horizon. The spread is slightly decreasing at longer horizons. Of course, at long enough

horizons the dispersion will disappear as expectations of all agents will converge to the unconditional mean of short rates.

**4.8. Conditional Sharpe Ratios.** Conditional Sharpe Ratios provide a measure of how trading opportunities varies over time. Another way to judge the plausibility of the model is thus to inspect the Sharpe Ratios implied by traders taking advantage of private information. Since information sets are partly trader specific, conditional Sharpe Ratios will differ across traders. Using the posterior distribution of the state  $p(X^T | \mathbf{y}^T)$  we can back out a time series of the conditional Sharpe Ratio associated with the median, or average, trader. The Sharpe Ratio of an  $n$  periods to maturity bond from the perspective of the average trader is then defined as

$$s_t^n \equiv \frac{\int \mathcal{P}_{t,j} \tilde{b}_{t+1}^{n-1} dj - \tilde{b}_t^n - \tilde{r}_t + \frac{1}{2} \text{var}(\tilde{b}_{t+1}^{n-1} | \Omega_t(j))}{\sqrt{\text{var}(\tilde{b}_{t+1}^{n-1} | \Omega_t(j))}} \quad (4.17)$$

The numerator of the conditional Sharpe Ratio (4.17) is the average trader's expected excess return of holding an  $n$  period bond for one period. The denominator is the conditional variance of excess returns which is common to all traders and do not vary over time. In order to compute a historical estimate of the Sharpe Ratio  $s_t^n$  we will use the approximation that  $\text{var}(\tilde{b}_{t+1}^{n-1} | \Omega_t(j)) \approx \text{var}(\tilde{b}_{t+1}^n | \Omega_t(j))$ . This approximation is necessary since not every maturity  $n$  is included in the estimated model.

The posterior median estimates of  $s_t^n$  for  $n = 12$  and  $n = 60$  are plotted in Figure 4 together with 95% (point wise) probability intervals. At the median, both Sharpe Ratios are quite volatile and the Sharpe Ratio of the 12 month bond is more volatile than that of the 60 month bond. At the median, the standard deviations are 1.51 and 1.24, respectively. Comparing the time series of the Sharpe Ratios here with the graphs in Sangvinatsos and Wachter (2005) suggest that the non-nested information model can generate as much variation in investment opportunities as the essentially affine model of Sangvinatsos and Wachter.<sup>7</sup> However, what is also clear from Figure 4 is that the probability intervals are very wide and for all maturities and all periods always include the zero line. One interpretation of this result is that it may be intrinsically difficult to get a precise estimate of excess returns due to private information using only publicly available yield data, even when the entire sample is exploited. Another possibility is that it is general property of term structure models that it is difficult to estimate Sharpe Ratios at a given point in time precisely, though I am not aware of any other study that reports probability intervals of conditional Sharpe Ratios.

**4.9. Hidden factors and predictable excess returns.** Duffee (2008) provide evidence of a "hidden" factor that is insignificant in explaining the cross-section of yields but important for predicting short rates and in extension, excess returns. Duffee estimates a 5 factor model

<sup>7</sup>Sangvinatsos and Wachter (2005) do not report the standard deviation of the maximum Sharpe Ratios of their model but a plotted time series.

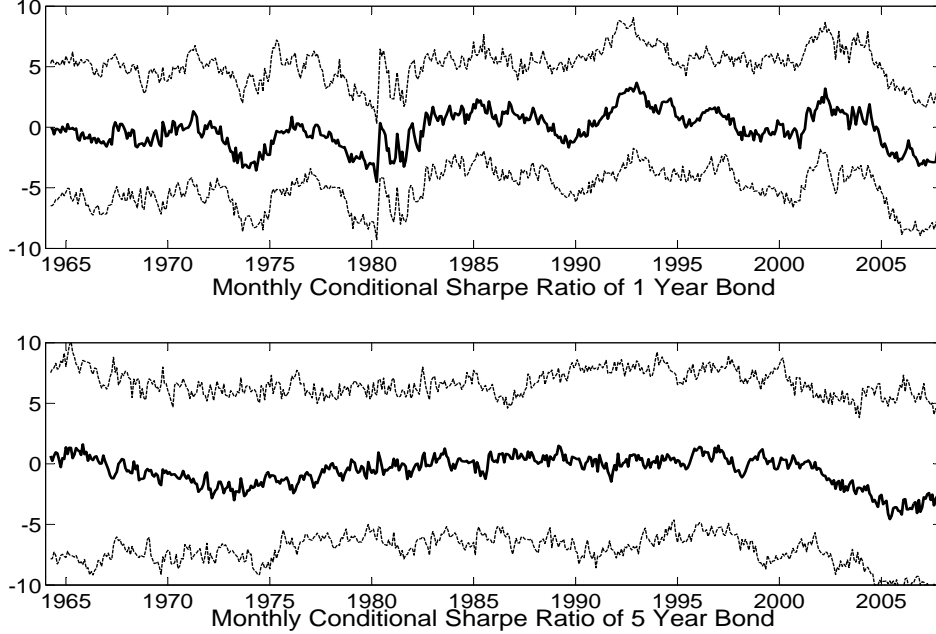


FIGURE 4. Posterior estimates of conditional Sharpe Ratios of the average trader with 90% confidence intervals.

of the form

$$x_t^\dagger = D^\dagger x_{t-1}^\dagger + \Sigma^\dagger \epsilon_t \tag{4.18}$$

$$y_t = A + B^\dagger x_t^\dagger + v_t^\dagger \tag{4.19}$$

on US bond data and the estimated model can be rotated to compute the implied principal components. Duffee finds that while the first three principal components explain almost all of the unconditional variation in yields, the fifth principal component is important for explaining expected future short rates. He illustrates this by impulse response functions of the 5 factors and their effect on the short rate. If a factor is unimportant for the cross section, but important for predicting short rates (and in extension, excess returns) it will manifest itself as an impulse response function of the short rate to the factor in question that originates at zero but then becomes positive (or negative).

We investigated whether the hidden factor found by Duffee is consistent with the model presented here by generating artificial data sets using parameter draws from the estimated posterior of our model with non-nested information sets. For each parameter draw, we simulated 528 months of data and then estimated Duffee’s five factor model by maximum likelihood and performed the required rotations to find the principal components. We then computed impulse responses of short rates to orthogonal innovations to the factors. This procedure was repeated 450 times. Figure 5 shows the median impulse response and the 5th

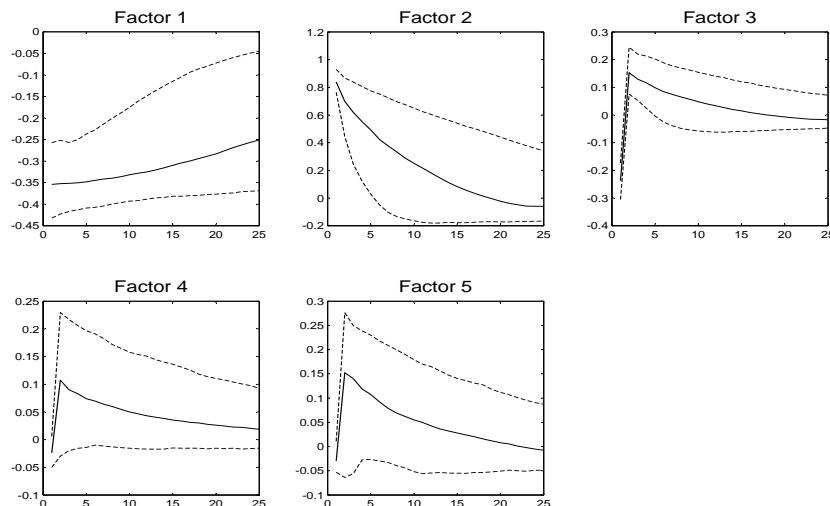


FIGURE 5. Impulse responses to orthogonal factor innovations, dashed lines are 5th and 95th percentile.

and 95th percentile.<sup>8</sup> As we can see, the fourth and fifth principal components have little effect on the short rate in the impact period, but becomes more important at longer time horizons. This is exactly what we should expect from a model where the term structure does not reveal all information about future short rates perfectly. If the state of the model would be revealed perfectly by the cross section of yields, no additional factors beyond the three (level, slope and curvature) that explains the cross sections would be useful to predict future yields. However, if the state is not revealed by the cross section, then by construction there must be additional factors that can help predict future yields.

## 5. CONCLUSIONS

Introducing private information in a model of bond pricing can give rise to speculative behavior in the sense of Harrison and Kreps (1978). For traders in the model to engage in speculative behavior it is sufficient that information sets are non-nested, or equivalently that there is dispersion across traders' forecasts of future short rates. Dispersed expectations are also sufficient for traders to be able to predict excess returns and the model provides a novel mechanism for time varying risk premia in the term structure of interest rates.

Theoretical models of private information in asset markets dates back at least to Grossman (1976). More recently, Allen, Morris and Shin (2006) presented a single risky asset, finite horizon model with an information structure similar to the one presented here. They show that a concern among market participants about other market participants opinions as

<sup>8</sup>The percentile refer to the percentiles of the point estimates from Duffee's model estimated on artificial data and can thus not be given a probabilistic interpretation. A full Bayesian posterior simulation for each draw of artificial data is too time consuming to be feasible.

described in Keynes' (1936) "beauty contest" metaphor of the stock market, can be present among fully rational traders if traders have access to private information. Here we have demonstrated that even if the estimated amount of private information is small in the sense that the dispersion of (first order) expectations across traders is low, speculative trade driven by attempts to exploit perceived market mispricing of bonds can be quantitatively important. We showed this by formulating an empirically plausible model of the term structure that was estimated using likelihood based methods. The fit of the model is good, and it is thus possible to construct dynamic models with privately informed agents that are rich enough to take directly to the data. We also demonstrated how a historical time series of the effect of speculative dynamics on implied forward rates can be estimated from public price data, in spite of the fact that speculation according to the model is orthogonal to real time public information.

The zero-coupon bonds traded in the present model have a known value at maturity. The uncertainty about future bond prices arise solely from the uncertainty about the discount rates that apply between the current period and the period when a bond matures. Arguably, these discount rates should matter also for the pricing of other assets that will be traded and pay dividends in the future. To the extent that there is additional uncertainty about dividend payments and returns on other classes of assets, speculative dynamics may be even more important in other markets than the bond market and thus worth investigating.

While the empirical results of the paper suggest that speculative dynamics can be quantitatively important, it is worth also discussing two potential shortcomings of the model that may lead us to overstate their role in explaining the observed history of bond yields.

First, recent studies have found that "unspanned" macro factors help predict excess returns, e.g. Ludvigson and Ng (2009) and Joslin, Priebsch and Singleton (2010) while the model here abstracts from public information not spanned by bond prices. Potentially, this could have implications for the estimated importance speculative dynamics. One possibility is that there are publicly observed unspanned factors that have exactly offsetting effects on short rate expectations and risk premia (as in the model of Joslin *et al* 2010). These factors would then appear not to be priced, i.e. these factors would not be spanned by current bond prices, but could still help predict future excess returns. The public information set of the model presented here would then be too small and the importance of speculative dynamics could be overstated. However, there is another interpretation of the results of Ludvigsson and Ng (2009) and Joslin *et al* (2010) that is more favorable to our model. In both studies cited above, the macro information is dated according to the calendar time of the quantity that it measures. As an example, the Consumer Price Index (CPI) for say January is dated as January in the bond pricing model and not dated as February, when the data is actually made public. In a perfect information setting like Joslin *et al*'s this distinction is unimportant as traders are assumed to observe the true state of the world at all times. However, in imperfect information models this distinction becomes important. To see why, note that a natural explanation why the predictive content of January CPI may appear not to be spanned by bond prices in January is that it is not observed until February. If interest rates moves at the time of the data release (i.e. in February), this will create the impression that January CPI in the bond price model helps predict excess returns. The fact that bond prices respond at the time of the many macro economic data releases (see for instance

Faust, Rogers, Wang and Wright 2007) lends support to the latter interpretation. Whether there is relevant public information that is unspanned by bond prices but captured by macro economic data that *is available in real time* is thus arguably still an open question.

Second, the model presented here abstracts from variation in risk premia not caused by speculative dynamics. We found that conditional on the model presented here, speculative dynamics were found to be quantitatively important. But in order to quantify the relative importance of speculative dynamics and other sources of risk premia we would ideally like to have a model that also incorporates preference based or conditional heteroscedasticity based variation in risk premia. While constructing such a model is beyond the scope of the present paper, the theoretical results presented here suggest that at least in principle, the fact that speculative dynamics must be orthogonal to public information in real time should allow us to identify speculation from other sources of time varying risk premia. The relative importance of time varying risk premia and speculative dynamics could then be addressed more directly.

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## APPENDIX A. SOME USEFUL PROPERTIES OF PROJECTIONS

This Appendix reproduces some results and properties of orthogonal projections on inner-product spaces that are used in Section 3 of the main text above. Proofs and more details can be found in for instance Brockwell and Davis (2006).

**Definition 1.** (The inner-product space  $L^2$ .) The inner product space  $L^2$  is the collection  $C$  of all random variables  $X$  with finite variance

$$EX^2 < \infty \tag{A.1}$$

and with inner-product

$$\langle X, Y \rangle \equiv E(XY) : X, Y \in L^2 \tag{A.2}$$

**Definition 2.** Let  $\Omega$  be a subspace of  $L^2$ . An orthogonal projection of  $X$  on  $\Omega$ , denoted  $\mathcal{P}_\Omega X$ , is the unique element in  $L^2$  satisfying

$$\langle X - \mathcal{P}_\Omega X, \omega \rangle = 0 \tag{A.3}$$

for any  $\omega \in \Omega$ .

Orthogonal projections have the following useful properties:

- (1) The projection  $\mathcal{P}_\Omega X$  coincides with the conditional expectation  $E[X | \Omega]$  in linear models with Gaussian shocks.
- (2) Let  $\Omega'$  be a subspace of  $\Omega$  and  $\Omega'^\perp$  its orthogonal complement in  $\Omega$ . Then each  $\omega \in \Omega$  has a representation as a sum of an element in  $\Omega'$  and an element of  $\Omega'^\perp$ , i.e.

$$\omega = \mathcal{P}_{\Omega'} \omega + \mathcal{P}_{\Omega'^\perp} \omega \tag{A.4}$$

- (3)  $X \in \Omega^\perp$  if and only if  $\mathcal{P}_\Omega X = 0$ , where  $\Omega^\perp$  is the orthogonal complement to  $\Omega$ .

- (4)  $\Omega_1 \subseteq \Omega_2$  if and only if  $\mathcal{P}_{\Omega_1} X = \mathcal{P}_{\Omega_1} \mathcal{P}_{\Omega_2} X$  for all  $X \in L^2$ .

Property (1) is obviously useful as it allows us to use property (2) - (4) to analyze traders's expectations in a model with linear constraints and Gaussian shocks. Property (2) was used in the proof of Proposition 4 where we decompose bond prices into a component that is the projection of future short rates on public information and into a component that is orthogonal to public information. Property (3) is used to show that individuals can predict average expectations errors when information sets are non-nested. Property (4) is used to show both that in the absence of supply shocks the expectations hypothesis holds in our

model with respect to a public information set and that individual traders can predict other traders' prediction errors as well as excess returns when information sets are non-nested.

## APPENDIX B. SOLVING THE MODEL

Solving the model implies substituting out the higher order expectations from the bond pricing equation (2.13). We are looking for a solution of the form

$$X_t = MX_{t-1} + Ne_t \quad (\text{B.1})$$

$$\mathbf{y}_t = BX_t + \widehat{\mathbf{v}}_t \quad (\text{B.2})$$

where

$$X_t \equiv \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(1)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k})} \end{bmatrix}, \mathbf{e}_t = \begin{bmatrix} \mathbf{u}_t \\ \widehat{\mathbf{v}}_t \end{bmatrix}$$

That is, to solve the model, we need to find the matrices  $M$ ,  $N$  and  $B$  as functions of the parameters governing the short rate process, the stochastic supply shocks and the idiosyncratic noise shocks. The integer  $\bar{k}$  is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation to the limit as  $\bar{k} \rightarrow \infty$ . Here, a brief overview of the method is given, but the reader is referred to Nimark (2010) for more details on the solution method.

First, common knowledge of the model can be used to pin down the law of motion for the vector  $X_t$  containing the hierarchy of higher order expectations of  $\mathbf{x}_t$ . Rational, i.e. model consistent, expectations of  $\mathbf{x}_t$  thus implies a law of motion for average expectations  $\mathbf{x}_t^{(1)}$  which can then be treated as a new stochastic process. Knowledge that other traders are rational, means that second order expectations  $\mathbf{x}_t^{(2)}$  are determined by the average across traders of the rational expectations of the stochastic process  $\mathbf{x}_t^{(1)}$ . Third order expectations  $\mathbf{x}_t^{(3)}$  are then the average of the rational expectation of the process  $\mathbf{x}_t^{(2)}$ , and so on. Imposing this structure on all orders of expectations allows us to find the matrices  $M$  and  $N$ . Section B.2 below describes how this is implemented in practice.

Second, the method exploits that the importance of higher order expectations are decreasing with the order of expectation. This has two components:

(i) The variance of higher order expectations of the factors  $\mathbf{x}_t$  are bounded by the variance of the true process, or more generally, the variance of  $k + 1$  order expectation cannot be larger than the variance of a  $k$  order expectation

$$E \left[ \mathbf{x}_t^{(k+1)} \mathbf{x}_t'^{(k+1)} \right] \leq E \left[ \mathbf{x}_t^{(k)} \mathbf{x}_t'^{(k)} \right] \quad (\text{B.3})$$

To see why, note that by the identity

$$\mathbf{x}_t^{(k)} \equiv \mathbf{x}_t^{(k+1)} + \varepsilon_t^{(k+1)} \quad (\text{B.4})$$

and the fact that since  $\mathbf{x}_t^{(k+1)}$  is the average of an optimal estimate of  $\mathbf{x}_t^{(k)}$  the  $k = 1$  order error  $\varepsilon_t^{(k+1)}$  must be orthogonal to  $\mathbf{x}_t^{(k+1)}$  we have that

$$E \left[ \mathbf{x}_t^{(k)} \mathbf{x}_t'^{(k)} \right] = E \left[ \mathbf{x}_t^{(k+1)} \mathbf{x}_t'^{(k+1)} \right] + E \left[ \varepsilon_t^{(k+1)} \varepsilon_t'^{(k+1)} \right]. \quad (\text{B.5})$$

Since  $E \left[ \varepsilon_t^{(k+1)} \varepsilon_t'^{(k+1)} \right]$  is a covariance

$$E \left[ \varepsilon_t^{(k+1)} \varepsilon_t'^{(k+1)} \right] \geq \mathbf{0} \quad (\text{B.6})$$

and the inequality (B.3) then follows immediately. (This is an abbreviated version of a more formal proof available in Nimark (2010).)

That the variances of higher order expectations of the factors are bounded is not sufficient for an accurate finite dimensional solution. We also need (ii) that the impact of the expectations of the factors on bond yields are decreasing “fast enough” with the order of expectation. To understand why this is the case, it helps to first note that the full information rational expectation equilibrium is a special case of the solution (B.1) - (B.2) where

$$\mathbf{x}_t^{(k)} = \mathbf{x}_t^{(k+1)} \forall t, k \quad (\text{B.7})$$

That is, under full information, all orders of expectations of the factors coincide with the true factors at all times. Define the full information solution, i.e. bond yields as a function of the true state  $\mathbf{x}_t^{(0)}$  as

$$\mathbf{y}_t = \tilde{B} \mathbf{x}_t^{(0)} + \hat{\mathbf{v}}_t \quad (\text{B.8})$$

where the rows of the matrix  $\tilde{B}$  can be found by stacking the matrices  $\frac{1}{n} \sum_{s=0}^{n-1} \left[ \mathbf{1}_{1 \times 3} \quad \mathbf{0} \right] A^{s-1}$  for the appropriate choices of  $n$ . By chance, for some  $t$ , all orders of expectations can coincide also in the non-nested information case. Yields in the non-nested information model should then be the same as the yields in the full information model: If everybody believes that everybody agrees about what the current “true” state  $\mathbf{x}_t^{(0)}$  is, they must also agree about their expectations about future short rates. For this to then result in the same yields as a function of the state as the full information solution, the sub-matrices of  $B$  must sum to  $\tilde{B}$  since

$$\begin{aligned} \tilde{B} \mathbf{x}_t^{(0)} &= B X_t \\ &= \left[ B_0 \quad B_1 \quad \cdots \quad B_{\bar{k}} \right] \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(0)} \\ \vdots \\ \mathbf{x}_t^{(0)} \end{bmatrix} \end{aligned} \quad (\text{B.9})$$

$$= \sum_0^{\infty} B_j \mathbf{x}_t^{(0)} \quad (\text{B.10})$$

if  $\mathbf{x}_t^{(0)} = \mathbf{x}_t^{(k)} \forall k$  and where the  $B_j$ s are the sub matrices of  $B$  in the non-nested information solution (B.2). Since  $\tilde{B}$  is finite, the sequence  $B_0, B_1, \dots, B_k, \dots$  must be a convergent series, which implies that  $\lim_{k \rightarrow \infty} B_k = 0$ .

Summing up, we can use common knowledge of the model to derive a law of motion for the hierarchy of expectations of the factors  $\mathbf{x}_t$ . We can then find an approximate solution for bond yields of the form

$$\mathbf{y}_t = \begin{bmatrix} B_0 & B_1 & \cdots & B_{\bar{k}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(1)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k})} \end{bmatrix} + Q\mathbf{e}_t \quad (\text{B.11})$$

where the coefficients  $B_k$  that tend to zero as  $k \rightarrow \infty$ , are multiplied with a  $k$  order expectation, where the variance of the expectations are bounded from above by the variance of the true state  $\mathbf{x}_t^{(0)}$ . The accuracy of the solution for the  $\bar{k}$  chosen in the estimated model is discussed below in section B.4.

**B.1. Bond yields as a function of the state.** For a given law of motion (4.7), bond prices can be derived using the average expectation operator  $H : \mathbb{R}^{\bar{k}} \rightarrow \mathbb{R}^{\bar{k}}$  that annihilates the lowest order expectation of a hierarchy so that

$$\begin{bmatrix} \mathbf{x}_t^{(1)} \\ \mathbf{x}_t^{(2)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k}+1)} \end{bmatrix} = H \begin{bmatrix} \mathbf{x}_t^{(0)} \\ \mathbf{x}_t^{(1)} \\ \vdots \\ \mathbf{x}_t^{(\bar{k})} \end{bmatrix} \quad (\text{B.12})$$

and where  $\mathbf{x}_t^{(k)} = 0 : k > \bar{k}$ . The matrix  $H$  is then given by

$$H = \begin{bmatrix} \mathbf{0}_{3\bar{k} \times 3} & I_{\bar{k}-3} \\ \mathbf{0}_{3 \times (\bar{k}-3)} & \end{bmatrix} \quad (\text{B.13})$$

Combing the operator  $H$  that moves expectations one step up in orders of expectations and the matrix  $M$  from the law of motion (4.7) that moves expectations one step forward in time allows us to compute the higher order expectation in the bond pricing equation (2.13) as

$$\tilde{b}_t^n = - \sum_{s=0}^{n-1} \begin{bmatrix} \mathbf{1}_{1 \times 3} & \mathbf{0} \end{bmatrix} (MH)^{s-1} X_t + \tilde{v}_t^n \quad (\text{B.14})$$

where  $\tilde{v}_t^n \equiv nv_t^n$ . The yield on a bond with  $n$  periods to maturity is then given by

$$y_t^n = \frac{1}{n} \sum_{s=0}^{n-1} \begin{bmatrix} \mathbf{1}_{1 \times 3} & \mathbf{0} \end{bmatrix} (MH)^{s-1} X_t + v_t^n \quad (\text{B.15})$$

By stacking the yield formula (B.1) for appropriate maturities  $n$  gives the matrix  $B$  in (B.2).

**B.2. The law of motion of higher order expectations of the factors.** To find the law of motion for the hierarchy of expectations  $X_t$  we use the following strategy. For a given  $M, N$  and  $B$  in (B.1) - (B.2) we will derive the law of motion for trader  $j$ 's expectations of

$X_t$ , denoted  $X_{t|t}(j) \equiv E[X_t | \Omega_t(j)]$ . First, write the vector of signals  $S_t(j)$  as a function of the state, the aggregate shocks and the idiosyncratic shocks

$$S_t(j) = [ \mathbf{z}'_t(j) \quad \tilde{r}_t \quad \mathbf{y}'_t ]' \quad (\text{B.16})$$

$$= DX_t + R \begin{bmatrix} \zeta_t(j) \\ \mathbf{u}_t \\ \widehat{\mathbf{v}}_t \end{bmatrix} \quad (\text{B.17})$$

where the matrix  $D$  is given by the definitions (4.1), (4.3) and (4.12)

$$D = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{1}_{1 \times 3} & \mathbf{0} \\ & B \end{bmatrix} \quad (\text{B.18})$$

and  $R$  can be partitioned conformably to the idiosyncratic and aggregate shocks

$$R = [ R_j \quad R_A ] .$$

Agent  $j$ 's updating equation of the state  $X_{t|t}(j)$  estimate will then follow

$$X_{t|t}(j) = MX_{t|t-1}(j) + K (S_t(j) - DMX_{t|t-1}(j)) \quad (\text{B.19})$$

Rewriting the observables vector  $S_t(j)$  as a function of the lagged state and taking averages across traders using that  $\int \zeta_t(j) dj = 0$  yields

$$X_{t|t} = MX_{t|t-1} + K (DMX_{t-1} + (DN + R_A) \mathbf{e}_t + -DMX_{t|t-1}(j)) \quad (\text{B.20})$$

$$= (M - KDM) X_{t|t-1} + KDMX_{t-1} + K (DN + R_A) \mathbf{e}_t \quad (\text{B.21})$$

Appending the average updating equation to the exogenous state gives us the conjectured form of the law of motion of  $\mathbf{x}_t^{(0;\bar{k})}$

$$\begin{bmatrix} \mathbf{x}_t \\ X_{t|t} \end{bmatrix} = M \begin{bmatrix} \mathbf{x}_{t-1} \\ X_{t-1|t-1} \end{bmatrix} + N \mathbf{e}_t$$

where  $M$  and  $N$  are given by

$$M = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0} \\ \mathbf{0} & [M - KDM]_- \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [KDM]_- \end{bmatrix} \quad (\text{B.22})$$

$$N = \begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ [K(DN + R_A)]_- \end{bmatrix} \quad (\text{B.23})$$

where  $[\cdot]_-$  indicates that the a last row or column has been canceled to make a the matrix  $[\cdot]$  conformable, i.e. implementing that  $\mathbf{x}_t^{(k)} = 0 : k > \bar{k}$ . The Kalman gain  $K$  in (B.19) is given by

$$K = (PD' + NR') (DPD' + RR')^{-1} \quad (\text{B.24})$$

$$P = M (P - (PD' + NR') (DPD' + RR')^{-1} (PD' + NR'))' M' + NN' \quad (\text{B.25})$$

The model is solved by finding a fixed point that satisfies (B.2), (B.22), (B.23), (B.24) and (B.25).

**B.3. Numerical issues.** It is well-known that due to finite machine precision, the Kalman filter may be unstable in applications where some state variables (or some linear combination of state variables) are estimated very precisely compared to others, see Simon (2006). This manifests itself as a Kalman filter covariance matrix  $P$  that is numerically not symmetric and/or positive semi-definite. In our model, there are three latent factors, but a linear combination of them (i.e. the short rate  $\tilde{r}_t$ ) is observed perfectly and solving the model as above may (and for some parameterizations, do) give rise to numerical inaccuracies. To avoid this problem, we can rewrite the short rate process as

$$\tilde{r}_t = x_t^1 + x_t^2 + x_t^3 \quad (\text{B.26})$$

$$= x_t^1 + x_t^2 + \rho_3 (\tilde{r}_{t-1} - x_{t-1}^1 - x_{t-1}^2) + [c_{31} \ c_{32} \ c_3] \mathbf{u}_t \quad (\text{B.27})$$

We can still solve the model as above if we redefine the state as

$$\hat{X}_t = \begin{bmatrix} \hat{\mathbf{x}}_t^{(0)} \\ \hat{\mathbf{x}}_t^{(1)} \\ \vdots \\ \hat{\mathbf{x}}_t^{(\bar{k})} \end{bmatrix}, \quad \hat{\mathbf{x}}_t^{(0)} \equiv \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{bmatrix}$$

to get a system of similar form as in Section B.2 above

$$\hat{X}_t = \hat{M}\hat{X}_{t-1} + \bar{N}\mathbf{e}_t \quad (\text{B.28})$$

$$\mathbf{y}_t = \hat{B}\hat{X}_t + R\mathbf{e}_t + B_r\tilde{r}_t \quad (\text{B.29})$$

but where yields now also depend on the current short rate directly. We also need to adjust the measurement equation for trader  $j$  to be

$$S_t(j) = D_1\hat{X}_t + D_2\hat{X}_{t-1} + D_r\tilde{r}_{t-1} + \hat{R} \begin{bmatrix} \zeta_t(j) \\ \mathbf{u}_t \\ \hat{\mathbf{v}}_t \end{bmatrix} \quad (\text{B.30})$$

where

$$\hat{R} = [R_j \ R_A] + \begin{bmatrix} \mathbf{0} \\ [ \mathbf{0} \ c_{31} \ c_{32} \ c_3 \ \mathbf{0} ] \\ \mathbf{0} \end{bmatrix} \quad (\text{B.31})$$

The updating equation of trader  $j$ 's state estimate is now given by

$$\hat{X}_{t|t}(j) = \hat{M}\hat{X}_{t|t-1}(j) + \hat{K} \left[ S_t(j) - D_1\hat{M}\hat{X}_{t-1|t-1}(j) - D_2\hat{X}_{t-1|t-1}(j) - D_r\tilde{r}_{t-1} \right] \quad (\text{B.32})$$

Since the additional term  $D_2\hat{X}_{t-1}$  in trader  $j$ 's measurement equation is a function of the lagged state  $\hat{X}_{t-1}$ , the innovation representation is non-standard. The Kalman gain  $\hat{K}$  in

(B.32) is the steady state ( $t \rightarrow \infty$ ) Kalman gain of the modified Kalman filter

$$\widehat{K} = P_{+1} D_1' \quad (\text{B.33})$$

$$\times \left[ \left( D_1 \widehat{M} + D_2 \right) P \left( D_1 \widehat{M} + D_2 \right)' + \left( D_1 \widehat{C} + \widehat{R} \right) \left( D_1 \widehat{C} + \widehat{R} \right)' \right]^{-1}$$

$$P = P_{+1} \quad (\text{B.34})$$

$$- \widehat{K} \left[ \left( D_1 \widehat{M} + D_2 \right) P \left( D_1 \widehat{M} + D_2 \right)' + \left( D_1 \widehat{C} + \widehat{R} \right) \left( D_1 \widehat{C} + \widehat{R} \right)' \right] \widehat{K}'$$

$$P_{+1} = \widehat{M} P \widehat{M}' + \widehat{C} \widehat{C}' \quad (\text{B.35})$$

For a derivation of the modified filter, see Nimark (2009).

$\widehat{M}$  and  $\widehat{N}$  are found in the same way as in (B.22), (B.23), i.e. by taking averages of the update equation (B.32) and amending it to the “new” true factor process

$$\begin{bmatrix} x_t^1 \\ x_t^2 \end{bmatrix} = \widehat{A} \begin{bmatrix} x_{t-1}^1 \\ x_{t-1}^2 \end{bmatrix} + \widehat{C} \mathbf{u}_t \quad (\text{B.36})$$

$$= \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} x_{t-1}^1 \\ x_{t-1}^2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ c_{21} & c_2 & 0 \end{bmatrix} \mathbf{u}_t \quad (\text{B.37})$$

The resulting system is algebraically equivalent to the system in B.2, but does not suffer from numerical instability.

**B.4. The accuracy of the solution.** While it can be shown that the impact coefficients of higher order expectations on yields are a convergent sum, exactly how many orders of expectations that are needed for an accurate solution in a given application depends on the parameters of the model. In the model estimated in Section 4  $\bar{k} = 25$ . Figure 8 displays the posterior distribution of the loadings onto the different orders of expectations of the factors for the 12, 36 and 60 month yields, that is, the posterior distributions of the odd (left column) and even (right column) numbered elements of the rows of the relevant rows of  $B$  in (4.12). As we can see, the loadings approaches zero quite quickly and the loadings onto the factors for the 12 month yield are approximately zero for  $k > 12$ . For longer maturities (36 and 60 months shown), the loadings are practically zero for  $k > 15$ . Choosing  $\bar{k} = 25$  thus seems more than sufficient.

Code and data used to estimate the model and generate figures are available at the author’s web page.

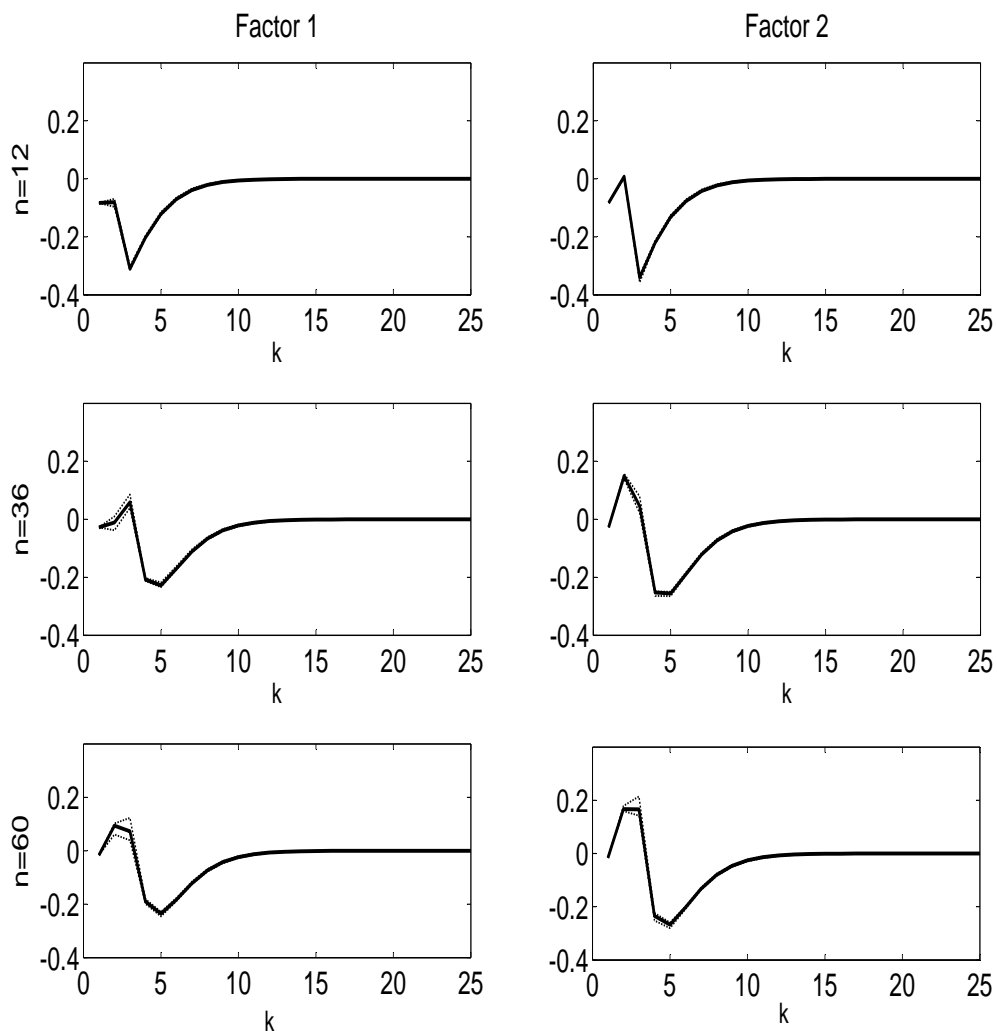


FIGURE 6. Estimated posterior (higher order) factor loadings, median (solid) and 95% probability interval (dotted). First column contains loadings onto the (higher order expectations of the) first factor  $x_t^1$ , with order of expectation  $k$  on the x-axis for yields of 1, 3 and 5 years to maturity. Second column contains the same information regarding the second factor  $x_t^1$