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Intergenerational justice when future worlds are uncertain☆

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Abstract

Let there be a positive (exogenous) probability that, at each date, the human species will disappear. We postulate an Ethical Observer (EO) who maximizes intertemporal welfare under this uncertainty, with expected-utility preferences. Various social welfare criteria entail alternative von Neumann- Morgenstern utility functions for the EO: utilitarian, Rawlsian, and an extension of the latter that corrects for the size of population. Our analysis covers, first, a cake-eating economy (without production), where the utilitarian and Rawlsian recommend the same allocation. Second, a productive economy with education and capital, where it turns out that the recommendations of the two EOs are in general different. But when the utilitarian program diverges, then we prove it is optimal for the extended Rawlsian to ignore the uncertainty concerning the possible disappearance of the human species in the future. We conclude by discussing the implications for intergenerational welfare maximization in the presence of global warming.

Keywords: Discounted utilitarianism, Rawlsian, sustainability, maximin, uncertainty, expected utility, von Neumann-Morgenstern, dynamic welfare maximization.
JEL Classification Numbers: D63, D81, O40, Q54, Q56.

1. Introduction

We study the problem of intergenerational welfare maximization when the existence of future worlds is uncertain. One of the major examples of this problem today concerns global warming, and how to structure resource use intertemporally in its presence. The theoretical issues raised by uncertainty are quite complex, and in the interest of clarity, we will study only two simple models in this article – and neither of them explicitly models the effect of production on the biosphere and global temperature. In a companion paper (Llavador, Roemer, and Silvestre, 2009), we study a more complex version of the second model of this article, which does take into account the biosphere as a renewable resource: but that paper studies only the case with no uncertainty

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concerning the existence of future generations. The conclusions of the present paper suggest some inferences for the more complex problem.

We study several (intergenerational) social welfare functions: utilitarian, Rawlsian, 'extended Rawlsian,' and 'Rawlsian with growth.' The Rawlsian function is identified with the view of *sustainability*, in a model with production.[2] Sustainability, in our parlance, means sustaining human welfare over time at the highest possible level. This is often called 'weak sustainability,' to be contrasted with 'strong sustainability', which advocates sustaining the physical stock of bio-resources – species variety, forests, and so on. (See, for instance, Neumayer, 2003, and the articles in Asheim, 2007.) In another dimension, it is to be contrasted with the discounted-utilitarian approach, which does not advocate sustaining human welfare over time, but rather the maximization of a weighted sum of generational welfare levels.

There is a literature on Rawlsian social choice in the dynamic context, beginning with Arrow (1973), Dasgupta (1974), Solow (1974) and (Phelps and Riley, 1978). As far as we know, however, there is no literature on the Rawlsian problem when the existence of future generations is uncertain.

In the next section, we introduce an Ethical Observer (EO) who has von Neumann – Morgenstern preferences over the future history of the world. These preferences can be utilitarian, Rawlsian or extended Rawlsian. We show that the EO's expected utility, evaluated at the lottery which specifies stochastically when the human species will come to an end, gives rise either to 'discounted utilitarianism' or 'discounted sustainabilitarianism,' depending on the EO's preferences. We apply these criteria to two alternative economies.

First (Section 3), we consider a 'cake-eating' model: there is a single non-produced consumption good that must be allocated over all future generations. The perhaps surprising result is that the sustainabilitarian and the utilitarian recommend exactly the same solution to the cake-eating problem (Theorem 1). Thus, these two apparently very different social preference orders do not differ in their optimal choice in this simple economy.

We introduce in Section 4 a generalization of the classical Solow economic growth model. There are two links between generations: investment, which determines the change in capital stock, and education, which determines the transmission of skill to the next generation. It is obvious that the utilitarian and sustainabilitarian cannot in general choose the same path in this model, for with some parameter values, the discounted utilitarian program diverges, while the discounted sustainabilitarian program always has a (finite) solution. Nevertheless, we show that if the discounted utilitarian program converges, and if the initial capital-labor ratio of the economy is sufficiently large, then the two programs do have the same solution (Corollary, Section 4.4). A fundamental result for this model is a Turnpike Theorem (Theorem 4), which we prove.

More important, perhaps, is the case when the discounted utilitarian program diverges – indeed, given the characterization of when this occurs (Theorem 5), this may be the empirically salient case. The remarkable result is that in this case, the solutions of the *discounted* sustainabilitarian program (in the sense of the extended Rawlsian EO) and *undiscounted* sustainabilitarian program are identical (Theorem 6). This case occurs when the economy is sufficiently productive, and the result says that great productivity renders it optimal for the sustainabilitarian EO to ignore the uncertainty concerning the possible disappearance of the human species in the future. We consider this the most important result of our analysis.

[2] Calling the intergenerational welfare function 'Rawlsian' may lead to some confusion. We mean 'maximin' applied to the society consisting of an infinite number of generations. It is well known that Rawls himself, however, did not advocate 'maximin' for the intergenerational problem.

Some readers may find ‘sustainability,’ as we model it, too stark, as it precludes the increase in the welfare of the representative generational agent over time. In Section 4.5, we introduce growth, and study optimal paths when it is specified that welfare should grow at some exogenously specified rate g over time.

As noted above, when the initial capital-labor ratio is above a certain lower bound, the discounted utilitarian and sustainabilitarian programs have the same solution. In the Appendix we compute an example showing how the optimal paths of these two programs differ when the initial capital-labor ratio is below this bound and the utilitarian program converges.

In Section 4.6, we focus upon the case when the discounted utilitarian program diverges, and we note that, if an overtaking criterion is applied to order divergent paths, then the EO would recommend almost starving the early generations. We contrast this with the discounted sustainabilitiarian, who in this case recommends equal utility for all future generations. We find the latter recommendation much more appealing.

Section 5 concludes and offers some conjectures about the generalization of our results to the problem of intertemporal distribution in presence of global warming.

2. Ethical Observers

Consider an economy that will exist for an infinite number of generations; there is one representative agent at each date. Denote the generic utility stream by $(u_1, u_2, \dots) \equiv \{u_t\}_{t=1}^{\infty}$.

Let $\mathbf{P}$ be an abstract set of feasible infinite utility streams, which may depend on a vector of initial conditions. A *social welfare function* is a real-valued function with domain $\mathbf{P}$. If the social welfare function of the planner, whom we call an Ethical Observer (EO), is $\Omega : \mathbf{P} \to \Re$, then she maximizes $\Omega(u_1, u_2, \dots)$ on $\mathbf{P}$.

For example, if the EO is *utilitarian*, then her maximization program is

Program *U*. $\max \sum_{t=1}^{\infty} u_t$ subject to $(u_1, u_2, \dots) \in \mathbf{P}$.

If the EO is a Rawlsian maximinner (i. e., sustainabilitarian), then her maximization program is

$$\max \inf\{u_1, u_2, \dots\} \text{ subject to } (u_1, u_2, \dots) \in \mathbf{P},$$

which can also be written:

Program *SUS*. $\max \Lambda$ subject to $(u_1, u_2, \dots) \in \mathbf{P}, u_t \geq \Lambda, \forall t \geq 1$.

“*SUS*” stands for *sustainability*: the economy is sustainable if it chooses a path that guarantees a certain level of human welfare forever. Note that in programs U and *SUS* there is no uncertainty concerning the existence of future generations.

We now introduce uncertainty by assuming that there is an exogenous probability $p \in (0, 1)$ that mankind will become extinct at each date, if it has not done so already.

The exogeneity of p is a simplifying assumption: in many realistic applications, such as climate change, the policies adopted may well alter the probabilities of survival of mankind. Our postulate of an exogenous p implies that the EO cannot influence the length T of human history, i. e., the size of population across time, allowing us to focus on choosing potential utility levels, while T is randomly variable but exogenous. Whether a generation exists or not is, in our model, independent of the choices of the EO, enabling us to sidestep the well-known dilemmas of population ethics (see, e. g., Parfit, 1982, 1984).

We suppose that the preferences of the EO satisfy the expected utility hypothesis. An outcome (or 'prize') is defined by a date T, interpreted as the last date before extinction, and a utility vector $(u_1, u_2, \ldots, u_T)$. Accordingly, her von Neumann-Morgenstern (vNM) utility function is defined on outcomes $(T; u_1, u_2, \ldots, u_T)$, with vNM utility values $W(T; u_1, u_2, \ldots, u_T)$. Under our assumption of exogenous probabilities, the EO's choice of a path $(u_1, u_2, \ldots) \in \mathbf{P}$ defines a lottery with expected utility

$$\begin{aligned} p\,W(1; u_1) \quad &+ \quad p\,(1-p)\,W(2; u_1, u_2) + p(1-p)^2 W(3; u_1, u_2, u_3) + \ldots \\ &= \quad p \sum_{t=1}^{\infty} (1-p)^{t-1} W(t; u_1, u_2, ., u_t). \end{aligned} \tag{1}$$

The vNM utility of a *utilitarian* EO if the world lasts T dates and she has chosen the path $(u_1, u_2, \ldots)$ is

$$W^U(T, u_1, \ldots, u_T) \equiv \sum_{t=1}^{T} u_t,$$

and the expected utility of $(u_1, u_2, \ldots)$ is

$$pu_1 + (1-p)p(u_1 + u_2) + (1-p)^2 p(u_1 + u_2 + u_3) + \ldots \tag{2}$$

By grouping the terms in (2), it becomes

$$\begin{aligned} &u_1 p(1 + (1-p) + (1-p)^2 + \ldots) + \\ &\quad u_2(1-p)p(1 + (1-p) + (1-p)^2 + \ldots) + \\ &\quad u_3(1-p)^2 p(1 + (1-p) + (1-p)^2 + \ldots) + \ldots \\ &\quad = \sum_{t=1}^{\infty} (1-p)^{t-1} u_t. \end{aligned} \tag{3}$$

This immediately justifies the view that the utilitarian Ethical Observer should be, in the presence of uncertain future worlds, a *discounted utilitarian*, with the following optimization program.

Program *DU*. $\max \sum_{t=1}^{\infty} \varphi^{t-1} u_t$ subject to $(u_1, u_2, \ldots) \in \mathbf{P}$, with $\varphi \equiv 1 - p$.

We believe this is, indeed, the most solid justification for the discounted-utilitarian ethic.[3] Note, however, that the discount factor, $\varphi \equiv 1 - p$, should be very close to one, assuming that p is very close to zero.[4] Indeed, we cannot justify, using this approach, the relatively small discount factors that are often used in intergenerational welfare economics.

[3] Many economists attempt to justify the use of a discount factor on the grounds that individuals discount the utility they will receive at a later period in their lives. This fact can only justify using such a (subjective) discount factor in the context of a model with an infinite number of generations if we view the problem as isomorphic to a problem in which there is a single, infinitely lived agent. We cannot accept the plausibility of such an isomorphism. Just because an individual may today discount his future utility does not imply that ethical observers, today, are entitled to discount the utility of future generations. This point was clearly stated by Ramsey (1928) in his pioneering work on the theory of saving, who wrote, "One point should be emphasized more particularly; we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from weakness of the imagination; we shall, however, in Section II, include such a rate of discount in some of our investigations."

[4] Indeed the Stern Review (2007) chooses $\varphi = 0.999$ per annum, which we believe is reasonable. Nordhaus (2008), on the contrary, uses the low discount factor 0.985.

On the other hand, suppose that the EO is Rawlsian (or sustainabilitarian): she wishes to maximize the minimum utility of all individuals who ever live. In this case her vNM utility function is

$$W^R(T; u_1, \ldots, u_T) = \min\{u_1, u_2, \ldots, u_T\}, \tag{4}$$

and her expected utility associated with the path $(u_1, u_2, \ldots)$ is $p \sum_{t=1}^{\infty} (1-p)^{t-1} \min\{u_1, \ldots, u_t\}$. Her optimization program is then the following one.

Program *R*. $\max p \sum_{t=1}^{\infty} (1-p)^{t-1} \min\{u_1, \ldots, u_t\}$ subject to $(u_1, u_2, \ldots) \in \mathbf{P}$.

Klaus Nehring, Andreu Mas-Colell and Geir Asheim have objected (in private communications) to (4) for the following reason. Interpreting the vNM values as *ex post* utilities, the EO will never *ex post* prefer a longer time span to a shorter one with the same utility values for the dates present in both, i. e., she will *ex post* weakly prefer the outcome $(T; \overline{u}_1, \ldots, \overline{u}_T)$ to the outcome $(T+\tau; \overline{u}_1, \ldots, \overline{u}_T, u_{T+1}, \ldots, u_{T+\tau})$, and she will actually prefer the shorter one if $u_t < \min\{\overline{u}_1, \ldots, \overline{u}_T\}$ for some $t > T$. Consider for instance the outcomes $(5; \overline{u}, \overline{u}, \overline{u}, \overline{u}, \overline{u} - \varepsilon)$ and $(4; \overline{u}, \overline{u}, \overline{u}, \overline{u})$. In the second case, humans disappear at date 5; in the first case, at date 6, and the last generation has almost the utility of the previous ones. Yet the EO under formulation (4) must *ex post* prefer the second, shorter outcome. Note that this preference violates the "mere addition" desideratum in Parfit's population ethics (Parfit, 1982).

As indicated, the difficulty is not critical under our assumption of an exogenous p, because our EO chooses, *ex ante*, lotteries with fixed probabilities, rather than outcomes. For instance, under our assumption of constant, exogenous probability, the EO would certainly choose the lottery $(\overline{u}, \overline{u}, \overline{u}, \overline{u}, \overline{u} - \varepsilon, 0, 0, \ldots)$ over the lottery $(\overline{u}, \overline{u}, \overline{u}, \overline{u}, 0, 0, 0, \ldots)$. But the problem would become serious were p endogenous. Indeed, the well-known criticisms of the maximin approach become more telling in the presence of an endogenously variable population.

Nehring's suggestion is that we modify the vNM utility function to be

$$W^N(T; u_1, \ldots, u_T) = T \min\{u_1, u_2, ., u_T\}. \tag{5}$$

Thus, in the example just given, the EO would *ex post* prefer the first outcome as long as $\varepsilon < \frac{\overline{u}}{5}$. Formulation (5) confers a powerful role to the length T of human history. But this too could be problematic were the probability of extinction endogenous and, accordingly, the EO could influence T: the resulting tradeoff between T and the sustainable utility level $\min\{u_1, u_2, ., u_T\}$ could then lead to Parfit's (1984) "repugnant conclusion."[5]

More generally, the EO may adopt a vNM utility function of the form

$$W^\beta(T; u_1, \ldots, u_T) = (1 + (T-1)\beta) \min\{u_1, u_2, ., u_T\}, \tag{6}$$

with $\beta \in [0, 1]$, which reduces to (4) when $\beta = 0$ and to (5) when $\beta = 1$. An EO with the vNM utility function of (6) will be called an *Extended Rawlsian EO*.

We study the optimization programs of the various EO's in two particular economic models: the cake-eating economy, and the education and capital economy, which yield quite different results. We will say that two programs are *equivalent* if one possesses a solution if and only if the other possesses a solution, and when both possess a solution, the solutions are the same.

[5] We are indebted to the referee for this comment.

Our main result in the cake-eating economy is the equivalence between programs *DU* and *R*: the Rawlsian (or sustainabilitarian) ethical observer and the utilitarian ethical observer make identical choices in the presence of uncertain future worlds.

In the education and capital economy, Program *DU* may diverge or converge: our main result there is that, if *DU* diverges, then, for any $\beta \in [0, 1]$, the EO's optimization problem under the vNM of (6), which, as noted, includes as special case Program *R*, is equivalent to the uncertainty-free program *SUS*: the Extended Rawlsian EO can then *ignore uncertainty*.

We conclude this section with a lemma.

Lemma 1. *If "*$(u_1^R, u_2^R, \dots)$ *solves Program R* $\Rightarrow u_t^R \geq u_{t+1}^R, \forall t \geq 1$*" and "*$(u_1^{DU}, u_2^{DU}, \dots)$ *solves Program DU* $\Rightarrow u_t^{DU} \geq u_{t+1}^{DU}, \forall t \geq 1$*," then Programs R and DU are equivalent.*

Proof. Note that $\min\{u_1, u_2, ., u_t\} = u_t$, $\forall t \geq 1$, if and only if $u_t \geq u_{t+1}, \forall t \geq 1$, in which case the objective function of Program *R* is $p \sum_{t=1}^{\infty}(1-p)^{t-1} u_t$, and Program *R* can be rewritten as

Program *CDU*. $\max p \sum_{t=1}^{\infty}(1-p)^{t-1} u_t$ subject to $u_t \geq u_{t+1}, \forall t \geq 1$ and $(u_1, u_2, \dots) \in \mathbf{P}$.

The objective function of Program *CDU* is that of Program *DU* multiplied by the positive constant p. If "$(u_1^{DU}, u_2^{DU}, \dots)$ solves Program *DU* $\Rightarrow u_t^{DU} \geq u_{t+1}^{DU}, \forall t \geq 1$," then the constraints $u_t \geq u_{t+1}$ can be added to Program *DU*, which then becomes equivalent to Program *CDU*. □

Remark. Lemma 1 cannot cover the Extended Rawlsian EO with $\beta > 0$, who has a different objective function.

3. The cake-eating economy

Postulate an economy with a single good, non-producible and initially available in the amount ω. A consumption path is written $(y_1, y_2, \dots)$, where y_t is the consumption of the agent (or generation) alive at date t. For $t = 1, 2, \dots$, the utility function of Agent t is denoted $\tilde{u} : \Re_+ \to \Re : y_t \mapsto \tilde{u}(y_t)$, and assumed to be increasing. Hence, a consumption path $(y_1, y_2, \dots)$ induces the utility path $(u_1, u_2, \dots) = (\tilde{u}(y_1), \tilde{u}(y_2), \dots)$. Taking $\omega = 1$, the set of feasible consumption paths is $\Im \equiv \{(y_1, y_2, \dots) \in \Re_+^{\infty} : \sum_{t=1}^{\infty} y_t \leq 1\}$, with the set of feasible utility paths $\mathbf{P} = \{(u_1, u_2, \dots) \in \Re^{\infty} : \exists (y_1, y_2, \dots) \in \Im$ such that $u_t = \tilde{u}(y_t), \forall t \geq 1\}$.

The *discounted utilitarian* program *DU* specializes to Program DU_1, as follows, in the cake-eating economy.

Program DU_1. $\max \sum_{t=1}^{\infty} \varphi^{t-1} \tilde{u}(y_t)$ subject to $\sum y_t \leq 1, y_t \geq 0, \quad \forall t \geq 1$.

Lemma 2. *If* $(y_1^{DU}, y_2^{DU}, \dots)$ *solves Program* DU_1*, then* $y_t^{DU} \geq y_{t+1}^{DU}, \forall t \geq 1$.

Proof. Suppose that for some $T, y_{T+1}^{DU} > y_T^{DU}$. Then switch these two terms, and the new policy strictly dominates $(y_1^{DU}, y_2^{DU}, \dots)$, because the coefficients of the objective function of DU_1 are strictly decreasing. Contradiction. □

The Rawlsian Program *R* becomes, in the cake-eating economy, Program R_1, as follows.

Program R_1.

$$\max \left\{ p\tilde{u}(y_1) + p(1-p)\min\{\tilde{u}(y_1), \tilde{u}(y_2)\} + p(1-p)^2 \min\{\tilde{u}(y_1), \tilde{u}(y_2), \tilde{u}(y_3)\} + \dots \right\}$$
$$\text{subject to } \sum y_t \leq 1, \; y_t \geq 0, \; \forall t \geq 1.$$

Lemma 3. *If $(y_1^R, y_2^R, \dots)$ solves Program R_1, then $y_t^R \geq y_{t+1}^R, \forall t \geq 1$.*

Proof. Appendix. □

Theorem 1. *Programs DU_1 and R_1 are equivalent, and $y_t \geq y_{t+1}, \forall t \geq 1$, at any solution.*

Proof. Immediate from Lemmas 1-3. □

Theorem 1, perhaps surprisingly, tells us that the Rawlsian EO behaves just like a discounted utilitarian –and uses the same discount factor.

We now analyze the (common) solutions to programs DU_1 and R_1.

Theorem 2. *Let $\tilde{u}$ be concave, differentiable on $\mathfrak{R}_{++}$ and increasing, and suppose that $\lim_{y \to 0} \tilde{u}'(y) = \infty$ (i. e., $\tilde{u}$ satisfies an "Inada condition"). If $(y_1^{DU}, y_2^{DU}, \dots)$ solves Program DU_1, then $y_t^{DU} > 0$ for all t.*

Proof. Appendix. □

Example 1 in the Appendix provides a utility function $\tilde{u} : \mathfrak{R}_{++}^2 \to \mathfrak{R}$ (concave, increasing, differentiable) for which programs DU_1 and R_1 do *not* possess a solution.

The next theorem studies the case when the derivative of $\tilde{u}$ at zero is finite.

Theorem 3. *Let $\tilde{u}$ be strictly concave, increasing, and differentiable on $\mathfrak{R}_+$, with $\tilde{u}'(0) = \gamma < \infty$. Then Program R_1 possesses a unique solution $(y_1^R, y_2^R, \dots)$, and there is a date T such that $y_t^R = 0$ for all $t \geq T$.*

Proof. Appendix. □

We may thus summarize as follows, for functions $\tilde{u}$ which are strictly concave, increasing, and differentiable except perhaps at zero:

1. When programs DU_1 or R_1 have a solution, then the solution is unique and identical: the Rawlsian EO and the discounted utilitarian EO make exactly the same recommendation (Theorems 1-3).
2. If $\tilde{u}'(0) < \infty$, then a solution to programs DU_1 and R_1 does exist. Furthermore, there is a T such that the optimal policy awards zero resource to all dates $t \geq T$: both the Rawlsian EO and the discounted utilitarian EO prescribe zero consumption for all sufficiently distant generations (Theorems 1-3).
3. If $\lim_{y \to 0} \tilde{u}'(y) = \infty$, and if there is a solution to programs DU_1 and R_1, then the solution implies $y_t > 0$ for all t: both the Rawlsian EO and the discounted utilitarian EO prescribe positive consumption for all generations (Theorems 1-2).
4. There are functions $\tilde{u} : \mathfrak{R}_{++}^2 \to \mathfrak{R}$ with $\lim_{y \to 0} \tilde{u}(y) = -\infty$ for which programs DU_1 or R_1 have no solution. But if $\tilde{u}'(y_t)$ does not approach infinity too fast as y_t approaches zero, then a solution exists (see Example 1 and its discussion in the Appendix).

4. An economy with education and capital

4.1. The model

At date t, the available amount of labor, measured in skill units and denoted x_t, is partitioned into three parts: leisure (x_t^l), labor used in the production of commodities (x_t^c) and labor used to educate the next generation (x_t^e). Utility depends on consumption (c_t) and leisure, and is given by the function u: when no confusion is likely, we will denote $u_t = u\left(c_t, x_t^l\right)$. Physical capital ($s_t^k$) and labor produce output according to the production function $f(s_t^k, x_t^c)$: output is partitioned into consumption and investment (i_t). The initial endowment is the pair of stocks $(x_0^e, s_0^k) \in \Re_{++}^2$. Given the initial endowment, a path for the economic variables is feasible if it satisfies the following inequalities for $\forall t \geq 1$:

$$
\begin{aligned}
&(1-\delta)s_{t-1}^k + i_t \geq s_t^k \text{ (law of motion of capital), where } \delta \in (0,1) \text{ is the depreciation rate,}\\
&f(s_t^k, x_t^c) \geq c_t + i_t \text{ (technology for the production of output),}\\
&\xi\, x_{t-1}^e \geq x_t^e + x_t^c + x_t^l \text{ (education technology).}
\end{aligned}
$$

The last inequality models the technology of education: the quantity of skilled labor at the next date t is simply a multiple ξ of the efficiency units of labor devoted to teaching at date $t-1$. Thus ξ , which will turn out to be a key parameter, is the rate at which skilled labor can reproduce itself intergenerationally, or, in another locution, the student-teacher ratio.

The problem is non-traditional in one way: utility depends not upon raw leisure but upon educated leisure. Thus, we assume that a person's leisure activities are more fulfilling, if she is more highly educated. One might challenge this as an elitist view, but we insist upon it, as we believe that education opens up for the individual increasing opportunities for the use of leisure. We may think of education as permitting the diversification of the leisure resource, which increases its usefulness. In the words of Martin Wolf (2007):

> The ends people desire are, instead, what makes the means they employ valuable. Ends should always come above the means people use. The question in education is whether it, too, can be an end in itself and not merely a means to some other end – a better job, a more attractive mate or even, that holiest of contemporary grails, a more productive economy.
>
> The answer has to be yes. The search for understanding is as much a defining characteristic of humanity as is the search for beauty. It is, indeed, far more of a defining characteristic than the search for food or for a mate. Anybody who denies its intrinsic value also denies what makes us most fully human.

On the role of education in production, we are reminded of the recent work of Goldin and Katz (2008), who argue that the main reason for the excellent performance of the American economy in the twentieth century was universal education. Similar points have been made with respect to South Korea and Japan. Of course, the Goldin-Katz claim is somewhat different from ours –theirs is based on the growth of consumption, while ours is based on the centrality of the educational technology for growth of welfare.

We impose the following assumption. The Cobb-Douglas hypotheses could be dispensed with in some of the results, but we adopt them for convenience and to shorten some of the arguments.

Assumption A**.**

(*a*) Cobb-Douglas Utility Function: $u(c, x^l) = c^\alpha (x^l)^{1-\alpha}, \alpha \in (0, 1)$;

(*b*) Cobb-Douglas Production Function: $f(s^k, x^c) = (s^k)^\theta (x^c)^{1-\theta}, \theta \in (0, 1)$;

(*c*) $\xi > 1$.

The sustainability program SUS specializes to Program $SUS_2[x_0^e, s_0^k]$, as follows, in the education and capital economy.

Program $\boldsymbol{SUS}_2[x_0^e, s_0^k]$**.**

$$
\begin{array}{lll}
\multicolumn{3}{l}{\max \Lambda \text{ subject to}} \\
(v_t) & u(c_t, x_t^l) \geq \Lambda, & t \geq 1, \\
(a_t) & (1-\delta)s_{t-1}^k + i_t \geq s_t^k, & t \geq 1, \\
(b_t) & f(s_t^k, x_t^c) \geq c_t + i_t, & t \geq 1, \\
(d_t) & \xi\, x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, & t \geq 1.
\end{array}
$$

We have written the dual variables in parentheses for future use.

We state a turnpike theorem for the SUS_2 program.

Theorem 4 (Turnpike Theorem)**.**

A. There is a ray $\Gamma \in \Re_+^2$ *such that, if* $(x_0^e, s_0^k) \in \Gamma$*, then the solution path of Program* $SUS_2[x_0^e, s_0^k]$ *is stationary.*

B. If $(x_0^e, s_0^k) \notin \Gamma$*, then along the solution path the sequence* $((x_1^e, s_1^k), (x_2^e, s_2^k), \ldots)$ *converges to a point in* Γ*.*

C. Along the solution path, all constraints hold with equality (in particular, utility is constant over t).

D. The solution to $SUS_2[x_0^e, s_0^k]$ *is unique.*

Proof. Appendix. □

Figure 1 illustrates the Turnpike Theorem. The solution path determined by initial conditions off ray Γ has constant utility, and it has the property that, along this path, the sequence converges to a point in Γ.

4.2. Discounted utilitarianism: the convergence condition $\varphi < 1/\xi$

The discounted utilitarian program DU of Section 2 specializes to program $DU_2[\varphi, x_0^e, s_0^k]$, as follows, for the education and capital economy.

Program $\boldsymbol{DU}_2[\varphi, x_0^e, s_0^k]$**.**

$$
\begin{gathered}
\max \sum_{t=1}^{\infty} \varphi^{t-1} u\left(c_t, x_t^l\right) \text{ subject to} \\
(1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1, \\
f\left(s_t^k, x_t^c\right) \geq c_t + i_t, \quad t \geq 1, \\
\xi\, x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \quad t \geq 1.
\end{gathered}
$$

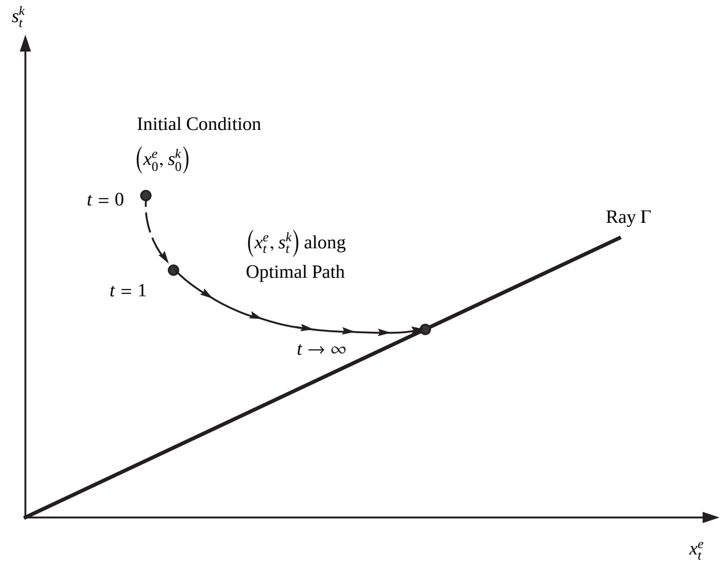

Figure 1: Convergence to ray Γ

Note that whether or not $DU_2[\varphi, x_0^e, s_0^k]$ converges depends only on φ and the initial 'capital-labor ratio' $\sigma = s_0^k / x_0^e$, by the homogeneity of the program. (The set of feasible paths is a convex cone.) We are interested in understanding the set $\left\{(\varphi, \sigma) \mid DU_2[\varphi, x_0^e, \sigma x_0^e] \text{ converges}\right\}$.

Theorem 5.

A. If $\varphi\xi > 1$, then Program $DU_2[\varphi, x_0^e, \sigma x_0^e]$ diverges for all $\left(x_0^e, \sigma x_0^e\right) \in \Re^2_{++}$.

B. If $\varphi\xi < 1$, then Program $DU_2[\varphi, x_0^e, \sigma x_0^e]$ converges for all $(x_0^e, \sigma x_0^e) \in \Re^2_{++}$.

Proof. Appendix. □

Theorem 5 is important for our theory, and perhaps surprising, for it says that the 'power' of the economy, in the sense of its capacity to cause the DU_2 program to diverge, depends only on the efficiency of the educational technology, namely, the coefficient ξ. In particular, we need no special assumptions on the technology f other than the standard ones in Assumption A.

The proof of Theorem 5 is not particularly transparent, and so we provide here a more intuitive argument. Let $x_0^e = 1$. Suppose we can find positive numbers (σ, c, i, x^c, x^l) such that the following equations hold for some given positive g:

$$(g + \delta)\sigma = i, \tag{7}$$

$$f(\sigma, x^c) = c + i, \tag{8}$$

$$\xi = 1 + g + x^c + x^l. \tag{9}$$

Then, from an initial endowment of $(x_0^e, s_0^k) = (1, \sigma)$, we can produce a balanced growth path in which all variables grow by a rate g at each period. Just notice that the investment defined by (7) will make $s_1^k = (1 + g)\sigma$, that equation (9) says that $x_1^e = (1 + g)x_0^e$, and that the solution

(c, i, x^c, x^l) will grow at rate g from date one onwards, invoking the fact that all three equations are homogeneous of degree one in the five variables. Now, in order to solve these equations, it is obviously necessary that $1+g < \xi$, for otherwise (9) would have no positive solution for (x^c, x^l). The interesting fact is that the converse is true as well: as long as $1+g < \xi$, we can produce the required solution, which would support a balanced growth path at growth rate g beginning at a capital-labor ratio σ. To see this, eliminate i using (7) (which will surely be positive for any positive σ); then we must find (σ, c, x^c, x^l) positive such that:

$$\begin{aligned} f(\sigma, x^c) &= c + (g+\delta)\sigma, \\ \xi - (1+g) &= x^c + x^l, \end{aligned}$$

which is equivalent to finding (σ, x^c) such that

$$\begin{gathered} f(\sigma, x^c) > (g+\delta)\sigma, \\ 0 < x^c < \xi - (1+g). \end{gathered}$$

But this can be accomplished if and only if there exists $\sigma > 0$ such that

$$f(\sigma, \xi - (1+g)) > (g+\delta)\sigma,$$

or, invoking the fact that f is one-homogeneous, if and only if:

$$f\left(1, \frac{\xi - (1+g)}{\sigma}\right) > g + \delta.$$

But since f increases without bound as we increase its labor argument, we can surely find σ sufficiently small that this is true. Let the value of such an admissible σ be denoted $\hat{\sigma}$.

Now beginning with an arbitrary positive endowment vector (x_0^e, s_0^k), we can reach the capital-labor ratio $\hat{\sigma}$ in a finite number of steps; from there we take off at any desired growth rate $g < \xi - 1$. Since utility is also homogenous of degree one in (c, x^l), it grows at that rate too. So the growth *factor* of utility is $(1+g) < \xi$. It is now clear that Program DU_2 diverges if and only if $\varphi\xi > 1$.

The reason the above argument is only an intuition for, rather than a proof of, Theorem 5, is that a proof cannot limit itself to studying only balanced growth paths.

We remind the reader that Theorem 5 depends, as well, on our assumption that the leisure argument of the utility function is measured in quality units, one that we strongly defend, although it may be somewhat controversial.

4.3. The divergence of discounted utilitarianism and the sustainability of the extended Rawlsian path

Consider the Extended Rawlsian EO, i. e., with vNM utility function given by (6) above, with $\beta \in [0, 1]$. Her optimization program for the education and capital economy can be written as follows.

Program $R^\beta[\varphi, x_0^e, s_0^k]$.

$$\max\left\{u_1 + \varphi(1+\beta)\min\{u_1, u_2\} + \varphi^2(1+2\beta)\min\{u_1, u_2, u_3\} + \dots\right\}$$

subject to $u_t \equiv u\left(c_t, x_t^l\right)$ and

$$\begin{aligned} &(1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1, \\ &f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1, \\ &\xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \quad t \geq 1. \end{aligned}$$

Lemma 4. *For any path* $(u_1, u_2, \dots) \in \Re_+^\infty$*, the sum* $\sum_{t=1}^{\infty} \varphi^{t-1} (1 + (t-1)\beta) \min\{u_1, \dots, u_t\}$ *converges.*

Proof. Appendix. □

Lemma 5. *If* $(u_1, u_2, \dots)$ *solves Program* $R^\beta[\varphi, s_0^k, x_0^e]$*, then* $u_t \geq u_{t+1}$ *for all t.*

Proof. Suppose to the contrary that $u_2 > u_1$. Then it follows that $u_1 = \min\{u_1, u_2\}$. We can distribute back a small amount of resources from date 2 to date 1: reduce by a small amount ε the value of x_1^e, increase x_1^l by ε, and decrease x_2^l by $\xi\varepsilon$, making the date 2 agent take the reduction of his skilled labor supply entirely in a reduction of leisure. This will increase the values of u_1 and $(1+\beta)\min\{u_1, u_2\}$ and will leave all other numbers $(1 + (t-1)\beta)\min\{u_1, \dots, u_t\}$ unchanged or possibly greater. Hence, since the objective was finite by Lemma 4, it is now increased, a contradiction. The general claim follows from an induction argument. □

We now state our main theorem:

Theorem 6. *Let* $(x_0^e, s_0^k) \in \Gamma$*. If* $\varphi\xi \geq 1$*, then for* $\beta \in [0, 1]$*, any solution to Program* $R^\beta[\varphi, x_0^e, s_0^k]$ *is the solution to Program* $SUS_2[x_0^e, s_0^k]$*. Since the solution to* $SUS_2[x_0^e, s_0^k]$ *is unique, so is the solution to* $R^\beta[\varphi, x_0^e, s_0^k]$*.*

Proof. Appendix. □

Combined with Theorem 5, we have that, if Program DU_2 diverges, then the Extended Rawlsian EO can *ignore uncertainty* in choosing the optimal path (at least in the case when the initial endowment vector lies on the ray Γ). We conjecture that Theorem 6 is true even if the initial endowment is not on the ray Γ.

4.4. The case where discounted utilitarianism converges

This section focuses on the case $\beta = 0$, for which Program R^β is just the application of the Rawlsian Program R of Section 2 above to the education and capital economy: let us refer to it as Program $R_2[\varphi, x_0^e, s_0^k]$, or simply Program R_2. We expect that, if $\varphi\xi < 1$, then the solution to Program R_2 will not be the solution to Program SUS_2, which is to say that the inequalities $u_t \geq u_{t+1}$ of Lemma 5 will not all be satisfied with equality. Thus, the solution to the Rawlsian EO's problem under uncertainty R_2 may involve decreasing utilities over time. Indeed this is true for φ sufficiently close to zero, as the following simple result shows.

Theorem 7. *Given* (x_0^e, s_0^k)*, there is a number* $\bar{\varphi} > 0$ *such that, if* $\varphi < \bar{\varphi}$*, then the solution to* $R_2[\varphi, x_0^e, s_0^k]$ *entails* $u_1 > u_2$ *on the solution path.*

Proof. Appendix. □

Moreover, a consequence of Theorem 8 below is that, under our Cobb-Douglas assumptions, for any $\varphi < 1/\xi$, if the capital-labor ratio s_0^k/x_0^e is sufficiently high, then utilities are strictly monotone decreasing on the optimal path.

We now ask: If the $DU_2[\varphi, x_0^e, s_0^k]$ program converges, is its solution the same as the solution to $R_2[\varphi, x_0^e, s_0^k]$? By lemmas 1 and 5, this will be the case if, at the solution to $DU_2[\varphi, x_0^e, s_0^k]$, utilities are weakly decreasing with time.

For an initial condition (s_0^k, x_0^e), define the 'capital-labor ratio' $\sigma_0 = s_0^k/x_0^e$. Recall that $u(c,x) = c^\alpha x^{1-\alpha}$, and $f(s,x) = s^\theta x^{1-\theta}$. Define the following variables:

$$E = (\varphi\xi)^{1/(\alpha\theta)}(1-\delta),$$
$$\tilde{x}_1^c = \frac{(\xi - E)\alpha(1-\theta)}{1-\alpha\theta}, \quad \tilde{x}_1^l = \frac{(\xi - E)(1-\alpha)}{1-\alpha\theta}, \quad \tilde{x}_t^e = E^t,$$
$$\tilde{c}_1 = \sigma_0^\theta(\tilde{x}_1^c)^{1-\theta}(1-\delta)^\theta,$$
$$\tilde{s}_t^k = (1-\delta)^t s_0^k, \quad \tilde{x}_t^c = \tilde{x}_1^c E^{t-1}, \quad \tilde{x}_t^l = \tilde{x}_1^l E^{t-1}, \quad \tilde{c}_t = \tilde{c}_1((1-\delta)^\theta E^{1-\theta})^{t-1}.$$

Theorem 8. *Suppose that $\varphi\xi < 1$, and that $s_0^k/x_0^e = \sigma_0 \geq \sigma^*$ where σ^* is the root of the equation*

$$1 - \theta\left(\frac{\tilde{x}_1^c}{(1-\delta)\sigma}\right)^{1-\theta} = \frac{\tilde{c}_1}{\tilde{x}_1^l}(\varphi\xi)^{1/(\alpha\theta)}\frac{\theta(1-\alpha)}{\sigma(1-\alpha\theta)}.$$

Then the solution to $DU_2[\varphi, x_0^e, s_0^k]$ is given by the geometric sequence: $s_t^k = \tilde{s}_t^k x_0^e$, $x_t^e = \tilde{x}_t^e x_0^e$, $x_t^l = \tilde{x}_t^l x_0^e$, $x_t^c = \tilde{x}_t^c x_0^e$, $i_t = 0$ for all $t \geq 1$.

Proof. Appendix. □

Corollary. *If $\varphi\xi < 1$ and $\sigma_0 \geq \sigma^*$, then Programs $DU_2[\varphi, x_0^e, s_0^k]$ and $R_2[\varphi, x_0^e, s_0^k]$ are equivalent.*

Proof. Along the solution to Program DU_2, we have that

$$\tilde{u}_t = \tilde{u}_1\left(\left((1-\delta)^\theta E^{1-\theta}\right)^\alpha E^{1-\alpha}\right)^{t-1},$$

where $\tilde{u}_1 = \tilde{c}_1^\alpha(\tilde{x}_1^l)^{1-\alpha}$; thus utilities are strictly decreasing with time because $E < 1$. The result then follows from lemmas 1 and 5. □

What happens when $\sigma_0 < \sigma^*$? The solution to DU_2 will not be the well-behaved solution of geometric decay of Theorem 8. Will, nevertheless, utilities still be monotone decreasing on the optimal path? Perhaps, surprisingly, the answer is in general negative. Example 2 in the Appendix has the property that, along the solution path to Program DU_2, $u_2 > u_1$, whereas the utilities from date 2 onwards decay geometrically as in Theorem 8.

How do the solutions to DU_2 and R_2 compare when they are different and DU_2 converges? To see this, we calculate the solution to R_2 for Example 2 in the Appendix. There, the Rawlsian EO gives higher utility to the first generation than the utilitarian EO, but the reverse is true for all dates after that. In fact, the ratio of utilities for the two programs is constant for dates 2 and later at 1.015, with the larger utility associated with DU_2: this is perhaps a surprise.

This concludes our discussion of the relationship between the DU_2 and R_2 programs in the case where DU_2 converges. Unlike the cake-eating problem, the solutions to these two programs are not always identical –although they are identical when the initial capital-labor ratio is sufficiently large.

Based on Example 2 in the Appendix, we may conjecture what the general solution to $DU_2[\varphi, x_0^e, s_0^k]$ looks like in the convergent case. There will be a sequence of numbers $\tilde{\sigma} > \sigma^* > \sigma_1 > \sigma_2 > \cdots > 0$, where σ^* is given in Theorem 8, where, if $\sigma_T > \sigma_0 > \sigma_{T+1}$, the first T dates will have $i_t > 0$, and at date $T+1$, the capital-labor ratio will be $\tilde{\sigma}$, at which point the geometric-decay solution of Theorem 8 takes over. The same pattern should be true in the solution to Program $R_2[\varphi, x_0^e, s_0^k]$, except that utility will be equal for all the dates when investment is positive.

4.5. *Growth*

Some may find sustainability, in the sense of program *SUS*, to be too stark, as it leads to a constant level of human welfare until the disappearance of the species. If, however, we treat resources, such as the biosphere, as of limited capacity, then sustainability may be the best we can hope for. Nevertheless, we now introduce a program which permits the growth of welfare.

Program g-***SUS***$[x_0^e, s_0^k]$**.**

$$\begin{array}{lll} & \max \Lambda \text{ subject to} & \\ [(r_t)] & u(c_t, x_t^l) \geq (1+g)^{t-1}\Lambda, & t \geq 1, \\ [(a_t)] & f(s_t^k, x_t^c) \geq c_t + i_t, & t \geq 1, \\ [(b_t)] & (1-\delta)s_{t-1}^k + i_t \geq s_t^k, & t \geq 1, \\ [(d_t)] & \xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, & t \geq 1. \end{array}$$

Program g-*SUS* maximizes date-1 welfare subject to assuring that welfare grows at rate g forever. Obviously, Program g-*SUS* becomes SUS_2 when $g = 0$.

What is the largest g for which Program g-*SUS* possesses a solution? We give a partial answer with the next theorem.

Definition. A *balanced growth path at rate* g is a path satisfying the (a_t), (b_t) and (d_t) constraints of Program g-*SUS*$[x_0^e, s_0^k]$ such that:

$$\begin{aligned} s_t^k &= (1+g)s_{t-1}^k \text{ and } x_t^e = (1+g)x_{t-1}^e, \text{ for } t \geq 1, \\ z_t &= (1+g)z_{t-1} \text{ for all other variables } z \in \left\{x^c, x^l, i, c\right\}, \text{ for } t \geq 2. \end{aligned}$$

Theorem 9. *Suppose that* $0 \leq g < \xi - 1$ *and* $x_0^e = 1$*. Then there exists a value* s_0^k *such that the solution to Program* g-*SUS*$[x_0^e, s_0^k]$ *is a balanced growth path at rate* g*. Conversely, if* $g \geq \xi - 1$*, then there exists no such path for any value of* s_0^k*.*[6]

Proof. Appendix. □

We expect that a turnpike theorem holds for the g-*SUS* model as well, and so, if and only if $0 \leq g < \xi - 1$, and given any value of s_0^k, Program g-*SUS* will possess a solution at which all constraints bind, which converges to a balanced growth path at rate g.

4.6. *Social choice when* $DU_2[\varphi, x_0^e, s_0^k]$ *diverges*

According to Theorem 5, $DU_2[\varphi, x_0^e, s_0^k]$ diverges when $\varphi\xi > 1$. The usual way of choosing among paths in the case of divergence is to use a version of the overtaking criterion: the latest proposal that we have seen along these lines is that of Basu and Mitra (2007). The utility path $(\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots)$ is *at least as good as the utility path* $(\bar{u}_1, \bar{u}_2, \ldots)$ *according to the overtaking criterion* if there exists a T such that $\sum_{t=1}^{T-1} \varphi^{t-1}\bar{\bar{u}}_t \geq \sum_1^{T-1} \varphi^{t-1}\bar{u}_t$ and $t \geq T \Rightarrow \bar{\bar{u}}_t \geq \bar{u}_t$. This defines a pre-order (i. e., an incomplete order) on feasible paths when a program diverges.

The proof of Theorem 9 showed that balanced growth paths exist for the education and capital economy as long as $g < \xi - 1$. The condition for a divergence of such a path in Program DU_2 is $\varphi(1+g) \geq 1$. This condition surely holds when g is close to $\xi - 1$ because $\varphi(1+(\xi-1)) = \varphi\xi > 1$.

[6] We are not interested in the problem with negative g.

Let $(\bar{u}_1, \bar{u}_2, \dots)$ and $(\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots)$ be two feasible balanced-growth paths for a given initial endowment (x_0^e, s_0^k) which grow at rates g_1 and g_2, respectively, where $g_2 > g_1$. It is easy to see that $(\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots)$ is better than the utility path $(\bar{u}_1, \bar{u}_2, \dots)$ according to the overtaking criterion. But it is also the case that utility will be smaller for the early date(s) on the preferred path. (To grow forever faster requires making early sacrifices.) This is interesting, because discounted utilitarianism is usually associated with implying that the later generations sustain low utility. *This, however, is only the case when the program converges*. Indeed, as the proof of Theorem 9 shows, as the growth rate g approaches its unattainable supremum $(\xi - 1)$ (and these high-growth-rate paths are the most desirable paths according to the overtaking criterion), the utility of the first generation approaches zero. We do not take this as a criticism of overtaking: rather, it is a criticism of discounted utilitarianism.

In contrast, as Theorem 6 showed, if $\varphi\xi \geq 1$, then the solution to Program $R_2[\varphi, x_0^e, s_0^k]$ entails constant utility for all generations, at the highest possible level at which such a level can be sustained. We find this distinctly superior, from the ethical viewpoint, to the recommendation of the discounted utilitarian.

Finally, we note that the case of divergence may be the salient one. By definition, $\xi = x_t / x_{t-1}^e = x_t / (\tau^e x_{t-1})$, where τ^e is the fraction of the labor force of generation $t-1$ that is devoted to teaching. As a rough approximation, assume that population growth is zero and that skill growth is zero; then $x_t = x_{t-1}$ and so, if $\tau^e \approx 0.05$, we have $\xi \approx 20$. Since we have suggested, following Stern (2007), that $\varphi = 0.999$ is appropriate, we have that $\varphi\xi$ is substantially larger than one.

5. Conclusion

In the cake-eating problem, we showed that two Ethical Observers, facing uncertain possible future worlds, who have utilitarian and Rawlsian von Neumann Morgenstern preferences over risk, respectively, would recommend the same allocation of the exhaustible resource over future generations. At first blush, it seems surprising that these two Observers, with apparently very different preferences, would agree on the recommended path. The best analogy we can think of is with the solution to the problem with *no uncertainty* concerning the existence of future generations, and a *finite* horizon. The utilitarian and the Rawlsian will recommend the same allocation of the exhaustible resource in this case –namely, split it equally among all generations. This solution is unique only if u is strictly concave –if u is linear, then the utilitarian is indifferent among all possible distributions of the resource.

We then introduced a generalization of the classical growth model, which includes an education sector. Moreover, we postulated that welfare depends on consumption and educated leisure. Now, the program of the utilitarian Ethical Observer, in the presence of uncertainty, does not always converge, while the program of the sustainabilitarian (i. e., Rawlsian) does. We characterized when the former program converges (Theorem 5), and we showed that when it does *not* converge, the (extended) sustainabilitarian proposes the same path as she would if there were no uncertainty (Theorem 6). We believe this is an important result, as parameter values in the real world are likely to be such that the discounted utilitarian program does not converge (see Section 4.6). Moreover, we argued that if this is the case, then the most desirable paths according to the discounted-utilitarian objective would leave the early generations with very low utility. (This conclusion is very different from the recommendation of discounted utilitarianism in the convergent case.) In contrast, when the discounted utilitarian program diverges, as we said, the sustainabilitarian recommends equal welfare for all generations.

Finally, we showed that when the discounted utilitarian program converges, it is not generally the case that the two Ethical Observers will recommend the same paths, although they do if the capital-labor ratio of the initial endowment vector is sufficiently large (Theorem 8 and its Corollary).

In our companion paper Llavador et al. (2009), we study a model which is a ramification of the model of Section 4 of the present paper, one which articulates the issue of global warming. In that model, production of the consumption-investment good affects negatively the quality of the biosphere (carbon emissions increase global temperature), and the quality of the biosphere enters into the utility of individuals. As well as a production and education sector, that model also contains an R&D sector, where research produces knowledge that both improves the technology of commodity production, and enters directly into the utility of people. (Knowledge and biospheric quality are global public goods.) We know that with appropriate parameter values, the discounted utilitarian program of the more ramified model diverges; we do not know whether analogues of the theorems presented here continue to hold. Naturally, we would be interested in eventually extending the present analysis to that model: we propose to think of the central results of the model of Section 4 as conjectures concerning the global-warming model. In particular, if the discounted-utilitarian objective function diverges on the set of paths defined for the global-warming model, then we conjecture that the sustainabilitarian can ignore the kind of uncertainty studied in the present paper (Theorem 6). However, we must say that there is another kind of uncertainty, not discussed here, which is more the focus of current discussions of global warming: the uncertainty about the relationship between atmospheric carbon and global temperature (biospheric quality). That kind of uncertainty involves quite different considerations from those studied here.

Appendix A. Proofs and Examples

Proof of Lemma 3

We claim that for every T, $y_T^R = \min\{y_1^R, y_2^R, \ldots, y_T^R\}$. For suppose this were not the case, for some T. Then let $\varepsilon = y_T^R - \min\{y_1^R, y_2^R, \ldots, y_{T-1}^R\}$. By hypothesis, $\varepsilon > 0$. Define the path $(\bar{y}_1, \bar{y}_2, \ldots)$ as follows:

$$\begin{aligned} \bar{y}_T &= y_T^R - \frac{\varepsilon}{2}, \\ \bar{y}_t &= y_t^R + \frac{\varepsilon}{2(T-1)}, \text{ for } 1 \le t \le T-1, \\ \bar{y}_t &= y_t^R, \text{ for } t > T. \end{aligned}$$

Obviously $(\bar{y}_1, \bar{y}_2, \ldots)$ is feasible for Program R_1. In the move from $(y_1^R, y_2^R, \ldots)$ to $(\bar{y}_1, \bar{y}_2, \ldots)$, the first T terms in the objective function of Program R_1 all (strictly) increase. Furthermore, all terms greater than the Tth term either increase or stay the same. Notice that $\bar{y}_T$ remains at least $\frac{\varepsilon}{2}$ greater than the minimum of $\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_t\}$ for all $t > T$, since that minimum is bounded above by $\min\{\bar{y}_1, \ldots, \bar{y}_{T-1}\}$. So $\tilde{u}(\bar{y}_T)$ is never the minimum in any of the terms of the objective with $t > T$. Consequently, the objective function of Program R_1 (obviously bounded) attains a higher value at $(\bar{y}_1, \bar{y}_2, \ldots)$ than at $(y_1^R, y_2^R, \ldots)$, a contradiction. □

Proof of Theorem 2

Step 0. Since $\tilde{u}(0)$ is finite, w.l.o.g., we take $\tilde{u}(0) = 0$.

Step 1. Let $(y_1^{DU}, y_2^{DU}, \dots)$ solve Program DU_1. Suppose there is a T such that $y_t^{DU} = 0$. Then T must be greater than one. For if $y_1^{DU} = 0$, simply define a new path $(\bar{y}_1, \bar{y}_2, \dots)$ by $\bar{y}_t = y_{t+1}^{DU}$ for all $t = 1, 2, \dots$ This path increases the value of the objective function in DU_1, an impossibility. Therefore $T > 1$.

Step 2. Now let T be the smallest date for which $y_t^{DU} = 0$. Then it must be the case that for any sufficiently small $\varepsilon > 0$, we have $\tilde{u}(y_{T-1}^{DU} - \varepsilon) + \varphi\tilde{u}(\varepsilon) \le \tilde{u}(y_{T-1}^{DU})$, for otherwise, a transfer of ε from date $T - 1$ to date T would increase the value of the objective function in Program DU_1. But this inequality can be written $\varphi\tilde{u}(\varepsilon) \le \tilde{u}(y_{T-1}^{DU}) - \tilde{u}(y_{T-1}^{DU} - \varepsilon)$. Dividing both sides by ε and letting ε approach zero, this implies that $\varphi \lim \frac{\tilde{u}(\varepsilon)}{\varepsilon} \le \tilde{u}'(y_{T-1}^{DU})$. But $\lim_{\varepsilon \to 0} \frac{\tilde{u}(\varepsilon)}{\varepsilon} = \infty$, which gives the desired contradiction. □

Example 1

This is an example of a function $\tilde{u}$ for which Program DU_1 has no solution. Consider the function $\tilde{u} : \Re_{++} \to \Re : \tilde{u}(y) = \int e^{1/y} dy - y$. We have $\tilde{u}'(y) = e^{1/y} - 1, \quad \tilde{u}''(y) = -\frac{e^{1/y}}{y^2}$. Thus, $\tilde{u}$ is an increasing, concave function on the positive real line, and the Inada condition holds. The function $\tilde{u}$ cannot be continuously defined at zero, as it approaches negative infinity as y approaches zero.

If the path $\left(y_1^{DU}, y_2^{DU}, \dots\right)$ solves problem DU_1 for this $\tilde{u}$, then $\left(y_1^{DU}, y_2^{DU}, \dots\right)$ must be strictly positive because the domain of $\tilde{u}$ is $\Re_{++}$. It follows that the first-order Kuhn-Tucker conditions hold –there is a number $\lambda > 0$ such that $e^{1/y_t} = 1 + \left(\lambda/\varphi^{t-1}\right)$ for all t. But this implies that $y_t = 1/\log\left[1 + \lambda/\varphi^{t-1}\right]$, and so it must be the case that $\sum_{t=1}^{\infty}\left\{1/\left(\log\left[1 + \lambda/\varphi^{t-1}\right]\right)\right\} = 1$. For large t, we can approximate the denominator in the terms in this series by $\log\lambda/\varphi^{t-1} = \log\lambda + (t-1)\log(1/\varphi)$. But these terms grow like $k(t-1)$, where $k = \log(1/\varphi)$, and so the series grows like $1/(k(t-1))$, and therefore it does not converge, a contradiction. Therefore there is no solution to program DU_1, and hence to the Program R_1, for this $\tilde{u}$.

The intuition here is that the derivative of $\tilde{u}$ is increasing too fast (exponentially) as y approaches zero. Let $V^R(y_1, y_2, \dots)$ be the value of the objective function of Program R_1 at path $(y_1, y_2, \dots)$. The result is perhaps surprising, because it is easy to see that the function $V^R(y_1, y_2, \dots)$ is bounded on the feasible set. Hence, it must be the case for this $\tilde{u}$ that the finite supremum of $\left\{V^R(y_1, y_2, \dots) \mid (y_1, y_2, \dots) \text{ is feasible}\right\}$ is never attained. It is easy to check that if $\tilde{u}(y) = \frac{y^r}{r}$, for any $r \in (-\infty, 1)$, then Program DU_1 has a solution. The Inada condition holds for these functions, and the first order-conditions can be solved for a positive path whose components sum to unity.

Proof of Theorem 3

Step 1. We introduce the following sequence of programs. Define Program DU^T as:

$$\max \sum_{t=1}^{T} \varphi^{t-1}\tilde{u}(y_t)$$

$$\text{subject to } \sum_{t=1}^{T} y_t \le 1,$$

$$y_t \ge 0, \forall t \ge 1.$$

Step 2. Note that for sufficiently large T, it must be the case that the solution $(z_1, z_2, \ldots, z_T)$ to Program DU^T, which of course exists, has $z_T = 0$. For if not, and $(z_1, z_2, \ldots, z_T) >> 0$, then there are first order conditions of the form:

$$\varphi^{t-1}\tilde{u}'(z_t) = \lambda, \text{ some positive number } \lambda.$$

Of course it follows, from the usual argument, that $(z_1, z_2, \ldots, z_T)$ is a weakly decreasing sequence, and consequently, by choosing a large T, we can guarantee that z_T is bounded above by an arbitrarily small number, because of the cake-eating constraint. Consequently λ must be very close to $\varphi^{T-1}\gamma$, and hence must be arbitrarily small. But since $\tilde{u}'(z_1) = \lambda$, this implies that z_1 becomes arbitrarily large, contradicting the fact that $\sum z_t = 1$. Thus there is a date T such that the solution to Program DU^T has $z_T = 0$.

Step 3. Now let T be the smallest date such that $z_T = 0$; denote the solution to Program DU^T by $(z_1, z_2, \ldots, z_T)$. We will assume that $z_t > 0$ for $t < T$, but the proof can be modified in an obvious way if this is not the case. Then the following Kuhn-Tucker (K-T) conditions must hold for the (concave) Program DU^T :

There are non-negative numbers λ, μ_T such that:

$$\begin{aligned} &\varphi^{t-1}\tilde{u}'(z_t) - \lambda = 0, \text{ for } t < T, \\ &\varphi^{T-1}\gamma - \lambda + \mu_T = 0. \end{aligned}$$

Step 4. We claim that the path $z^{T+1} \equiv (z_1, \ldots, z_T, 0)$ is the solution to Program DU^{T+1}. To see this, write down the K-T conditions for this program, namely:

There are non-negative numbers $(\lambda, \mu_T, \mu_{T+1})$ such that:

$$\begin{aligned} &\varphi^{t-1}\tilde{u}'(z_t) - \lambda = 0, \text{ for } t < T, \\ &\varphi^{T-1}\gamma - \lambda + \mu_T = 0, \\ &\varphi^{T}\gamma - \lambda + \mu_{T+1} = 0. \end{aligned}$$

We note that the values of λ and μ_T continue to solve these FOCs, for the vector z^{T+1}, and we define the new shadow price by

$$\mu_{T+1} = \lambda - \varphi^T\gamma > 0.$$

Thus, since we have a concave program, we have shown that z^{T+1} is its solution.

Step 5. We continue in this manner to show that the vector $z^S = (z, 0, 0, \ldots, 0)$ is the solution for Program DU^S for any $S > T$. The new Lagrangian multiplier at each step is defined by:

$$\mu_S = \lambda - \varphi^{S-1}\gamma,$$

and so we note, for use below, that $\lim_{S\to\infty} \mu_S = \lambda$.

Step 6. We now claim that the vector $(z_1^\infty, z_2^\infty, \ldots) \equiv (z, 0, 0, \ldots.)$ solves Program DU_1. We proceed by contradiction. Denote by $V^{DU}(y_1, y_2, \ldots)$ the value of the objective function of Program DU_1 at the path $(y_1, y_2, \ldots)$. Suppose the claim were false, and there is a path $(y_1, y_2, \ldots)$ which $V^{DU}(y_1, y_2, \ldots) > V^{DU}\left(z_1^\infty, z_2^\infty, \ldots\right)$. Write $y_t = z_t^\infty + g_t$ for all t; of course, $\sum g_t = 0$. We define a function $H : \Re \to \Re$ as follows:

$$H(\varepsilon) = \sum_{t=1}^{T-1} \varphi^{t-1}\tilde{u}(z_t^\infty + \varepsilon g_t) + \sum_{t=T}^{\infty} \varphi^{t-1}\tilde{u}(0 + \varepsilon g_t) + \lambda\left(1 - \sum_{t=1}^{\infty}(z_t^\infty + \varepsilon g_t)\right) + \sum_{t=T}^{\infty} \mu_t(0 + \varepsilon g_t).$$

Verify that $H(0) = V^{DU}(z^\infty)$ and that $H(1) \geq V^{DU}(y_1, y_2, \ldots)$, which follows from the fact that $(y_1, y_2, \ldots)$ is feasible and that the Lagrangian multipliers are all non-negative. Suppose we can show that H is maximized at zero: then we will know that $H(0) \geq H(1)$, which implies that $V^{DU}(z^\infty) \geq V^{DU}(y_1, y_2, \ldots)$, which is the desired contradiction.

Step 7. It therefore remains to show that zero maximizes H. Note that H is a concave function, so it suffices to show that $H'(0) = 0$. We compute:

$$H'(0) = \sum_{t=1}^{T-1} \varphi^{t-1}\tilde{u}'(z_t^\infty)g_t + \sum_{t=T}^{\infty} \varphi^{t-1}\gamma g_t - \lambda \sum_{t=1}^{\infty} g_t + \sum_{t=T}^{\infty} \mu_t g_t.$$

Grouping together all terms associated with the same g_t, we see that for $t < T$, the coefficient of g_t is $\varphi^{t-1}\tilde{u}'(z_t^\infty) - \lambda = 0$, and for $t \geq T$ the coefficient of g_t is $\varphi^{t-1}\gamma - \lambda + \mu_t = 0$. Thus the derivative vanishes at zero, as required.

Step 8. There is a final, transversality condition: We must show that the function H is well-defined on the interval [0,1]. The only term that might cause concern is the last one, which is $\varepsilon \sum_{t=1}^{\infty} \mu_t g_t$. But since $\mu_t \to \lambda$ and $g_t \to 0$ and $\sum_{t=T}^{\infty} g_t = -\sum_{t=1}^{T-1} g_t$, it follows that $\sum_{t=1}^{\infty} \mu_t g_t$ converges, and the proof is complete.

Step 9. The uniqueness of the solution follows from the strict concavity of $\tilde{u}$. □

Proof of Theorem 4 (The Turnpike Theorem)

The program

Recall that we aim at finding the maximum level of sustainable utility for a fairly simple infinitely lived economy. Formally:

Program SUS_2

$$\begin{aligned} &\max \Lambda \quad \text{subject to} \\ &(P1) \quad c_t^\alpha (x_t^l)^{1-\alpha} \geq \Lambda, \quad t \geq 1, \\ &(P2) \quad \xi x_{t-1}^e \geq x_t^e + x_t^l + x_t^c, \quad t \geq 1, \\ &(P3) \quad (s_t^k)^\theta (x_t^c)^{1-\theta} \geq c_t + i_t, \quad t \geq 1, \\ &(P4) \quad (1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1. \end{aligned}$$

The initial endowment is a vector (x_0^e, s_0^k).

The *value function* of the program maps the initial endowment into the value Λ; thus we write $V(x_0^e, s_0^k) = \Lambda$.

Define $F^\Lambda = \{(x_0^e, s_0^k) \mid V\left(x_0^e, s_0^k\right) = \Lambda\}$. This is the set of initial endowments that generate the same value for SUS_2.

We define a *feasible path* as a set of sequences $\{x_t^e\}_{t=0,1,2,...}, \{s_t^k\}_{t=0,1,2,...}$ and all other variables beginning at $t = 1$, such that inequalities $(P2)$, $(P3)$, and $(P4)$ hold. Denote the set of feasible paths by $\mathcal{P}$.

Denote the set of feasible paths beginning at a given initial vector (x_0^e, s_0^k) by $P(x_0^e, s_0^k)$.

Proposition 1. *The set $\mathcal{P}$ is a closed convex cone. The set $P(x_0^e, s_0^k)$ is closed and convex.*

Proof. Easy. □

Proposition 2. *At any solution to Program SUS_2, all the constraints $(P1)$-$(P4)$ bind at all dates. The solution to SUS_2 is unique.*

Proof. Step 1. It is obvious that $(P2)$-$(P4)$ bind. What requires proof is that $u(c_t, x_t^l) = \Lambda$ for all t. We first prove this is the case for $t = 1$. Suppose, to the contrary, that at an optimal solution, $u(c_1, x_1^l) > \Lambda$. Reduce x_1^l by ε and increase each of x_1^e and x_1^c by $\frac{\varepsilon}{2}$ so that $u(c_1, x_1^l - \varepsilon) \equiv \Lambda' > \Lambda$. Now define $\left(i_1', s_1^{k'}\right)$ to be the simultaneous solution of the two equations:

$$c_1 + i_1' = f\left(s_1^{k'}, x_1^c + \frac{\varepsilon}{2}\right),$$
$$(1-\delta)s_0^k + i_1' = s_1^{k'}.$$

Obviously, $s_1^{k'} > s_1^k$; therefore $(x_1^e + \frac{\varepsilon}{2}, s_1^{k'}) >> (x_1^e, s_1^k)$. It follows that, with this altered vector of endowments, $(x_1^e + \frac{\varepsilon}{2}, s_1^{k'})$, for the program beginning at date 2, the value of the program beginning at date 2 is greater than Λ, since the value function of the program is homogeneous of degree one in its endowment vector. Let the value of the program, beginning at date 2, be $\Lambda^* > \Lambda$. We have now produced a feasible path where for all t, $u(\hat{c}_t, \hat{x}_t^l) \geq \min(\Lambda', \Lambda^*) > \Lambda$. This contradiction proves that $u(c_1, x_1^l) = \Lambda$.

Step 2. Assume now that in any optimal solution, for $1 \leq t < T$, $u\left(c_t, x_t^l\right) = \Lambda$, but there is an optimal solution for which $u(c_T, x_T^l) > \Lambda$. Reduce x_{T-1}^e by ε/ξ and increase x_{T-1}^l by the same amount, increasing utility at date $T-1$, which is now greater than Λ. This decreases $x_T \equiv x_T^e + x_T^l + x_T^c$ by ε, and let this decrease be implemented by decreasing x_T^l by ε, which may be chosen small enough that utility at date T is still greater than Λ. We have now produced an optimal path for the program for which $u(c_{T-1}, x_{T-1}^l) > \Lambda$, which contradicts the induction hypothesis. This proves that for all t, $u_t = \Lambda$.

Step 3. We next show that the solution to SUS_2 is unique. Any two solutions must have the same values of $\{(c_t, x_t^l)\}$: for if not, take any non-trivial convex combination of the two solutions, producing another optimal solution for which the constraints $(P1)$ do not bind (using the Cobb-Douglas form of u); this contradicts what has been proved above. In like manner, the values $\{(x_t^e, s_t^k)\}$ must be the same in the two solutions, since otherwise a convex combination of them would produce an optimal solution in which the constraints $(P3)$ do not bind. But if the dated capital-stocks are identical in the two solutions, so must be the dated investments. Since the values $\{(x_t^c, x_t^l)\}$ are identical in the two solutions, we see, by iteration, that the values of $\{x_t^e\}$ are also identical. This proves the claim. □

Proposition 3.

A. Let $(\tilde{x}_0^e, \tilde{s}_0^k) >> (x_0^e, s_0^k)$. Then $V(\tilde{x}_0^e, \tilde{s}_0^k) >> V(x_0^e, s_0^k)$.

B. Along the optimal path beginning at (x_0^e, s_0^k), there is no T such that $(x_T^e, s_T^k) >> (x_0^e, s_0^k)$.

C. Let $(x_{0j}^e, s_{0j}^k) \in F^\kappa$ be an infinite sequence of points in F^κ, for some fixed κ, such that $x_{0j}^k \to \infty$. Then $s_{0j}^k \to 0$.

Proof. *A.* If $(\tilde{x}_0^e, \tilde{s}_0^k) >> (x_0^e, s_0^k)$, then there is a positive number δ^* such that $\left(\tilde{x}_0^e, \tilde{s}_0^k\right) >> (1+\delta^*)\left(x_0^e, s_0^k\right)$. Since $\mathcal{P}$ is a cone, and the utility of Generation t is homogenous of degree 1 in its arguments, it follows immediately that $V\left(\tilde{x}_0^e, \tilde{s}_0^k\right) > (1+\delta^*)\, V\left(x_0^e, s_0^k\right)$.

B. Suppose that there is a T such that $(x_T^e, s_T^k) >> (x_0^e, s_0^k)$. Let the value of the program be κ. By Part A, the value of the sub-program *that begins at date T* is strictly greater than κ. This contradicts the fact that the constraints $(P1)$ are binding for all t.

C. Suppose the premise were false; then there is a subsequence $s^k_{0j} \to S > 0$, some S. We can choose a number $\hat{S} > S$ and a number $\hat{x}$ such that $V(\hat{x}, \hat{S}) = \hat{\kappa} > \kappa$. We can also choose an index j such that the program beginning with the endowments (x^e_{0j}, s^k_{0j}) possesses a feasible path that, at its first step, has three properties:

$$\begin{array}{ll} (i) & s^k_1 > \hat{S}, \\ (ii) & x^e_1 > \hat{x}, \\ (iii) & c^\alpha_1 (x^l_1)^{1-\alpha} > \hat{\kappa}. \end{array}$$

(This is obvious from examining the technology.) It therefore follows that $V(x^e_1, s^k_1) > \hat{\kappa}$: invoke Part *A* of this proposition. But this is a contradiction, because $V(x^k_{0j}, s^k_{0j}) = \kappa < \hat{\kappa}$. □

Since all the constraints of SUS_2 bind, we can write down the Kuhn-Tucker conditions for this concave program. It turns out that these conditions imply only three new pieces of information, which are:

$$(D1) \quad \frac{x^l_t}{c_t} = \frac{1-\alpha}{\alpha(1-\theta)} \frac{x^c_t}{c_t + i_t}, \quad t \geq 1;$$

$$(D2) \quad \frac{x^l_{t+1}}{c_{t+1}} = \frac{x^l_t}{c_t} \frac{\xi}{1-\delta} \left(1 - \frac{\theta(c_t + i_t)}{s^k_t} \right), \quad t \geq 1.$$

$$(D3) \quad \sum_t \left(\frac{1}{\xi} \right)^t x^l_t \text{ converges.}$$

The other Kuhn-Tucker conditions just define the various Lagrangian multipliers, which are all non-negative.

It follows that: *A feasible path and a number κ for which all the primal constraints bind at all t, and for which (D1) ,(D2) and (D3) hold, is an optimal solution.*[7]

The stationary ray

We ask: Is there a ray of initial endowments in $\Re^2_+$ for which the optimal solution is *stationary*, that is, for which all variables are constant over time? We study this by writing down the primal constraints and equations (D1) and (D2) for a hypothetical stationary ray, and see what they imply. Indeed, we can solve them: there is a unique such ray for the initial condition. The ray passes through the following point:

$$x^e_0 = 1, s^k_0 = (\xi - 1) \left(\frac{\xi\theta}{\xi + \delta - 1} \right)^{\frac{1}{1-\theta}} x^{c*},$$
$$\text{where } x^{c*} = \frac{\alpha(1-\theta)(\xi + \delta - 1)}{\alpha(1-\theta)(\xi + \delta - 1) + (1-\alpha)(\xi + \delta - 1 - \xi\delta\theta)}.$$

Indeed, we can compute the values of all the variables on this ray. Call these the *stationary state values.* Of course they are defined up to a multiplicative constant. Let us denote this ray by Γ.

[7] One may ask, conversely: Does the optimal solution have to satisfy these equations? The answer to this must be affirmative: there is an infinite dimensional version of the Kuhn-Tucker theorem, using the Hahn-Banach theorem, which tells us that this is so.

The Turnpike Theorem

It is very difficult to actually compute the optimal path, if we begin from an endowment vector off the stationary ray Γ. We shall, however, prove (Proposition 4 below) that from any initial vector (x_0^e, s_0^k), the optimal solution to SUS_2 converges to a point on Γ.

In the following, given any two variables a_t and b_t, we use the notation for ratios: $\frac{a_t}{b_t} = \left(\frac{a}{b}\right)_t$.

Lemma 6. *Suppose that, in the optimal solution, the limit of the sequence* $\left\{\left(x^l/c\right)_{t=1,2,\ldots}\right\}$ *exists and is finite. Then the solution converges to the stationary state values.*

Proof. Step 1. Denote the limit of the sequence $\left\{\left(x^l/c\right)_{t=1,2,\ldots}\right\}$ by $\overline{\lambda}$. We first argue that $\overline{\lambda} \neq 0$. If $\overline{\lambda} = 0$, then $\lim\left(c/x^l\right)_t = \infty$. By $(D1)$, $\lim\left(x^c/\left(c+i\right)\right)_t = 0$, and so $\lim\left(x^c/s^k\right)_t = 0$, by invoking $(P3)$. Now $\theta\left(c_t + i_t\right)/s_t^k = \theta\left(x_t^c/s_t^k\right)^{1-\theta}$, so $\lim \theta\left(c_t+i_t\right)/s_t^k = 0$, which means, by $(D2)$, that $\left(c/x^l\right)_{t+1} \Big/ \left(c/x^l\right)_t \to \left(\left(1-\delta\right)/\xi\right) < 1$, because $\xi > 1$. It is therefore impossible that $\lim\left(\frac{c}{x^l}\right)_t = \infty$. Therefore $\overline{\lambda} > 0$.

Step 2. By $(P1)$, $x_t^l\left(c_t/x_t^l\right)^{\alpha} = \kappa$ for all t. Therefore $\lim x_t^l = \kappa\overline{\lambda}^{\alpha}$ and so $\lim c_t = \kappa\overline{\lambda}^{\alpha-1}$. From $(D2)$, it also follows that $\frac{\xi}{1-\delta}\lim\left(1 - \frac{\theta(c_t+i_t)}{s_t^k}\right) = 1$; therefore $\lim\left(\left(c+i\right)/s^k\right)_t$ has the value of the ratio of $(c+i)/s^k$ in the stationary state. Therefore $\lim\left(x^c/s^k\right)_t$ has the same value as the ratio of those variables in the stationary state. By $(D1)$ it now follows that $\overline{\lambda}$ is also the ratio of x^l/c in the stationary state.

Step 3. Suppose that there were a subsequence of $\{s_t^k\}$ that diverged to infinity. Since $\lim\left(x^c/s^k\right)_t$ is finite, it follows that the same subsequence of $\{x_t^c\}$ diverges to infinity. It follows from $(P2)$ that the same subsequence of $\{x_t^e\}$ diverges to infinity. In particular, there exists a T such that $(x_T^e, s_T^k) >> (x_0^e, s_0^k)$. But this contradicts Part B of Proposition 3. Therefore the sequence $\{s_t^k\}$ is bounded. It immediately follows that the sequence $\{x_t^c\}$ is bounded, since $\lim\left(x^c/s^k\right)_t$ exists and is finite; and since $\lim\left((c+i)/s^k\right)_t$ also exists and is finite, the sequence $\{i_t\}$ is bounded.

Thus all the sequences of variables, except possibly for $\{x_t^e\}$, are bounded. Therefore we can choose a single subsequence of all the variables (except possibly of $\{x_t^e\}$) which converges to values $(\overline{s}^k, \overline{x}^c, \overline{i})$ and we have already shown that $\{x_t^l\}, \{c_t\}$ converge to values $\overline{x}^l$ and $\overline{c}$. Furthermore we know that $\{s_t^k\}$ converges to a positive number, because $\lim\left(\theta(c_t+i_t)/s_t^k\right)$ has the value of the same ratio in the stationary state and $\{c_t\}$ converges to a positive number.

It now follows, by invoking Proposition 3, Part C, that $\{x_t^e\}$ does not diverge to infinity –since $(x_t^e, s_t^k) \in F^{\kappa}$ for all t. So there is a subsequence of the original sequence such that *all* variables converge.

We proceed to show that this subsequence of variables converges to stationary state values. Denote the limits:

$$\overline{\lambda}_1 = \lim \frac{c_t + i_t}{x_t^c} = \lim \frac{\overline{c} + i_t}{x_t^c}, \tag{A.1}$$

$$\overline{\lambda}_2 = \lim\left(\frac{s^k}{x^c}\right)_t. \tag{A.2}$$

We have shown that $\overline{\lambda}_1$ and $\overline{\lambda}_2$ are the values of the corresponding ratios in the stationary state. Now from ($P3$) we have:

$$x_t^c \overline{\lambda}_2^\theta - i_t \to \overline{c}. \tag{A.3}$$

Note that equations (A.1) and (A.3) comprise two simultaneous equations, in the limit, for the limits of the variables x^c and i. Hence the sequences $\{x_t^c\}$ and $\{i_t\}$ must converge, and to stationary state values, since these same two equations hold for the stationary state variables. We therefore have, by (A.2), that $\{s_t^k\}$ also converges to the appropriate stationary state value. Likewise with $\{x_t^e\}$.

Finally, indeed the *whole* sequence of variables converges to the same stationary state: for if not, there would be another limit point approached simultaneously by some other subsequence of the variables, to a stationary state. But since the stationary ray is unique, that limit of (x_t^e, s_t^k) must also be on the ray Γ. However, we cannot have two subsequences approaching different points on the ray: that would violate Proposition 3, Part B. □

Proposition 4. *From any initial vector* (x_0^e, s_0^k)*, the optimal solution to* SUS_2 *converges to a point on* Γ.

Proof. [8]
Step 1. On the optimal path, the sequence $\{(\frac{x^l}{c})_{t=1,2,...}\}$ does not diverge to infinity. Suppose it did diverge to infinity. Then from ($D1$), the sequence $x_t^c/\,(c_t + i_t)$ diverges to infinity also. But, invoking ($P3$), $x_t^c/\,(c_t + i_t) = \left(x_t^c/s_t^k\right)^\theta$, and so $x_t^c/s_t^k \to \infty$. Now $\theta\,(c_t + i_t)\,/s_t^k = \theta\left(x_t^c/s_t^k\right)^{1-\theta}$, and so it follows that $\theta\,(c_t + i_t)\,/s_t^k$ diverges to infinity. But this contradicts ($D2$), for it would mean that eventually the ratio x_t^l/c_t is negative.

Step 2. Hence it follows that, on the optimal path, the sequence $\{(\frac{x^l}{c})_{t=1,2,...}\}$ has a (finite) limit point. If the sequence $\{(\frac{x^l}{c})_{t=1,2,...}\}$ indeed converges to this limit point, then the theorem is proved, by Lemma 6.

Step 3. Thus, the remainder of the proof will show that the limit point of the sequence $\{(\frac{x^l}{c})_{t=1,2,...}\}$ is unique, and hence it is the limit of the sequence.

By exploiting equations ($D1$) and ($P3$), we can rewrite ($D2$) as follows:

$$(D2^*) \qquad \left(\frac{x^l}{c}\right)_{t+1} = \left(\frac{x^l}{c}\right)_t \frac{\xi}{1-\delta}\left(1 - \theta\left(\frac{\alpha(1-\theta)}{1-\alpha}\right)^{\frac{1-\theta}{\theta}}\left(\frac{x^l}{c}\right)_t^{\frac{1-\theta}{\theta}}\right).$$

It will be convenient to define the function: $f^*(x) = ax(1 - bx^r)$, where $a = \frac{\xi}{1-\delta}$, $b = \theta\left(\frac{\alpha(1-\theta)}{1-\alpha}\right)^{\frac{1-\theta}{\theta}}$, and $r = (1-\theta)/\theta$. Thus ($D2^*$) says that

$$f^*\left(\frac{x_t^c}{c_t}\right) = \frac{x_{t+1}^c}{c_{t+1}} \text{ for all } t.$$

Compute that $\frac{d^2 f^*}{dx^2} = -rab(1+r)x^{r-1}$, and so f^* is a concave function on $\Re_+$. Let A^* be the value of the ratio $\frac{x^l}{c}$ in the stationary state. Then we have: $f^*(A^*) = A^*$ and $f^*(0) = 0$. The first claim follows since the equation ($D2^*$) holds, of course, at the stationary state as well.

Finally, note that another root of f^* is given by $x^* = (1/b)^{1/r}$. Concavity implies that f^* has only the two fixed points 0 and A^*.

[8]Thanks to Cong Huang, who completed and simplified this proof.

Because $\left\{\left(\frac{x^l}{c}\right)_{t=1,2,\ldots}\right\}$ is bounded, it possesses a lim inf and a lim sup. For convenience, denote $y_t = \left(\frac{x^l}{c}\right)_t$, and define

$$\sigma = \liminf y_t, \quad \sigma^* = \limsup y_t.$$

Since $f^*(y_t) = y_{t+1}$, we have $\inf f^*(y_t) = \sigma$, and by the continuity of f^*, $\inf f^*(y_t) = f^*(\inf y_t) = f^*(\sigma) = \sigma$, so σ is a fixed point of f^*. In like manner, σ^* is a fixed point of f^*.

If we can establish that $\sigma \neq 0$, then we must have $\sigma = A^* = \sigma^*$, and hence the limit of $\{y_t\}$ exists. But this is established by an argument that mimics Step 1 of the proof of Lemma 6, as follows.

If $\sigma = 0$, then, by $(D1)$, $\liminf (x^c/(c+i))_t = 0$, and so $\liminf \left(x^c/s^k\right)_t = 0$, by invoking $(P3)$. Now $\theta\,(c_t + i_t)\,/s_t^k = \theta\left(x_t^c/s_t^k\right)^{1-\theta}$, so $\liminf \theta\,(c_t+i_t)\,/s_t^k = 0$, which means, by $(D2)$, that $\liminf y_{t+1}/y_t = \xi/\,(1-\delta) > 1$, because $\xi > 1$. But this immediately implies that $\liminf y_t > 0$, a contradiction. Therefore $\sigma \neq 0$, and Proposition 4 is proved. □

The proof of the Turnpike Theorem follows from the previous discussion, in particular from propositions 2 and 4. □

Proof of Theorem 5

Part A

Step 1. Let $x_0^e = 1$. We claim that for any small $\varepsilon > 0$, we can find values σ and i such that:

$$\begin{aligned} i &= (\xi - \varepsilon + \delta - 1)\sigma, \\ i &= f\left((\xi - \varepsilon)\sigma, \varepsilon\right). \end{aligned}$$

By plotting the graphs of these two functions in (σ, i) space, we can observe that they cross at the origin and at some positive value of i –by assumption $A(b)$.

Step 2. Let $\varepsilon < (\varphi\xi - 1)/\varphi$, and let σ be chosen to satisfy the equations in Step 1, thus defining investment at date 1 when

$$x_1^c = \varepsilon, \quad x_1^e = \xi - \varepsilon, \quad c_1 = 0 = x_1^l.$$

Note from Step 1 that we may take $s_1^k = (\xi - \varepsilon)\sigma$. Let $V(x_0^e, s_0^k)$ be the value function of Program $DU_2[x_0^e, s_0^k]$, if it converges. Then we must have, by consideration of the choice of date 1 values above, $V\,(1,\sigma) \geq 0 + (\xi - \varepsilon)\,\varphi V\,(1,\sigma)$. But $(\xi - \varepsilon)\,\varphi > \xi\varphi - (\xi\varphi - 1) = 1$, implying that the last equation stated cannot hold, and hence Program DU_2 must diverge beginning with endowment $(1, \sigma)$.

Step 3. It immediately follows that Program DU_2 diverges for $\hat{\sigma} > \sigma$. (Just throw away some capital at date 1 and reduce the capital-labor ratio to σ.) Moreover, the program must diverge for $0 < \hat{\sigma} < \sigma$ as well (at date 1, invest very little in education, thus increasing the capital-labor ratio at date 2 to a value $s_1^k/x_1^e \geq \sigma$). □

Part B

Step 1. Let $x_0^e = 1$. The largest possible investment that can be made at date 1 if $s_0^k = \sigma$ is given by $I(\sigma)$, defined by the equation:

$$f((1-\delta)\sigma + I(\sigma), \xi) = I(\sigma),$$

because $x_1^c \leq \xi$. Define σ^* such that:

$$f((1-\delta), \frac{\xi}{\sigma^*})/(\xi - (1-\delta)) = 1 - f_1((1-\delta)\sigma^*, \xi),$$

where $f_1(s,x) = \frac{\partial f}{\partial s}(s,x)$. A monotonicity argument, invoking the intermediate value theorem, shows that σ^* exists uniquely.

Let $m = f_1((1-\delta)\sigma^*, \xi)$ and note that $0 < m < 1$.

Step 2. The graph of the function

$$z(i) = f((1-\delta)\sigma^* + i, \xi)$$

lies everywhere on or below the graph of the function

$$y(i) = f((1-\delta)\sigma^*, \xi) + mi,$$

and $y(0) = z(0)$ (by the concavity of f). The second graph meets the 45^o ray in (i, y) space at the point $i = f\left((1-\delta)\sigma^*, \xi\right)/(1-m)$. Therefore

$$I(\sigma^*) \leq \frac{f((1-\delta)\sigma^*, \xi)}{1-m}.$$

Step 3. Hence, beginning at $s_0^k = \sigma^*$:

$$\begin{aligned} s_1^k &\leq (1-\delta)\sigma^* + I(\sigma^*) \leq (1-\delta)\sigma^* + f((1-\delta)\sigma^*, \xi)/(1-m) \\ &\leq (1-\delta)\sigma^* + \sigma^*(\xi - (1-\delta)) = \xi\sigma^*. \end{aligned}$$

Therefore:

$$u(c_1, x_1^l) \leq u(f(s_1^k, \xi), \xi) \leq u(f(\xi\sigma^*, \xi), \xi) \leq \xi N,$$

where $N \equiv u(f(\sigma^*, 1), 1)$.

Step 4. For any number $\psi > 1$, we have:

$$f((1-\delta)\psi\sigma^* + \psi I(\sigma^*), \xi) < f((1-\delta)\psi\sigma^* + \psi I(\sigma^*), \psi\xi) = \psi I(\sigma^*).$$

Consider the function $\Psi(x) = x - f((1-\delta)\psi\sigma^* + x, \xi)$; note that $\Psi'(x) > 0$ (since $m < 1$). We have (from the above) that $\Psi(\psi I(\sigma^*)) > 0$, and by definition, $\Psi(I(\psi\sigma^*)) = 0$. It follows that $I(\psi\sigma^*) < \psi I(\sigma^*)$.

Step 5. Now compute that

$$\begin{aligned} s_2^k \leq (1-\delta)s_1^k + I(s_1^k) \quad &\leq (1-\delta)\xi\sigma^* + I(\xi\sigma^*) \quad \leq (1-\delta)\xi\sigma^* + \xi I(\sigma^*), \\ &= \xi((1-\delta)\sigma^* + I(\sigma^*)) \quad \leq \xi^2\sigma^*, \end{aligned}$$

which follows by invoking the definition of $I(\cdot)$, and steps 3 and 4.

By induction we have $s_t^k \leq \xi^t\sigma^*$. But $x_t^c \leq \xi^t$ and $x_t^l \leq \xi^t$ as well, and so $u\left(c_t, x_t^l\right) \leq u\left(f\left(s_t^k, \xi^t\right), \xi^t\right) \leq \xi^t N$. It follows that $\sum \varphi^{t-1} u_t \leq \xi \sum (\varphi\xi)^{t-1} N < \infty$.

Step 6. Now suppose that $\sigma > \sigma^*$; let $\sigma = \psi\sigma^*, \quad \psi > 1$. Then beginning at $s_0^k = \sigma$:

$$\begin{aligned} s_1^k &\leq (1-\delta)\sigma + I(\sigma) = (1-\delta)\psi\sigma^* + I(\psi\sigma^*) \\ &< \psi((1-\delta)\sigma^* + I(\sigma^*)) \qquad \text{[by Step 4]} \\ &\leq \psi\xi\sigma^* = \xi\sigma. \end{aligned}$$

And so $u(c_t, x_t^l) \leq u(f(s_t^k, \xi^t), \xi^t) \leq \xi^t u(f(\sigma, 1), 1)$, and as before:

$$\sum (\varphi\xi)^{t-1} u_t < \infty.$$

Step 7. Therefore DU_2 converges for $\sigma \geq \sigma^*$. *A fortiori*, it converges for $\sigma < \sigma^*$, by the free disposal of capital. □

Proof of Lemma 4

For any $(u_1, u_2, \ldots)$, $\min\{u_1, \ldots, u_t\} \geq \min\{u_1, \ldots, u_{t+1}\}$. Therefore

$$\begin{aligned}
\sum_1^\infty \varphi^{t-1} (1 + (t-1)\beta) \min\{u_1, \ldots, u_t\} &\leqslant u_1 \sum_1^\infty \varphi^{t-1} (1 + (t-1)\beta) \\
&= u_1[\varphi^0 + \varphi^1 + \varphi^2 + \varphi^3 + \ldots \\
&\qquad + \beta\varphi^1 + \beta\varphi^2 + \beta\varphi^3 + \ldots \\
&\qquad\qquad + \beta\varphi^2 + \beta\varphi^3 + \ldots \\
&\qquad\qquad\qquad + \beta\varphi^3 + \ldots \\
&\qquad\qquad\qquad\qquad + \ldots] \\
&= u_1 \left(\sum_1^\infty \varphi^{t-1} + \beta[\sum_2^\infty \varphi^{t-1} + \sum_3^\infty \varphi^{t-1} + \ldots \right) = u_1 \left(\frac{1}{1-\varphi} + \beta \left(\frac{\varphi}{1-\varphi} + \frac{\varphi^2}{1-\varphi} + \ldots \right) \right) \\
&= u_1 \left(\frac{1}{1-\varphi} + \frac{\beta}{1-\varphi} \left(\frac{\varphi}{1-\varphi} \right) \right) < \infty.
\end{aligned}$$

Hence, the sum $\sum_{t=1}^\infty \varphi^{t-1}[1 + (t-1)\beta] \min\{u_1, \ldots, u_t\}$ of nonnegative terms converges. □

Proof of Theorem 6

Consider the *constrained discounted utility program* $CDU_2[\varphi, x_0^e, s_0^k]$ (which specializes program *CDU* in the proof of Lemma 1 to the education and capital economy), as follows.

Program $CDU_2[\varphi, x_0^e, s_0^k]$.

$$\begin{aligned}
\max \sum_{t=1}^\infty \varphi^{t-1} u(c_t, x_t^l) \text{ subject to :} \\
(1-\delta)s_{t-1}^k + i_t \geq s_t^k, \\
f(s_t^k, x_t^c) \geq c_t + i_t, \\
\xi x_{t-1}^e \geq x_t^e + x_t^c + x_t^l, \\
u(c_t, x_t^l) \geq u(c_{t+1}, x_{t+1}^l), \quad t \geq 1.
\end{aligned}$$

Note that Program CDU_2 is not concave, because of the last constraint, which is not quasi-concave. (The last constraint is quasi-concave only if u is linear.) Hence we cannot immediately use concave optimization theory to analyze Program CDU_2.

Lemma 7. *The solution to $R_2[\varphi, x_0^e, s_0^k]$ is also the solution to $CDU_2[\varphi, x_0^e, s_0^k]$.*

Proof. Immediate from Lemma 5. □

But the solution to CDU_2 is in general *different* from the solution to DU_2, the last being sometimes unbounded, while CDU_2 is surely bounded.

Now if $DU_2[\varphi, x_0^e, s_0^k]$ diverges, then utility is unbounded above over time. It seems reasonable to conjecture that, in this case, the last constraint of $CDU_2[\varphi, x_0^e, s_0^k]$ will bind at every date. But if this is the case, then the solution to $CDU_2[\varphi, x_0^e, s_0^k]$ is just the solution to $SUS_2[x_0^e, s_0^k]$, which means that the egalitarian ethical observer in the environment with uncertain worlds will behave just as if there were no uncertainty.

We now prove that this conjecture is true. To do so we make use of the following program.

Program $\boldsymbol{PP}[\varphi, x_0^e, s_0^k]$**.**

$$\max\left\{\frac{\Lambda}{1-\varphi} - \sum_2^\infty \varphi^{t-1}\lambda_t\right\} \text{ subject to:}$$

$$\begin{aligned}
&\lambda_1 \equiv 0,\\
(v_t)\quad & u(c_t, x_t^l) \geq \Lambda - \lambda_t, \quad t \geq 1,\\
(m_t)\quad & \lambda_{t+1} \geq \lambda_t, \quad t \geq 1,\\
(a_t)\quad & f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,\\
(b_t)\quad & (1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,\\
(d_t)\quad & \xi x_{t-1}^e \geq x_t^c + x_t^l + x_t^e, t \geq 1.
\end{aligned}$$

Dual variables are stated to the left of the constraints. The primal variables in Program *PP* are all the usual economic variables, plus the variables $\Lambda, \lambda_2, \lambda_3, \ldots$. We call the usual economic variables of a feasible path in Program *PP* the *economic part* of the path. Note that *PP* is a concave program, so it may be solved with traditional methods.

Lemma 8. *Let* $(x_0^e, s_0^k) \in \Gamma$.[9] *If* $\varphi\xi \geq 1$, *then the solution to Program* $SUS_2[x_0^e, s_0^k]$ *forms the economic part of the solution to Program* $PP[\varphi, x_0^e, s_0^k]$.

Proof.

Step 1. We first write down the Kuhn-Tucker conditions which characterize the solution to Program $SUS_2\left[x_0^e, s_0^k\right]$. These are:

$$\begin{aligned}
(SUS1)\quad (\partial\Lambda):&\quad 1 = \sum_1^\infty v_t,\\
(SUS2)\quad (\partial c_t):&\quad v_t u_1[t] = a_t,\\
(SUS3)\quad (\partial x_t^l):&\quad v_t u_2[t] = d_t,\\
(SUS4)\quad (\partial s_t^k):&\quad a_t f_1[t] + b_{t+1}(1-\delta) - b_t = 0,\\
(SUS5)\quad (\partial i_t):&\quad a_t = b_t,\\
(SUS6)\quad (\partial x_t^c):&\quad a_t f_2[t] = d_t,\\
(SUS7)\quad (\partial x_t^e):&\quad \xi d_{t+1} = d_t,
\end{aligned}$$

[9] Recall the definition of Γ in the statement of the Turnpike Theorem.

where we use the notation $u_1[t] \equiv \frac{\partial}{\partial c_t} u(c_t, x_t^l)$, $u_2[t] \equiv \frac{\partial}{\partial x_t^l} u(c_t, x_t^l)$, etc. At the solution to SUS_2, non-negative dual variables satisfying the above conditions exist and all the primal constraints are binding. Denote the primal (economic) variables at the solution by $\left(\hat{\Lambda}, \{\hat{c}_t, \hat{x}_t, \ldots\}_{t=1}^{\infty}\right)$. If $\left(x_0^e, s_0^k\right) \in \Gamma$, then, because the solution is stationary, $u_1[t] = u_1[1]$ for all t, and likewise for the other derivatives of u. and f.

Step 2. Define $\hat{\lambda}_t = 0$, for all $t \geq 1$. We wish to show that $\hat{\Phi} = \left(\hat{\Lambda}, \{\hat{c}_t, \hat{x}_t, ...\}_{t=1}^{\infty}, \left\{\hat{\lambda}_t\right\}_{t=1}^{\infty}\right)$ is the solution to Program $PP[\varphi, x_0^e, s_0^k]$. Let $\Phi = (\Lambda, \{c_t, x_t, ...\}_{t=1}^{\infty}, \{\lambda_t\}_{t=1}^{\infty})$ be the purported optimal path for Program $PP[\varphi, x_0^e, s_0^k]$. Denote the difference between these two paths by:

$$\Delta\Lambda = \Lambda - \hat{\Lambda},\ \Delta c_t = c_t - \hat{c}_t,\ \Delta x_t = x_t - \hat{x}_t, \ldots,\ \Delta\lambda_t = \lambda_t - \hat{\lambda}_t = \lambda_t,$$

that is, schematically, $\Delta\Phi \equiv \Phi - \hat{\Phi}$.

Define:

$$\begin{aligned} \hat{a}_t &= a_t/(1-\varphi), \quad t \geq 1, \\ \hat{b}_t &= b_t/(1-\varphi), \quad t \geq 1, \\ \hat{v}_t &= v_t/(1-\varphi), \quad t \geq 1, \\ \hat{d}_t &= d_t/(1-\varphi), \quad t \geq 1. \end{aligned}$$

Now define the following function of a real variable:

$$\begin{aligned} \Theta(\varepsilon) = {} & \frac{\hat{\Lambda} + \varepsilon\Delta\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1}(\hat{\lambda}_t + \varepsilon\Delta\lambda_t) + \sum_1^{\infty} \hat{v}_t \left(u(\hat{c}_t + \varepsilon\Delta c_t, \hat{x}_t^l + \varepsilon\Delta x_t^l) - (\hat{\Lambda} + \varepsilon\Delta\Lambda) + (\hat{\lambda}_t + \varepsilon\Delta\lambda_t)\right) \\ & + \sum_1^{\infty} m_t((\hat{\lambda}_{t+1} + \varepsilon\Delta\lambda_{t+1}) - (\hat{\lambda}_t + \varepsilon\Delta\lambda_t)) + \sum_1^{\infty} \hat{a}_t \left(f(\hat{s}_t^k + \varepsilon\Delta s_t^k, \hat{x}_t^c + \varepsilon\Delta x_t^c - (\hat{c}_t + \varepsilon\Delta c_t) - (\hat{i}_t + \varepsilon\Delta i_t)\right) \\ & + \sum_1^{\infty} \hat{b}_t \left((1-\delta)(\hat{s}_{t-1}^k + \varepsilon\Delta s_{t-1}^k) + (\hat{i}_t + \varepsilon\Delta i_t) - (\hat{s}_t^k + \varepsilon\Delta s_t^k)\right) \\ & + \sum_1^{\infty} \hat{d}_t \left(\xi(\hat{x}_{t-1}^e + \varepsilon\Delta x_{t-1}^e) - (\hat{x}_t^e + \varepsilon\Delta x_t^e) - (\hat{x}_t^c + \varepsilon\Delta x_t^c) - (\hat{x}_t^l + \varepsilon\Delta x_t^l)\right). \end{aligned}$$

All the variables in this function are defined except for the sequence of numbers $(m_1, m_2, \ldots)$. Note that Θ is a concave function, a consequence of the concavity of u and f. Note that Θ is defined on $[0, 1]$, since the feasible set of Program PP is convex. Suppose we can produce a non-negative sequence $(m_1, m_2, \ldots)$ such that the derivative of Θ exists and is zero at $\varepsilon = 0$. Then Θ will be maximized at zero, and so in particular, $\Theta(0) \geq \Theta(1)$. Now note that $\Theta(0) = \frac{\hat{\Lambda}}{1-\varphi}$, which is the value of the objective function of Program PP at the path $\hat{\Phi}$; all the other terms vanish, since all the primal constraints of Program SUS_2 are binding on this path, and $\hat{\lambda}_t = 0$ for all t. We also have: $\Theta(1) = \frac{\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1}\lambda_t +$ non-negative terms. It will therefore follow that

$$\frac{\hat{\Lambda}}{1-\varphi} \geq \frac{\Lambda}{1-\varphi} - \sum_2^{\infty} \varphi^{t-1}\lambda_t,$$

proving that the value of the objective function of Program PP at $\hat{\Phi}$ weakly dominates the value at any other feasible path, and hence $\hat{\Phi}$ is a solution to Program PP.

Step 3. We now evaluate $\Theta'(0)$, by taking the derivative of Θ w. r. t. ε term by term, gathering terms together. Indeed, what we are doing is re-deriving the Kuhn-Tucker conditions:

we are going through this process because there is a step at which we must deviate from the usual procedure. We compute:

$$\begin{aligned}\Theta'(0) &= \Delta\Lambda\left(\frac{1}{1-\varphi} - \sum_{1}^{\infty}\hat{v}_t\right) + \sum_{1}^{\infty}\Delta c_t\left(\hat{v}_t u_1[t] - \hat{a}_t\right) + \sum_{1}^{\infty}\Delta x_t^l\left(\hat{v}_t u_2[t] - \hat{d}_t\right)\\ &+ \sum_{1}^{\infty}\Delta x_t^c\left(\hat{a}_t f_2[t] - \hat{d}_t\right) + \sum_{1}^{\infty}\Delta x_t^e\left(\xi\hat{d}_{t+1} - \hat{d}_t\right) + \sum_{1}^{\infty}\Delta i_t\left(\hat{b}_t - \hat{a}_t\right)\\ &+ \sum_{1}^{\infty}\Delta s_t^k\left((1-\delta)\hat{b}_{t+1} - \hat{b}_t + \hat{a}_t f_1[t]\right) + \left\{\sum_{1}^{\infty} m_t\left(\Delta\lambda_{t+1} - \Delta\lambda_t\right) - \sum_{2}^{\infty}\varphi^{t-1}\Delta\lambda_t + \sum_{1}^{\infty}\hat{v}_t\Delta\lambda_t\right\}.\end{aligned}$$

Notice that all terms on the r. h. s. of this equation except the last bracketed term vanish by conditions ($SUS1$)-($SUS7$) of Step 1, and the definition of the ˆ dual variables. Furthermore, it is legitimate to collect and recombine terms as we have, because all the relevant series converge. The point at which care must be taken is *not* to attempt to recombine terms in the bracketed term, because the series in the bracketed term may not converge.

<u>Step 4.</u> It follows that we will have shown $\Theta'(0) = 0$ if we can produce a non-negative sequence $(m_1, m_2, \ldots)$ such that

$$\sum_{1}^{\infty} m_t\left(\Delta\lambda_{t+1} - \Delta\lambda_t\right) - \sum_{2}^{\infty}\varphi^{t-1}\Delta\lambda_t + \sum_{2}^{\infty}\hat{v}_t\Delta\lambda_t = 0,$$

which is the same equation as:

$$\sum_{1}^{\infty} m_t(\lambda_{t+1} - \lambda_t) - \sum_{2}^{\infty}\varphi^{t-1}\lambda_t + \sum_{2}^{\infty}\hat{v}_t\lambda_t = 0, \tag{A.4}$$

since $\Delta\lambda_t = \lambda_t$ for all $t \geq 1$.

If the sequence $(\lambda_1, \lambda_2, \ldots)$ is identically zero, then obviously any choice of $(m_1, m_2, \ldots)$ will guarantee (A.7). So suppose this is not the case. Then for some $T \geq 1$, $(\lambda_{T+1} - \lambda_T) > 0$ (recall that $\lambda_1 = 0$) and all terms $(\lambda_{t+1} - \lambda_t) \geq 0$ (see the constraint in Program PP). Consequently, by choosing $m_T \geq 0$ appropriately, and $m_t = 0$ for all $t \neq T$, we can make the sum $\sum_1^\infty m_t(\lambda_{t+1} - \lambda_t)$ equal *any* desired non-negative number. Hence we can solve (A.4) if (and only if):

$$-\sum_{2}^{\infty}\varphi^{t-1}\lambda_t + \sum_{1}^{\infty}\hat{v}_t\lambda_t = \sum_{2}^{\infty}\lambda_t(\hat{v}_t - \varphi^{t-1}) \leq 0. \tag{A.5}$$

Note that both series on the l. h. s. of (A.5) converge, since $(\lambda_1, \lambda_2, \ldots)$ is bounded above by Λ (since if $\lambda_t > \Lambda$ for any t, then one can replace λ_t with Λ, and the new path remains feasible while the objective function of Program *PP increases*), and $\hat{v}_t$ is a geometric series converging to zero (see below), so it is permissible to add these two series together term-wise.

We now invoke the premise that the solution $\hat{\Phi}$ is stationary. Using this fact, we can solve the Kuhn-Tucker conditions in Step 1 and compute that $\hat{v}_t = \left(\frac{1}{\xi}\right)^t \frac{\xi-1}{1-\varphi}$.

Now observe that

$$\begin{aligned}\sum_{2}^{\infty}(\hat{v}_t - \varphi^{t-1}) &= \sum_{2}^{\infty}\left(\left(\frac{1}{\xi}\right)^t\frac{\xi-1}{1-\varphi} - \varphi^{t-1}\right)\\ &= \frac{\xi-1}{\xi(1-\varphi)}\frac{1/\xi}{1-1/\xi} - \frac{\varphi}{1-\varphi} = \frac{1/\xi}{1-\varphi} - \frac{\varphi}{1-\varphi} < 0,\end{aligned}$$

where the last inequality follows because $\varphi\xi > 1$.[10] Note that the terms in this sum are surely positive for small values of t (at least for $t = 1$), but eventually they turn negative and stay negative forever. This is clear if we note that the sign of the t th term is the same as the sign of

$$\frac{\xi - 1}{(1-\varphi)\xi} - (\xi\varphi)^{t-1},$$

which becomes negative at some t because $(\xi\varphi)^{t-1}$ grows without bound.

Let us denote $\zeta_t = \left(\left(\frac{1}{\xi}\right)^t \frac{\xi-1}{1-\varphi} - \varphi^{t-1}\right)$. We have shown that $\sum_2^\infty \zeta_t < 0$. Let T be the largest integer for which ζ_t is non-negative. Then we may write:

$$\sum_2^\infty \lambda_t\zeta_t = \sum_2^T \lambda_t\zeta_t + \sum_{T+1}^\infty \lambda_t\zeta_t \leq \sum_2^T \lambda_T\zeta_t + \sum_{T+1}^\infty \lambda_T\zeta_t = \lambda_T \sum_2^\infty \zeta_T \leq 0,$$

where we have invoked the fact that $(\lambda_1, \lambda_2, \dots)$ is a weakly increasing non-negative sequence. This proves (A.5), and hence the lemma, except for the case $\varphi\xi = 1$.

If $\varphi\xi = 1$, then $\zeta_t = 0$ for all t, and (A.5) obviously holds. □

Lemma 9. *If the solution to Program $SUS_2[x_0^e, s_0^k]$) is the economic part of the solution to Program PP $\left[\varphi, x_0^e, s_0^k\right]$, then it is also the solution to Program $CDU_2\left[\varphi, x_0^e, s_0^k\right]$.*

Proof. Denote the solution to Program SUS_2 by $\hat{\Phi}$, as in the proof of Lemma 8. Denote the solution to Program CDU_2 by $\tilde{\Phi} = \{\tilde{c}_t, \tilde{x}_t, \dots\}$. We can extend the path $\tilde{\Phi}$ to a feasible path for Program *PP* by defining $\tilde{\Lambda} = \tilde{u}_1$ and $\tilde{\lambda}_t = \tilde{u}_1 - \tilde{u}_t$. The path $\hat{\Phi}$ is extended in like manner to a feasible path for Program SUS_2 (and in fact its solution path, by the premise) by letting $\hat{\lambda}_t = 0$ for all t. If the solution to Program CDU_2 were not the economic part of the solution to Program *PP*, then we would have:

$$\frac{\hat{u}_1}{1-\varphi} > \frac{\tilde{u}_1}{1-\varphi} - \sum_2^\infty \varphi^{t-1}(\tilde{u}_1 - \tilde{u}_t) = \sum_1^\infty \varphi^{t-1}\tilde{u}_t,$$

for the left-hand side of this inequality is the value of *PP*, by the premise of the lemma, and the right-hand side is the value of the objective of *PP* at a non-optimal, feasible solution. But note that this inequality says:

$$\frac{\hat{u}_1}{1-\varphi} > \sum_1^\infty \varphi^{t-1}\tilde{u}_t.$$

However the solution to SUS_2 –path $\hat{\Phi}$– is a feasible path for CDU_2; thus, the last equation contradicts the optimality of the $\tilde{\Phi}$ path for CDU_2 . This contradiction proves the lemma. □

Lemma 10. *Let $(x_0^e, s_0^k) \in \Gamma$. If $\varphi\xi \geq 1$, then the solution to Program $R_2[\varphi, x_0^e, s_0^k]$ is the solution to Program $SUS_2[x_0^e, s_0^k]$.*

Proof. Follows immediately from lemmas 7-9. □

[10]We deal with the boundary case $\varphi\xi = 1$ below.

We now proceed to the proof of Theorem 6.

Step1. From Lemma 5, we can write Program R^{β} as follows:

$$\max \quad \sum_{t=1}^{\infty} \varphi^{t-1} (1 + (t-1)\beta) \; u_t$$
$$\text{subject to } u \in P,$$
$$u_t \geq u_{t+1}, \text{ for } t \geq 1.$$

Since the value of the program is finite (by Lemma 4), we can break up the series in the objective function, and write it as:

$$\begin{aligned} & u_1 + \varphi u_2 + \varphi^2 u_3 + \varphi^3 u_4 + \ldots \\ & + \beta\varphi u_2 + \beta\varphi^2 u_3 + \beta\varphi^3 u_4 + \ldots \\ & + \beta\varphi^2 u_3 + \beta\varphi^3 u_4 + \ldots \\ & + \beta\varphi^3 u_4 + \ldots \\ & = \sum_{1}^{\infty} \varphi^{t-1} u_t + \beta \left[\sum_{2}^{\infty} \varphi^{t-1} u_t + \sum_{3}^{\infty} \varphi^{t-1} u_t + \sum_{4}^{\infty} \varphi^{t-1} u_t + \ldots \right]. \end{aligned} \tag{A.6}$$

Step 2. Suppose, contrary to the claim, that $(u_1^*, u_2^*, \ldots)$ solves Program $R^{\beta}[\varphi, x_0^e, s_0^k]$ but not Program $SUS_2[x_0^e, s_0^k]$, whose solution has constant utilities at the level denoted by Λ^*. Because Program $DU_2[x_0^e, s_0^k]$ diverges, we know by lemmas 4 and 10 that the solution to Program SUS_2 is the same as the solution to program R_2, which, by Lemma 5, is equivalent to Program $CDU_2[x_0^e, s_0^k]$:

$$\max \sum_1^{\infty} \varphi^{t-1} u_t$$
$$\text{subject to } u \in P,$$
$$u_t \geq u_{t+1}.$$

Hence, the solution to Program CDU_2 is $(\Lambda^*, \Lambda^*, \ldots)$. The assumption that $(u_1^*, u_2^*, \ldots)$ is not the solution to Program SUS_2 then implies that $(u_1^*, u_2^*, \ldots)$ is not the solution to Program $CDU_2[x_0^e, s_0^k]$ either, i. e.,

$$\sum_{1}^{\infty} \varphi^{t-1} u_t^* < \sum_{1}^{\infty} \varphi^{t-1} \Lambda^* = \frac{\Lambda^*}{1 - \varphi} \tag{A.7}$$

(since $(u_1^*, u_2^*, \ldots)$ is feasible for Program SUS_2 and the solution to that program is unique), i. e., the first term in (A.6) evaluated at $(u_1^*, u_2^*, \ldots)$ is less than $\frac{\Lambda^*}{1-\varphi}$.

Step 3. The proof will be completed after showing that (A.7) implies that the value of the objective function of R^{β} at $(\Lambda^*, \Lambda^*, \ldots)$ is higher than at $(u_1^*, u_2^*, \ldots)$, and, hence, $(u_1^*, u_2^*, \ldots)$ does not solve R^{β}, contrary to hypothesis. If $\beta = 0$, then from (A.6) the value of the objective function of R^{β} at $(u_1^*, u_2^*, \ldots)$ is $\sum_1^{\infty} \varphi^{t-1} u_t^*$, by (A.7) less than $\sum_1^{\infty} \varphi^{t-1} \Lambda^*$, which is the desired contradiction. So let $\beta > 0$. Again by (A.7), the first term of (A.6) is less than $\frac{\Lambda^*}{1-\varphi}$. Suppose now that the second term in (A.6) evaluated at $(u_1^*, u_2^*, \ldots)$ is greater than $\beta \frac{\varphi}{1-\varphi} \Lambda^*$, which, because $\beta > 0$, implies that

$$\sum_{2}^{\infty} \varphi^{t-1} u_t^* > \sum_{1}^{\infty} \varphi^{t} \Lambda^*. \tag{A.8}$$

If we had that $u_1^* \geq \Lambda^*$, then, by (A.8), $u_1^* + \sum_2^\infty \varphi^{t-1} u_t^* > \Lambda^* + \sum_1^\infty \varphi^t \Lambda^* = \Lambda^* + \frac{\varphi}{1-\varphi}\Lambda^* = \frac{1}{1-\varphi}\Lambda^*$, contradicting (A.7). Thus, $u_1^* < \Lambda^*$ and, therefore, by Lemma 5, $u_t^* \leq u_1^* < \Lambda^*$ for all t, and so (A.8) would be false. Therefore the second term of (A.6) when evaluated at $(u_1^*, u_2^*, \dots)$ is $\beta \sum_{t=2}^\infty \varphi^{t-1} u_t^*$, which is less than or equal to $\beta \frac{\varphi}{1-\varphi}\Lambda^*$. By induction, we see that for all values $\tau \geq 2$:

$$\beta \sum_{t=\tau}^{\infty} \varphi^{t-1} u_t^* \leq \beta \sum_{t=\tau}^{\infty} \varphi^{t-1} \Lambda^*.$$

Hence, $(\Lambda^*, \Lambda^*, \dots)$ dominates $(u_1^*, u_2^*, \dots)$ in Program R^β while satisfying its constraints, a contradiction which establishes the theorem. □

Proof of Theorem 7

Step 1. It is obvious that if $\varphi = 0$, then the solution to Program R_2 requires simply maximizing the utility of the first generation. In particular, this will require $x_1^e = 0$ and hence $u_2 = 0$.

Step 2. More generally, suppose that in the solution to Program R_2, we have $u_1 = u_2 > 0$. Then if we reduce x_2^l by $\xi\varepsilon$, we can increase x_1^l by ε. This leaves all variables after date 2 unchanged, since Generation 2 continues to pass down the same endowment to Generation 3. It therefore must be the case that this change does not increase the value of $u_1 + \varphi u_2$; therefore we must have:

$$\frac{\partial u(c_1, x_1^l)}{\partial x_1^l} - \varphi \frac{\partial u(c_2, x_2^l)}{\partial x_2^l} \xi \leq 0.$$

Choosing

$$\bar{\varphi} = \frac{\partial u(c_1, x_1^l)}{\partial x_1^l} \bigg/ \left(\frac{\partial u(c_2, x_2^l)}{\partial x_2^l} \xi \right)$$

therefore proves the theorem. □

Proof of Theorem 8

Step 1. Without loss of generality, we assume that $x_0^e = 1$, and so $s_0^k = \sigma_0$. Since the set of feasible paths is a convex cone, the primal variables at the solution of the general problem where $x_0^e \neq 1$ are simply the ones computed here, multiplied by x_0^e.

We write the DU_2 program with its dual variables:

$$\max \sum_{1}^{\infty} \varphi^{t-1} u\left(c_t, x_t^l\right) \text{ subject to}$$

$$\begin{aligned}
&(C1): \quad (1-\delta)s_{t-1}^k + i_t \geq s_t^k, \qquad (a_t)\\
&(C2): \quad f(s_t^k, x_t^c) \geq c_t + i_t, \qquad (b_t)\\
&(C3): \quad \xi x_{t-1}^e \geq x_t^e + x_t^l + x_t^c, \qquad (d_t)\\
&(C4): \quad i_t \geq 0. \qquad (e_t)
\end{aligned}$$

The Kuhn-Tucker conditions for a solution to this program where all the constraints bind are:

$$\begin{aligned}
&(KT1) \quad (\partial c_t): \quad \varphi^{t-1}u_1[t] = b_t,\\
&(KT2) \quad (\partial x_t^l): \quad \varphi^{t-1}u_2[t] = d_t,\\
&(KT3) \quad (\partial x_t^e): \quad d_t = (1/\xi)^{t-1}d_1,\\
&(KT4) \quad (\partial x_t^c): \quad b_t f_2[t] = d_t,\\
&(KT5) \quad (\partial s_t^k): \quad (1-\delta)a_{t+1} = a_t - b_t f_1[t],\\
&(KT6) \quad (i_t): \quad a_t = b_t - e_t,
\end{aligned}$$

where all equations hold for $t = 1, 2, 3, \ldots$. Again, $u_j[t]$ and $f_j[t]$ are the jth partial derivatives of the utility function u and the production function f for $j = 1, 2$.

We will show that there exist non-negative dual variables such that the proposed path satisfies all the Kuhn-Tucker constraints. All the relevant infinite series converge, so that the satisfaction of the K-T constraints suffices to prove optimality of this infinite program.

Step 2. Our method will be to substitute the values on the proposed solution path into the primal and dual constraints, and to show that non-negative values of all dual variables can be computed. To this end, the educational constraint $(C3)$ gives us:

$$\xi - E = x_1^l + x_1^c, \tag{A.9}$$

recalling that $x_0^e = 1$.

Step 3. The dual K-T constraints imply the following:

$$u_2[t] = f_2[t]u_1[t], \tag{A.10}$$

$$\varphi\xi u_2[t+1] = u_2[t], \tag{A.11}$$

$$e_t - (1-\delta)e_{t+1} = (1 - f_1[t])b_t - (1-\delta)b_{t+1}. \tag{A.12}$$

The remaining dual constraints simply define (non-negative) values of the dual variables.

Step 4. Equation (A.11) says that

$$\frac{1-\alpha}{\alpha}\frac{c_t}{x_t^l} = (1-\theta)\left(\frac{(1-\delta)^t\sigma_0}{E^{t-1}x_1^c}\right);$$

substituting $\tilde{c}_t$ for c_t allows us to reduce this equation to:

$$\frac{(1-\alpha)x_1^c}{\alpha(1-\theta)} = x_1^l. \tag{A.13}$$

Equations (A.12) and (A.13) comprise two linear equations in (x_1^c, x_1^l), which solve to give

$$x_1^c = \tilde{x}_1^c, \quad x_1^l = \tilde{x}_1^l,$$

as required.

Step 5. We next analyze equation (A.12), which says:

$$(\varphi\xi)\left(\frac{c_{t+1}}{x_{t+1}^l}\right)^\alpha\left(\frac{x_t^l}{c_t}\right)^\alpha = 1.$$

Substituting in the values $\tilde{c}_t$ and $\tilde{x}_t^l$ gives us an equation in the variable E:

$$\varphi\xi\left(\frac{1-\delta}{E}\right)^{\alpha\theta} = 1,$$

which solves to give the prescribed value for E. Note that $E < 1$ since $\varphi\xi < 1$.

Step 6. The prescribed values of all primal variables have been verified. The Kuhn-Tucker equations (KT1-3) give us non-negative solutions for b_t and d_t. It is left only to solve for e_t and to show that for all t, $b_t \geq e_t$, which will give non-negative values for a_t.

Step 7. Define the new variables:

$$m_t = (1 - f_1[t])\varphi^{t-1}u_1[t] - (1-\delta)\varphi^t u_1[t+1].$$

We show in this step that there exists a number $\hat{\sigma}$ such that if $\sigma_0 \geq \hat{\sigma}$, then $m_t \geq 0$ for all $t \geq 1$. The desired result is equivalent to:

$$(\forall t \geq 1) \quad 1 - \theta\left(\frac{\tilde{x}_1^c E^{t-1}}{\sigma(1-\delta)^t}\right)^{1-\theta} \geq^? (1-\delta)\varphi\left(\frac{E}{1-\delta}\right)^{(1-\alpha)\theta}. \tag{A.14}$$

Since $\frac{E}{1-\delta} < 1$, the l. h. s. of (A.14) is increasing in t; thus we need only verify (A.14) for $t = 1$, which is to say, to verify that:

$$1 - \theta\left(\frac{\tilde{x}_1^c}{\sigma(1-\delta)}\right)^{1-\theta} \geq^? (1-\delta)\varphi\left(\frac{E}{1-\delta}\right)^{\theta(1-\alpha)},$$

an inequality which holds for sufficiently large σ if and only if:

$$1 > (1-\delta)\varphi\left(\frac{E}{1-\delta}\right)^{\theta(1-\alpha)},$$

which is immediately seen to be true from the definition of E.

Step 8. Now note that equation (A.12) can be written

$$e_t - (1-\delta)e_{t+1} = m_t, \quad t \geq 1.$$

This system of difference equations yields the following solution:

$$e_T = \frac{e_1}{(1-\delta)^{T-1}} - \sum_{t=1}^{T-1} m_t(1-\delta)^{t-T}, \quad T = 2, 3, \ldots$$

Now choose $e_1 = \sum_{t=1}^{\infty}(1-\delta)^{t-1}m_t$. (We note that this series converges.) To verify that $e_T \geq 0$ for all $T \geq 1$ we must show that

$$T \geq 2 \Rightarrow e_1 \geq \sum_{t=1}^{T-1} m_t(1-\delta)^{t-1},$$

a fact which follows from the definition of e_1 and the fact that $(m_1, m_2, \ldots)$ is a non-negative sequence.

Step 9. The final step is to show that $a_t \geq 0$ where $a_t = b_t - e_t$. It suffices to show that for all $T \geq 1$, $(1-\delta)^{T-1} b_T \geq (1-\delta)^{T-1} e_T$, or that:

$$(1-\delta)^{T-1} b_T \geq^{?} \sum_{t=T}^{\infty} m_t (1-\delta)^{t-1}.$$

The r. h. s. of this inequality can be shown (with some algebra) to equal

$$((1-\delta)\varphi)^{T-1}(1 - f_1[T])u_1[T] - \sum_{T+1}^{\infty} f_1[t]((1-\delta)\varphi)^{t-1} u_1[t];$$

since $b_T = \varphi^{T-1} u_1[T]$, our desired inequality reduces to showing that

$$((1-\delta)\varphi)^{T-1} u_1[T] \geq^{?} ((1-\delta)\varphi)^{T-1}(1 - f_1[T])u_1[T] - \text{a positive term},$$

which is surely true. This concludes the demonstration that all the K-T conditions hold with the dual variables as defined.

Step 10. Finally, we derive the critical value σ^*. The infinite-series expression for e_1 can be expanded and reduced (with much algebra) to show that

$$e_1(\sigma_0) = (1 - f_1[1])u_1[1] - \left(\frac{\tilde{c}_1}{\tilde{x}_1^l}\right)^{\alpha} \frac{\alpha\theta}{\sigma_0} (\varphi\xi)^{\frac{1}{\alpha\theta}} \frac{1-\alpha}{1-\alpha\theta}, \tag{A.15}$$

which we write as a function of the initial capital-labor ratio. The reader should note, from the K-T conditions (KT1-6) in Step 1 that the dual variables are functions only of the *marginal* utilities and productivities at the various dates, which are, for the Cobb-Douglas case, functions of *ratios* of the primal variables. Therefore the dual variables are independent of the scale of the endowment vector (i. e., the value of x_0^e).

The critical value of σ_0 is that number σ^* for which $e_1(\sigma^*) = 0$: for if $e_1(\sigma_0) > 0$ then a slight decrease in σ_0 will still deliver a positive value of e_1, and all the other e_t. But this would mean that investment is identically zero on the optimal path. The zero of equation (A.15) is the solution to the equation in the statement of the theorem, which concludes the proof. □

Example 2

This is an example of an education and capital economy where, along the solution path to Program DU_2, $u_2 > u_1$, whereas the utilities from date 2 onwards decay geometrically. The example is presented in lemmas 11 and 12 below.

Lemma 11. *Let* $(\alpha, \theta, \delta, \xi, \varphi) = (0.66, 0.25, 0.1, 1.1, 0.9)$ *and* $(x_0^e, s_0^k) = (1, 0.15)$*. In particular,* $\varphi\xi < 1$*. Then* $\sigma^* = 0.186198$ *and so* $\sigma_0 = 0.15 < \sigma^*$*. The solution to* DU_2 *is given by* $(c_1, x_1^l, x_1^c, x_1^e, i_1, s_1^k) = (0.192294, 0.0482943, 0.870989, 0.154375, 0.0746361, 0.114795)$*. We have* $\sigma_1 = 0.1979 > \sigma^*$ *and the variables from date 2 onwards are given by:*
$t \geq 2: \quad i_t = 0, s_t^k = x_1^e \tilde{s}_t^k, x_t = x_1^e \tilde{x}_t, c_t = x_1^e \tilde{c}_t$*. In particular,* $u_1 = 0.1138$ *and* $u_2 = 0.1169 > u_1$*. The utilities from date 2 onwards decay geometrically as in Theorem 8.*

Proof. Step1. We will produce the example by finding an initial endowment vector (x_0^e, s_0^k) such that $\sigma_0 < \sigma^*$ and the solution to $DU_2[\varphi, x_0^e, s_0^k]$ has the following property: on the optimal path, at date 1, we have $\sigma_1 = s_1^k / x_1^e > \sigma^*$. For we then know what the optimal path is from date 1 onwards: it is just the path stipulated in Theorem 8. Our strategy will be to find such values of (x_0^e, s_0^k), where, on the optimal path, we have $u_1 < u_2$.

We write down the program we wish to solve, where (x_0^e, s_0^k) is now an unknown endowment.

Program $PP^*[\varphi, x_0^e, s_0^k]$.

$$\max \sum_1^\infty \varphi^{t-2} u(c_t, x_t^l) \text{ subject to :}$$

$$\begin{aligned}
&(a_t) \quad (1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1,\\
&(b_t) \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1,\\
&(d_t) \quad \xi x_{t-1}^e \geq x_t \equiv x_t^e + x_t^l + x_t^c, \quad t \geq 1,\\
&(e_t) \quad i_t \geq 0, \quad t \geq 1.
\end{aligned}$$

Note that we have factored out φ from the usual statement of the objective function. Of course this makes no difference to the solution. The reason for doing so will become apparent momentarily.

We are searching for a solution such that $\sigma_1 > \sigma^*$, $i_1 > 0$, and it follows (by Theorem 8) that $i_t = 0, t \geq 2$. Hence all constraints of program PP^* will bind except for the first investment constraint. This gives the following K-T conditions:

$$\begin{aligned}
(\partial c_t): &\quad \varphi^{t-2} u_1[t] = b_t, \quad t \geq 1,\\
(\partial x_t^l): &\quad \varphi^{t-2} u_2[t] = a_t, \quad t \geq 1,\\
(\partial x_t^c): &\quad b_t f_2[t] = d_t, \quad t \geq 1,\\
(\partial x_t^e): &\quad \xi d_{t+1} = d_t, \quad t \geq 1,\\
(\partial i_t): &\quad a_1 = b_1,\\
&\quad a_t = b_t - e_t, \quad t > 1,\\
(\partial s_t^k): &\quad (1-\delta) a_{t+1} = a_t - b_t f_1[t], \quad t \geq 1.
\end{aligned}$$

Step 2. In Theorem 8, we solved the DU_2 problem with the normalization $x_0^e = 1, \sigma_0 = s_0^k$. Recall from Step 10 of the proof of Theorem 8 that the values of the dual variables of that program are functions *only* of σ_0: that is, they depend only on the capital-labor ratio at date 0, not on the scale of the initial endowment vector.

Step 3. Program PP^* beginning at date 1 (not date 0) is *exactly* the program solved in Theorem 8. (That is why we factored out φ from the objective.) Since $\sigma_1 > \sigma^*$ in the solution we are looking for, it follows that the dual variables from date 1 on in Program PP^* are exactly the dual variables computed in Theorem 8, where the initial capital – labor ratio is σ_1, and the primal variables from date 1 are exactly the tilde primal variables of Theorem 8, multiplied by x_1^e, whatever that turns out to be.

Denote the dual variables computed in the proof of Theorem 8 with tildes –$\tilde{a}_t(\sigma), \tilde{b}_t(\sigma)$, etc., where σ is the initial capital-labor ratio of that program.

Step 4. We now compute what information is contained in the K-T constraints for Program PP^*. First, we know that $d_2 = \tilde{d}_1(\sigma_1)$: this follows from the above discussion. But $d_2 = \frac{1}{\xi} d_1 = \varphi^{-1} u_2[1]$ and therefore:

$$u_2[1] = \varphi \xi \tilde{d}_1(\sigma_1). \tag{A.16}$$

From Theorem 8, we know that $\tilde{d}_1(\sigma_1) = \tilde{u}_2[1] = (1-\alpha)\left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l(\sigma_1)}\right)^\alpha$, and we therefore can write, manipulating equation (A.16):

$$\frac{c_1}{x_1^l} = (\varphi\xi)^{1/\alpha}\left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l(\sigma_1)}\right), \tag{A.17}$$

where c_1, x_1^l are the date 1 values on the optimal path for Program PP^*.

Our second equation is

$$\frac{u_2[1]}{u_1[1]} = f_2[1],$$

which comes from the first three K-T constraints of Program PP^*. This gives:

$$\frac{1-\alpha}{\alpha}\frac{c_1}{x_1^l} = (1-\theta)\frac{f(s_1^k, x_1^c)}{x_1^c} = (1-\theta)\left(\frac{s_1^k}{x_1^c}\right)^{\theta}. \tag{A.18}$$

The next three equations simply restate the primal constraints:

$$(1-\delta)s_0^k + i_1 = s_1^k, \tag{A.19}$$

$$\xi x_0^e = x_1^e + x_1^l + x_1^c, \tag{A.20}$$

$$f(s_1^k, x_1^c) = c_1 + i_1. \tag{A.21}$$

Equation (A.21) comes from the (∂s_1^k) K-T condition. As before, we know that $b_2 = \tilde{b}_1(\sigma_1)$ and $e_2 = \tilde{e}_1(\sigma_1)$ and so $a_2 = b_2 - e_2 = \tilde{b}_1(\sigma_1) - \tilde{e}_1(\sigma_1)$. Thus, we may write that K-T condition as:

$$(1-\delta)(\tilde{b}_1(\sigma_1) - \tilde{e}_1(\sigma_1)) = \alpha\varphi^{-1}\left(\frac{x_1^l}{c_1}\right)^{1-\alpha}\left(1-\theta\left(\frac{x_1^c}{s_1^k}\right)^{1-\theta}\right). \tag{A.22}$$

The six equations (A.17)-(A.22) are equations in the six unknowns $x_1^e, x_1^l, x_1^c, i_1, c_1, s_1^k$ when the endowment (x_0^e, s_0^k) is given. Of course, $\sigma_1 = s_1^k/x_1^e$. We know the expressions for all the tilde variables from Theorem 8, as functions of σ_1.

Indeed, these six equations contain all the new information about the solution to Program PP^* –the remaining K-T conditions simply emulate the solution of the program from date 1 onwards, which we know from Theorem 8.

We now show how to solve these six equations. Define two new variables:

$$A = \frac{c_1}{x_1^l}, \quad B = \frac{s_1^k}{x_1^c}$$

Note that equations (A.17), (A.18) and (A.22) above are simultaneous equations in the three unknowns A, B and σ_1. Hence we can solve for these three variables (which we will do in an example, given below). Now, knowing these three variables, we can write all the information remaining in the six equations as the following system of six linear equations in the six unknowns:

$$\begin{aligned}
&s_1^k = \sigma_1 x_1^e,\\
&(1-\delta)s_0^k + i_1 = s_1^k,\\
&x_1^c f(B,1) = c_1 + i_1,\\
&c_1 = Ax_1^l,\\
&\xi x_0^e = x_1^e + x_1^l + x_1^c,\\
&s_1^k = Bx_c^1.
\end{aligned}$$

We write these equations in matrix form $Mz = Q$, where

$$M = \begin{pmatrix} 0 & 0 & 0 & -\sigma_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & f(B,1) & 0 & -1 & 0 \\ -1 & A & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 0 \\ (1-\delta)s_0^k \\ 0 \\ 0 \\ \xi x_0^e \\ 0 \end{pmatrix}.$$

(The order of the variables is $(c_1, x_1^l, x_1^c, x_1^e, i_1, s_1^k)$.) Hence we can compute the solution $z = M^{-1}Q$. *If we insert an endowment vector* (x_0^e, s_0^k) *with* $\sigma_0 < \sigma^*$ *and the solution Q generated is a positive vector, and* $\frac{s_1^k}{x_1^e} \geq \sigma^*$, *then we have a solution to* PP^* *of the required form.* For it will immediately follow that all the dual variables are non-negative, from the K-T conditions, and so we have produced a path where all the K-T conditions hold —by again invoking Theorem 8.

Step 5. Examine the *Mathematica* program (available from the authors) which calculates this solution for several numerical values. In particular, an instance is provided in which $u_1 < u_2$ on the optimal path, which proves the lemma. □

Lemma 12. *The solution to* $R_2[\varphi, x_0^e, s_0^k]$ *with the data of the premise of Lemma 11 is given by:* $\left(s_1^k, i_1, c_1, x_1^e, x_1^c, x_1^l, \sigma_1\right) = (0.167583, 0.0325828, 0.131227, 0.847974, 0.162572, 0.894544, 0.197627)$ *and for* $t > 1$*:* $i_t = 0$, $s_t^k = x_1^e \tilde{s}_t^k$, $x_t = x_1^e \tilde{x}_t$, $c_t = x_1^e \tilde{c}_t$. *At the solution,* $u_1 = u_2$. *Indeed, the utilities at the solutions of* DU_2 *and* R_2 *for this economy are given by:*

	u_1	u_2	$u_t, t > 2$
DU_2	0.1138	0.1169	*geometric decay*
R_2	0.1152	0.1152	*geometric decay*

Proof. Step 1. We will find a solution to Program $PP[\varphi, x_0^e, s_0^k]$: this will also be a solution to $CDU_2[\varphi, x_0^e, s_0^k]$ and hence to $R_2[\varphi, x_0^e, s_0^k]$. Recall that Program $PP[\varphi, x_0^e, s_0^k]$is:

$$\max \quad \left\{ \frac{\Lambda}{1-\rho} - \sum_2^\infty \varphi^{t-1} \lambda_t \right\} \quad \text{subject to}$$

$$\begin{aligned} &(v_t) \quad u(c_t, x_t^l) \geq \Lambda - \lambda_t, \quad t \geq 1, \\ &(m_t) \quad \lambda_{t+1} \geq \lambda_t, \quad t \geq 1, \\ &(b_t) \quad f(s_t^k, x_t^c) \geq c_t + i_t, \quad t \geq 1, \\ &(a_t) \quad (1-\delta)s_{t-1}^k + i_t \geq s_t^k, \quad t \geq 1, \\ &(d_t) \quad \xi x_{t-1}^e \geq x_t^c + x_t^l + x_t^e, \quad t \geq 1, \\ &(e_t) \quad i_t \geq 0, \quad t \geq 1, \end{aligned}$$

where $\lambda_1 \equiv 0$. For the specified economy, we conjecture a solution where $u_1 = u_2 > u_3 > \ldots$ and where the geometric-decay solution begins at date 2. Thus, of the set of m_t constraints, only the m_1 constraint will bind, and so $m_t = 0$ for $t > 1$. The e_1 constraint will be slack, since we conjecture that $i_1 > 0$. All other constraints will bind at the solution. The K-T conditions are therefore:

$$
\begin{aligned}
(\partial\Lambda): &\qquad \frac{1}{1-\varphi} = \sum_{1}^{\infty} v_t,\\
(\partial\lambda_2): &\qquad -\varphi + m_1 + v_2 = 0,\\
(\partial\lambda_t): &\qquad v_t = \varphi^{t-1}, \quad t > 2,\\
(\partial c_t): &\qquad v_t u_t = b_t, \quad t \geq 1,\\
(\partial x_t^l): &\qquad v_t u_t = d_t, \quad t \geq 1,\\
(\partial x_t^c): &\qquad b_t f_2[t] = d_t, \quad t \geq 1,\\
(\partial x_t^e): &\qquad \xi d_{t+1} = d_t, \quad t \geq 1,\\
(\partial s_t^k): &\qquad a_{t+1}(1-\delta) - a_t + b_t f_1[t] = 0, \quad t \geq 1,\\
(\partial i_1): &\qquad a_1 = b_1,\\
(\partial i_t): &\qquad a_t = b_t - e_t, \quad t > 1.
\end{aligned}
$$

Step 2. We can reduce the first three dual K-T conditions to the equations:

$$v_2 = 1 + \varphi - v_1, \quad m_1 = v_1 - 1,$$

thus eliminating the variables v_2 and m_1. We must, after finding a value for v_1, check that v_2 and m_1 are non-negative.

For $t \geq 2$, we define all the dual variables to equal the dual variables of the geometric-decay solution which begins at date 2 with the endowment (x_1^e, s_1^k), multiplied by λ. Denoting the latter variables with tildes, we therefore define for $t \geq 2$:

$$a_t = \varphi\tilde{a}_{t-1},\ b_t = \varphi\tilde{b}_{t-1}, \quad d_t = \varphi\tilde{d}_{t-1}, \quad e_t = \varphi\tilde{e}_{t-1}.$$

Then all the dual constraints which involve these variables are satisfied where the primal variables for dates $t \geq 2$ are given by the geometric-decay solution to Theorem 8. For this to be a solution, we must check that $\sigma_1 \equiv s_1^k / x_1^e \geq \sigma^*$. We are left only with the dual constraints associated with date 1, which are:

$$
\begin{aligned}
&v_1 u_1[1] = b_1,\\
&v_1 u_2[1] = d_1,\\
&u_2[1] = f_2[1] u_1[1],\\
&\xi d_2 = d_1,\\
&a_2(1-\delta) = b_1(1 - f_1[1]).
\end{aligned}
$$

The first two of the above constraints simply define b_1 and d_1. Thus we are left with three substantive equations. Substituting in for the values of d_2 and a_2, these become:

$$u_2[1] = f_2[1] u_1[1], \tag{A.23}$$

$$\varphi\xi\tilde{d}_1 = v_1 u_2[1], \tag{A.24}$$

$$\varphi(\tilde{b}_1 - \tilde{e}_1)(1-\delta) = v_1 u_1[1]\,(1 - f_1[1])\,. \tag{A.25}$$

Recall from the proof of Theorem 8 that the expressions for $\tilde{d}_1, \tilde{b}_1, \tilde{e}_1$ are known functions of $\sigma_1 \equiv s_1^k / x_1^e$. In particular, we have:

$$\tilde{b}_1 = \alpha \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l} \right)^{\alpha-1}, \tilde{d}_1 = (1-\alpha) \left(\frac{\tilde{c}_1(\sigma_1)}{\tilde{x}_1^l} \right)^{\alpha},$$

while the expression for $\tilde{e}_1$ is given as equation (A.15) in Step 10 of the proof of Theorem 8.

In addition we have the primal constraints:

$$u(c_1, x_1^l) = x_1^e u\left(\tilde{c}_1(\sigma_1), \tilde{x}_1^l\right) \quad \text{(i. e. }, u_1 = u_2), \tag{A.26}$$

$$x_1^c f(s_1^k / x_1^c, 1) = c_1 + i_1, \tag{A.27}$$

$$(1-\delta) s_0^k = s_1^k - i_1, \tag{A.28}$$

$$\xi x_0^e = x_1^e + x_1^c + x_1^l. \tag{A.29}$$

The seven equations (A.23)-(A.29) define a system of seven equations in the seven unknowns $\left(s_1^k, i_1, c_1, x_1^e, x_1^c, x_1^l, v_1\right)$.

Step 3. We proceed to solve these equations as follows. Recall that $A = c_1/x_1^l$, $B = s_1^k/x_1^c$. Rewriting equation (A.23) as

$$\frac{1-\alpha}{\alpha} A = (1-\theta) B^{\theta}, \tag{A.30}$$

allows us express A as a function of B:

$$A[B] = \frac{\alpha(1-\theta)}{1-\alpha} B^{\theta}. \tag{A.31}$$

We define the following mapping. Begin with an arbitrary positive value for B. Then compute A by (A.31). Now equations (A.24) and (A.25) comprise two simultaneous equations in (σ_1, v_1). Solve them. This leaves us with the four equations (A.26)-(A.29), which are now linear equations in the primal variables, once A, B and σ_1 are specified constants. To these, append the equations:

$$\sigma_1 x_1^e = s_1^k, \quad A x_1^l = c_1.$$

We now have a linear system of six equations in the six date-one primal variables. Solve them, and define $\hat{B} = s_1^k / x_1^c$. A fixed point of the mapping $B \to \hat{B}$ generates a solution to the seven equations (A.23-A.29) in the six primal variables plus v_1.

We find the fixed point of this mapping for the stipulated economy. (See the available *Mathematica* program.) We find that $v_1 = 1.01304$, and it follows that v_2 and m_1 are positive and $\sigma_1 = 0.1976 > \sigma^*$. Hence we have a solution to all the Kuhn-Tucker conditions, and hence, since PP is a concave program, to Program PP. The solution is reported in the lemma's statement. □

Proof of Theorem 9

Step 1. We first write down the Kuhn-Tucker conditions for a solution to Program *g-SUS*.

$$\begin{aligned}
(\partial \Lambda): &\quad 1 = \textstyle\sum_1^{\infty} r_t (1+g)^{t-1}, \\
(\partial c_t): &\quad r_t u_1[t] = a_t, \\
(\partial x_t^l): &\quad r_t u_2[t] = d_t, \\
(\partial x_t^e): &\quad \xi d_{t+1} = d_t, \\
(\partial x_t^c): &\quad a_t f_2[t] = d_t, \\
(\partial s_t^k): &\quad a_t f_1[t] + (1-\delta) b_{t+1} = b_t, \\
(\partial i_t): &\quad a_t = b_t.
\end{aligned}$$

In addition, let all the primal constraints hold with equality. We shall attempt to solve all these equations for a balanced growth path.

On such a path, $u_j[t] = u_j[1]$ and $f_j[t] = f_j[1]$ for $j = 1, 2$ and $t \geq 1$. The primal and dual equations yield the following substantive relations on a balanced growth path for the economic variables:

$$i_1 = (g + \delta)s_0^k, \tag{A.32}$$

$$\xi - (1 + g) = x_1^c + x_1^l, \tag{A.33}$$

$$f_2[1] = \frac{u_2[1]}{u_1[1]}, \tag{A.34}$$

$$\xi = \frac{1 - \delta}{1 - f_1[1]}, \tag{A.35}$$

$$f((1 + g)s_0^k, x_1^c) = c_1 + i_1. \tag{A.36}$$

The other dual constraints simply define non-negative dual variables in terms of the primal variables, with one exception: we must verify that the series in the $(\partial\Lambda)$ constraint converges. Thus, given g, if we can solve the five equations (A.32)-(A.36) for $(s_0^k, x_1^c, x_1^l, c_1, i_1)$ and the series in $(\partial\Lambda)$ converges, then the balanced growth path at rate g defined by these values, along with the associated dual variables, solves the Kuhn-Tucker constraints. Modulo transversality conditions, which we will comment upon below, and since *g-SUS* is a concave program, the theorem will be demonstrated.

Step 2. From the dual K- T conditions, we deduce that $r_t = \frac{d_1}{u_2[1]}\left(\frac{1}{\xi}\right)^{t-1}$. Consequently the series in the $(\partial\Lambda)$ K- T condition defines a value for d_1 if and only if $\frac{1+g}{\xi} < 1$. This is true because by hypothesis, $g < \xi - 1$.

Step 3. Thus, it remains to solve the five equations (A.32)-(A.36). Specializing to Cobb-Douglas, we re-write the five equations as follows.

$$i_1 = (g + \delta)s_0^k, \tag{A.37}$$

$$\xi - (1 + g) = x_1^c + x_1^l, \tag{A.38}$$

$$(1 - \theta)\left(\frac{(1 + g)s_0^k}{x_1^c}\right)^{\theta} = \frac{(1 - \alpha)c_1}{\alpha x_1^l}, \tag{A.39}$$

$$\theta\left(\frac{x_1^c}{(1 + g)s_0^k}\right)^{1-\theta} = \frac{\xi - (1 - \delta)}{\xi}, \tag{A.40}$$

$$((1 + g)s_0^k)^{\theta}(x_1^c)^{1-\theta} = c_1 + (g + \delta)s_0^k. \tag{A.41}$$

Step 4. Now denote $X = \frac{x_1^c}{s_0^k}$, $Y = \frac{c_1}{x_1^l}$. Solve (A.39) and (A.40) for X and Y:

$$X = (1 + g)\left(\frac{\xi-(1-\delta)}{\theta\xi}\right)^{1/(1-\theta)},$$
$$Y = \frac{\alpha(1-\theta)}{1-\alpha}\left(\frac{\xi-(1-\delta)}{\beta\xi}\right)^{-\theta/(1-\theta)}.$$

Next, divide equation (A.41) through by s_0^k, giving:

$$\frac{c_1}{s_0^k} = (1 + g)^{\theta}X^{1-\theta} - (g + \delta), \tag{A.42}$$

which generates a *necessary condition*:

$$(1+g)^{\theta}X^{1-\theta} > (g+\delta). \tag{A.43}$$

Now, noting that $XY = \frac{c_1}{s_0^k}\frac{x_1^c}{x_1^l}$, and using (A.43), we have:

$$Z\frac{x_1^c}{x_1^l} = XY, \text{ where } Z \equiv (1+g)^{\theta}X^{1-\theta} - (g+\delta),$$

or $x_1^c = \frac{x_1^l XY}{Z}$. Using (A.38), and substituting this value for x_1^c, we can solve for x_1^l:

$$x_1^l = \frac{Z}{Z+XY}(\xi - (1-\delta)).$$

Consequently, from (A.38), $x_1^c = \frac{XY}{XY+Z}(\xi - (1-\delta))$. Thus both x_1^c and x_1^l are positive numbers. We can now use the equations to solve quickly for positive values of s_0^k, i_1 and c_1.

Step 5. We now verify (A.43). Define the function $\Upsilon(g) = (1+g)\frac{\xi-(1-\delta)}{\theta\xi} - (g+\delta)$. Check that $\Upsilon(0) > 0$ if and only if $\xi > \frac{1-\delta}{1-\theta\delta}$; but this is true because $\xi > 1$. Check that $\Upsilon(\xi-1) = (\xi-(1-\delta))\frac{1-\theta}{\theta} > 0$. Since Υ is linear, it follows that $\Upsilon(g) > 0$ on the interval $[0, \xi-1]$, demonstrating (A.43).

Step 6. We finally remark that all the transversality conditions hold because each sequence of dual variables (e. g., $(a_1, a_2, \ldots)$) converges to zero geometrically. This proves the first direction of the theorem.

Step 7. To prove the converse, let $g = \xi - 1$. On a balanced growth path, we therefore require $x_1^e = (1+g)x_0^e = \xi x_0^e$, which implies that $x_1^c = x_1^l = 0$. So no balanced growth path can be supported at the rate $g = \xi - 1$. It is obvious, *a fortiori*, that no such path exists for $g > \xi - 1$. □

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