# Catchment Areas and Access to Better Schools

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**Abstract**

We compare popular school choice mechanisms in terms of children's *access to better schools* (ABS) than their catchment area school, in districts with school stratification and where priority is given for residence in the catchment area of the school. In a large market model with two good schools and one bad school, we calculate worst-case and best-case bounds of the Boston Mechanism (BM). We find that both BM and DA convey a non-negligible risk that catchment area priority fully determines the final assignment regardless parents' preferences. Top-Trading Cycles is an alternative that provides more access to better schools than DA.

**Key words**: priorities, bad school, school choice
**JEL classification numbers**: D78 C40

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# 1 Introduction

The importance of allowing for school choice has been greatly emphasized in the literature and in the policy debate. In the past, children were systematically assigned to their neighborhood school. School choice, then, referred to residential choice or Tiebout choice– see Hoxby(2003), Black (1999), Cullen, Jacob and Levitt (2006). A large fraction of OECD countries has expanded choice in various ways in the last two decades. As described in the Friedman Foundation for Educational Choice's website: "School Choice is a common sense idea that gives parents the power and freedom to choose their child's education [...] School Choice is a public policy that allows parents/guardians to choose a school regardless of residence and location". Hence, school choice aims to improve *Access to Better Schools* (ABS) for families beyond their neighborhood school.

The main problem when introducing school choice is resolving excess demands for certain schools and defining the alternatives for applicants rejected from those schools. In most cases local authorities define priority rules that determine how applicants shall be prioritized in case of overdemand. Priority is often given to families that have a sibling in the school, that live in the school's catchment area or that have particular socioeconomic circumstances. The norms describing how applicants rejected from their first choice are reallocated define the different mechanisms that the market design literature on school choice, starting with Abdulkadiroğlu and Sönmez (2003), has analyzed extensively.

In this paper we study the most common mechanisms, the Gale Shapley Deferred Acceptance (DA), the Boston mechanism (BM), and the Top Trading Cycles (TTC), with coarse priorities defined by residence in the neighborhood of the school to break ties for overdemanded schools.[1] We introduce vertical differentiation between schools: there is a *bad* school, a school that all families believe is the worst. We study the extent to which families can move away from their neighborhood school, the school they are given priority for by the authorities (the default school when there is no school choice). For this purpose we define *Access to Better School* (ABS), which is *the expected fraction of individuals who are allocated a school that is better than their neighborhood school.*

We show that priorities may limit ABS drastically, both under DA and BM. But the instances under which DA and BM do worst in terms of ABS differ. BM does worst when the bad school is (cardinally) substantially worse than the other schools, forcing families to apply for their neighborhood school to avoid the bad school. But when the bad school is not (cardinally) too different, then truthful revelation can be optimal, leading to maximal ABS. Instead, under DA the cardinality of the bad school plays no role, but the number of children living in the good neighborhood as compared to capacity in the school does. On one extreme, when both good schools are *overprioritized* (i.e. they have no less students living in the neighborhood than capacity), *all* students will, at best, be allocated to their neighborhood school, independently of their preferences. To understand why DA fails in

[1]In some cities only a fraction of the seats are reserved for prioritized students. The seats for which no priorities apply, overdemands are resolved randomly and so our results will be relaxed. We discuss this case in the discussion at the end of the paper.

this case, consider the simple example where all schools have equal capacity and equal mass of prioritized students, that is 1/3. Clearly, no student in the catchment area of a good school can end up in the bad school under DA (access to neighborhood school is guaranteed). In other words, all student living in the catchment area of the bad school are condemned to stay there, regardless of their lottery number. Students with priority in the different good schools may want to "exchange" their slots. Nevertheless, when applying for their preferred school, since they do not have priority for it, they will have to get a higher lottery number than any of the individuals living in the bad neighborhood. In a big economy, the individual with highest lottery number in the bad neighborhood would systematically win, blocking ABS for individuals in the good neighborhoods. Only when the bad school is more *prioritized* than both good schools (i.e. it has a larger students living in the neighborhood-capacity ratio), will there be increased ABS. Students with the best lottery numbers from the bad neighborhood obtain access to a good school, and this in turn allows for an increased exchange of slots. However, DA never reaches the upper bound in ABS that BM does. This result is true for any stable mechanism, a property that has been greatly emphasized in the literature– see Roth (2008). Importantly, the result in BM stems from risk avoidance and not from stability.[2]

On the other hand TTC can be proven to dominate DA in terms of ABS. In TTC individuals preferring each others' schools can always trade (this is precisely how the mechanism operates), and so a minimum level of ABS, always higher than that under DA, is guaranteed. As will become clear TTC does not always dominate BM.

Empirical evidence on the performance of these mechanisms is scarce. The main challenge is that preferences are unobservable. When DA or TTC are implemented strategyproofness facilitates inferring preferences, but under the BM preferences need to be estimated.[3] He (2014) uses data from Beijing to perform such exercise, but his framework is not useful for our purpose since there are no residential priorities. Calsamiglia and Güell (2014) show that priorities play a large role in determining the list submitted by parents under the BM. They exploit a change in the definition of neighborhoods in the city of Barcelona to identify that a large fraction of parents apply for the neighborhood school, independently of their preferences. Calsamiglia, Fu and Güell (2014) model and structurally estimate the preferences of individuals in Barcelona and do counterfactual analysis of the allocation that would result if DA or TTC were implemented instead. Table 18 in their paper shows the results from their simulations for specific subgroups of the population. We are particularly interested in the last line which presents the assignment for families who's favorite school is not their neighborhood school. As we can see both BM and DA assign them to their neighborhood school more often then TTC. For DA, despite the fact

[2]Recall that the set of NE assignments in BM necessarily coincides with the set of stable assignments only if priorities are strict.

[3]Dur, Kominers, Pathak and Sönmez (2014) analyze data from the city of Boston, where the DA is used with neighborhood priorities, but only for half of the seats in each school. There, all schools fill their prioritized seats with neighborhood students except for the worst three schools. This is obviously no proof of our theory, but provides some indication that our results may be relevant.

that families can be truthful and reveal that they want to move out of their neighborhood, the mechanism assigns them more often to the neighborhood school, more than BM does. This is so despite the fact that incentives under BM induce a substantial fraction of families to exclude their preferred school and apply to neighborhood. On the other hand we also see that TTC clearly facilitates families moving out of their neighborhood, more than DA and BM. In the particular case of Barcelona, then, the loss due to overassignment to neighborhood school induced by both DA and BM limits the power of families' preferences to determine the allocation of students to schools.

The Cost of the Elimination of Justified Envy (Calsamiglia, Fu and Güell (2014))

| | Assigned in Zone (%) | | | Assigned to Favorite (%) | | |
|---|---|---|---|---|---|---|
| | BM | DA | TTC | BM | DA | TTC |
| All Households | 65.5 | 69.7 | 58.5 | 79.4 | 75.0 | 80.7 |
| Favorite is in Zone | 95.3 | 95.4 | 89.9 | 92.6 | 89.8 | 83.8 |
| Favorite is out of Zone | 19.5 | 30.0 | 10.0 | 58.9 | 52.0 | 76.0 |

The mechanism design literature on school choice, starting with Abdulkadiroğlu and Sönmez (2003), has studied the problem of allocating students to a set of schools. This constitutes what the literature refers to as a two sided matching problem, but with the special feature that schools, in this case composing one of the two sides of the market, cannot express their preferences over students. School preferences are substituted by priority orders determined by the central administration according to residence and other socioeconomic characteristics of the family. These centralized processes involve families submitting a list with a ranking of schools and a set of rules defining how these preferences, together with priorities, determine the final allocation. These priorities have been taken as given by the school choice literature and they are viewed as a constraint that the mechanism should respect. The focus has been on designing norms that provide parents with incentives to submit preferences to allocate children most effectively, but taking priorities as given.

The literature has emphasized different properties of the norms characterizing the mechanism: strategy-proofness, stability and efficiency. The first property consists of providing incentives to reveal true preferences independently of what other families do, referred to as the mechanism being strategy proof. The Boston mechanism (BM), one of the most widely used but also criticized mechanisms, described later in the text, lacks this property. This implies that families can get a better allocation by stating a different ranking from that defined by their true preferences. Alternative mechanisms, such as the Gale-Shapley Deferred Acceptance algorithm (DA), also described later, do have this property and therefore elicit true preferences. This greatly simplifies matters for families. DA is also valued because its resulting allocation is *stable*. Stability requires the final allocation to be such that we cannot simultaneously have 1) an individual who prefers a given school to her assigned school, and 2) the preferred school has another individual admitted with lower priority than she has for that school. Importantly, the results on DA in this paper apply to any stable mechanism. But the DA allocation is not Pareto efficient except for some specific priority structures (Ergin, 2002). Pareto-efficiency is defined as the lack of an alternative allocation that makes an individual better off without making another individual worse off. The Top Trading Cycles (TTC), also described in the next section, is strategy proof and efficient, but is not stable. There is no mechanism that has the three properties. But the efficiency costs of DA, as measured in experiments, such as Chen and Sönmez (2006), are small and so DA has actually been adopted in cities like New York and Boston, substituting the former mechanism, referred to as the Boston Mechanism.[4] Both DA and Boston, or a combination of the two (see Chen and Kesten (2013)) are the most debated alternatives (Abdulkadiroğlu and Sönmez (2003); Abdulkadiroğlu, Pathak, Roth and Sönmez (2006); Ergin and Sönmez (2006); Miralles (2008); Pathak and Sönmez (2008); Abdulkadiroğlu, Che and Yasuda (2011)). TTC was only used in New Orleans for the year 2012.

This paper suggests that, unless certain priorities are questioned, the choice between these two main mechanisms may be less important, given that in both cases the final allocation of students is largely determined by priority rules. Similarly to Miralles (2008), Abdulkadiroğlu, Che and Yasuda (2011), we follow Auman (1964) and assume that there is a continuum of individuals to be allocated to a finite number of seats in schools. Nevertheless, we include a preliminary section containing a leading example with a finite number of students. There it can be seen that our insights hold even for relatively small assignment problems.

Our base model contains a binomial priority structure (families either have priority or not) that we intuitively connect to residential priorities (or walking-zone priorities).[5] This kind of reasonable coarse priority structures lie in between two rather extreme models in the theoretical literature: the strict priority model (e.g. Ergin and Sönmez, 2006; Pathak and Sönmez, 2008) and the no-priorities model (Abdulkadiroğlu, Che and Yasuda, 2011; Miralles, 2008).[6] In our model all seats follow the same priority structure. In cities like Boston or New Orleans only a fraction of the seats are prioritized. For the remaining seats, overdemands are resolved randomly. We do not consider these non-prioritizes seats explicitly in our model, but clearly the larger the fraction of non-prioritized seats, the less relevant are our results.[7] The objective in this paper is to illustrate that priorities create problems that the literature had not emphasized enough. Hence, for ease of exposition we focus on the case where all seats are prioritized.

The key additional element in our model is the existence of some degree of vertical

[4]Experiments evaluating the efficiency cost have been done in the lab, and the simulated environments used did not contain *bad schools*, as we model them here or are found in the data. This paper suggests that under the presence of bad schools efficiency losses may be very large, since preferences may have a rather small effect on the final allocation.

[5]Extensions including sibling priorities and low-income / bad-neighborhood priorities can easily be included and are available upon request.

[6]All these papers discuss their models beyond the adopted extreme assumption, yet their most illustrative proofs rely on them. See Ergin and Erdil (2008) for an exception that formally analyzes weak priority structures.

[7]We explicitly discuss the the case with a fraction of prioritized seats at the end of the paper.

differentiation across schools, so that there is at least one bad school that is homogeneously thought to be the worst. We argue that this is an element that we typically find in school choice realities. In most cities where school choice is implemented there is a set of so-called failing schools, schools that are explicitly considered *bad* by local authorities. In the US the requirement of the federal No Child Left Behind Public Choice Program requires that local school districts allow students in academically unacceptable schools (F-rated schools) to transfer to higher performing, non-failing schools in the district– if there is capacity available.[8]

This paper emphasizes that the two most debated and used mechanisms in the literature and in the policy debate may both be very limited in their capacity to allocate children according to their preferences whenever there are coarse priorities to break ties that impose a different ordering across different schools. Priorities limit the extent to which families' preferences determine the final allocation. The mechanism design literature should incorporate the design of the priority structure explicitly, since it may deem crucial for the properties of the final allocation.

The results of this paper are also important for the empirical literature in the economics of education that evaluates the impact of school choice on school outcomes– see Lavy (2010), Hastings, Kane and Staiger (2010). This literature assumes that implementing choice implies that preferences will affect the allocation of children to schools. But these empirical papers ignore how allocation mechanisms are affected by the priority structure and therefore they may be attributing the effects to the wrong source of variation.

The results in this paper are presented in a rather stylized model for ease of exposition. The model should facilitate understanding the intuition for the results without harming its perceived robustness. Next we present the mechanisms and an example that illustrates the main intuitions underlying our general results . Section 4 presents the main results, evaluating the performance of the different mechanisms. Section 5 provides some discussion and section 6 concludes. Appendix A contains a discussion of the discrete model and Appendix B contains all proofs.

# 2 The Mechanisms

The mechanisms we compare are the Deferred Acceptance (DA), the Boston Mechanism (BM) and the Top-Trading Cycles (TTC). In all these mechanisms, parents (students) are requested to submit a ranked list of schools. The student's strategy space is the set of all rankings among the schools. Each student may belong to the catchment area of a school. Belonging to a school's catchment area is the main priority criterion when resolving excess demands. Additionally, a unique lottery number per agent breaks any other eventual tie. The outcome of the lottery is uncertain at the moment students submit their lists.

[8]See Title I Public School Choice for schools identified as Low Performing: http://www.ncpie.org/nclbaction/publicchoice.html.

**Deferred Acceptance (DA):**

- In every round, each student applies for the highest school in its submitted list that has not rejected her yet.
- For every round $k$, $k \geq 1$: Each school tentatively assigns seats to the students that apply to it or that were preaccepted in the previous round following its priority order (breaking ties through a fair lottery)[9]. When the school capacity is attained the school rejects any remaining students that apply to it in that round.
- The DA mechanism terminates when no student is rejected. The tentative matching becomes final.[10]

**Boston Mechanism (BM):**

- In every round, each students applies for the highest school in its submitted list that has not rejected her yet.
- For every round $k$, $k \geq 1$: Each school assigns its remaining seats to the students that apply to it following its priority order, and breaking ties through a random lottery when necessary. If the school capacity is or was attained, the school rejects any remaining students that point to it.
- The Boston mechanism algorithm terminates when all students have been assigned to a school, in at most three rounds.

**Top-Trading Cycles (TTC):**

- In each round, we find a cycle as follows. Taking a school $s$, we choose its first student in the priority list, $i$. This student points at her most preferred school $s'$, which points at its highest-priority student $i'$, etc. A cycle is always found because there is a finite number of schools.
- We assign to each student of the cycle a slot of the school she points at. We remove these students and slots.
- We repeat the process round by round (having erased completely filled schools from students' lists and assigned students from schools' priority lists) until we have assigned all the students.

[9] We assume that there is a single tie-breaker that serves to break ties when necessary at all schools. We justify this assumption on Abdulkadiroğlu, Che and Yasuda (2014), which shows that a multiple tie-breaker (one for each school) in DA would lead to Pareto inefficient assignments.

[10] Abdulkadiroğlu, Che and Yasuda (2014) show that this algorithm converges to an assignment in big continuum economies, even though not necessarily in finite time.

# 3 A Finite Economy Example

Our main model uses a continuum economy for ease of exposition. In this section we illustrate the main insights of the paper through an example with a finite set of individuals. We have three neighborhoods with $n$ families living in each of them, and each with a school of capacity $n$. Let $i \in \{i_1, i_2, i_3\}$ denote that the individual lives in the neighborhood of school $s \in \{1, 2, 3\}$, and therefore has priority at school $s$. Ties in priorities given by residence are broken through a unique fair lottery. Individual preferences can be representen through the following von Neumann - Morgenstern valuations for schools, where $v \in (0, 1)$:

| *type**school* | 1 | 2 | 3 |
|---|---|---|---|
| $i_1$ | $v$ | 1 | 0 |
| $i_2$ | 1 | $v$ | 0 |
| $i_3$ | 1 | $v$ | 0 |

Note that we have constructed an example where potential for choice benefiting students is maximal. That is, we have assumed all students would rather be assigned to a different school than their neighborhood school. Any Pareto Efficient Allocation in this example involves having no pair of students in school 1 and 2 be assigned to their own neighborhood school, since they would be better off by trading their seats (individuals living in neighborhood 1 prefer attending $s_2$ and viceversa).

In Deferred Acceptance, students with priority at a good school have guaranteed assignment to a good school. This implies that all students of the $i_3$ type are eventually assigned to the worst school. Students $i_1$ and $i_2$ would like to "exchange" their guaranteed slots. But through the DA, since they do not have priority for their preferred school, this "exchange" will only happen if two individuals, an $i_1$ and an $i_2$, get a better lottery number than the $i_3-$student with the highest lottery number. The student $i_3$ with the highest lottery number can block each trade with a probability higher or equal to 2/3. The probability of blocking the $x$-th exchange rapidly increases with $x$, since not doing so requires both $x$-th best lottery numbers in $i_1$ and $i_2$ to beat the best lottery number in $i_3$. In the Appendix we provide the exact method of calculation of the chances to obtain exactly $x = 1, .., n$ exchanges. Thus, Table 1 presents the results from calculate the expected proportion of students that obtain access to a better school than the catchment area school. This percentage rapidly decreases to zero as $n$ grows large. Even with $n$ being small, the percentage is dramatically low (1.77% with twenty students per school).

This trade-blocking cannot happen under TTC. School 1 points at a student with type $i_1$, who points at school 2, which points at a student with type $i_2$, who points at school 1, and the cycle is closed. Students from types $i_1$ and $i_2$ are assigned to a school that is better than their priority-giving schools regardless of how lucky students of type $i_3$ are with their lottery numbers.

As for BM, we distinguish two different cases. In case 1, where $v = 0.9$, the bad school is much worse than the second-best school for every student. The unique Nash equilibrium (in undominated strategies) involves students of types $i_1$ and $i_2$ ranking their respective neighborhood school first, and each student is assigned his neighborhood school. Hence ABS in this case is equal to 0.

In case 2, where $v = 0.1$ the bad school is similar to the second-best school for every student. Even though the former Nash equilibrium still exists, another Nash equilibrium (in undominated strategies) exists in which all students submit a truthful ranking over schools. For any $n$, all students of type $i_1$ obtain a slot at school 2, while students of types $i_2$ and $i_3$ have fifty per cent chances of obtaining a slot at school 1, and fifty per cent to obtain a slot at the worst school. In case 2 then, an equilibrium exists in which all the slots at good schools are given to students who prefer these more than their priority-giving schools. Hence, in this case ABS is maximal and equal to 67%.

Table 1 summarizes the expected number of students who obtain a better placement than the school for which they had priority (weighted by the size $3n$ of the market).

**Table 1**: Expected percentage of students who get Access to Better School (ABS)

| Mechanism \ n | 1 | 2 | 5 | 10 | 20 | $\infty$ |
|---|---|---|---|---|---|---|
| BM ($v = 0.9$) | 0 | 0 | 0 | 0 | 0 | 0 |
| DA | 22 | 13.3 | 6.3 | 3.4 | 1.77 | 0 |
| BM ($v = 0.1$) | 66.7 | 66.7 | 66.7 | 66.7 | 66.7 | 66.7 |
| TTC | 66.7 | 66.7 | 66.7 | 66.7 | 66.7 | 66.7 |

BM (case 1) obtains the worst performance possible in terms of access to a better school. The assignment is fully determined by the priority structure. BM (case 1) performs worse than DA, although insignificantly so as $n$ goes large. In fact both perform extremely bad. Notice how the expected percentage of students who improve upon their catchment areas rapidly decreases to zero under DA. With ten students per school, only 3.4% of them are expected to obtain a better assignment outside their catchment areas. With twenty students per school, this percentage is 1.77%. This exemplifies that the bad results obtained by DA later in the model are not an artifact of the continuum.

In contrast, BM (case 2), which performs as good as TTC, obtain maximum access to better schools for any $n$. The comparison between DA and BM crucially depends on the intensity of students' preferences between the second-best school and the worst school. In case 1, preferences lead to a (bad) equilibrium in BM where agents manipulate preferences and use safe options. In case 2, students do not strategize, hence reaching a "good" equilibrium that outperforms DA.

Although TTC and BM (case 2) are similar in terms of ABS, they are very different in other respects. While TTC ensures that students with priority at good schools do not worsen their positions, BM (case 2) involves a risk of ending in the worst school. At the same time, BM (case 2) gives chances to $i_3$, the student with priority the bad school, of getting access to a better school. In the discussion at the end of the paper we comment on the limitations of our ABS measure.

# 4 The Model

We present a simple model that sufficiently illustrates our insights.[11] We have a unit mass of students $i \in [0,1]$, each of them to be allocated to one of three schools. Two of the schools are "good" and one is "bad", in the sense that all students rank it worse. Good schools are labelled 1 and 2, respectively, whereas the bad school is labelled $w$. Schools have capacities $C_1$, $C_2$ and $C_w$ that add up to one. Students have cardinal preferences over the schools. We represent them by a measurable vNM valuation for the second-best school $v : [0,1] \to (0,1)$. We normalize valuations so that each student's preferred school has valuation 1 and the least preferred school ($w$) has valuation 0. No student is indifferent between any two schools. Ordinal preferences are more extensively explained below.

There is a measurable catchment area function $\pi : [0,1] \to \{1,2,w\}$. Each student has a unique catchment area where she has priority over students outside the catchment area. Other ties are resolved when needed using a *fair* lottery outcome $n : [0,1] \to [0,1]$ that assigns one number to each student. We apply the convention that a lower lottery number beats a higher lottery number.

There is a mass $N_1$, $N_2$ and $N_w$ of students for the catchment areas of schools 1, 2 and $w$, respectively. Belonging to the catchment area of a school gives priority there over students outside its catchment area. Without loss of generality we assume that students in the catchment area of the good school actually prefer the other good school.[12] Regarding the students at the bad school's catchment area, $N_{w1}$ students prefer school 1, and $N_{w2} = N_w - N_{w1}$ students prefer school 2.

For each school $s$, define $\rho_s = C_s/N_s$. We say school $s$ is *overprioritized* if $\rho_s < 1$ (capacity is smaller than the number of individuals with priority in the school), and underprioritized in the opposite case. Notice that we cannot have the three schools being either all overprioritized or all underprioritized, since we have assumed that total capacity is equal to total mass of students. For two schools $s$ and $s'$ we say that $s$ is more prioritized than $s'$ if $\rho_s < \rho_{s'}$. This variable comes out to be important when comparing the performance of the studied mechanisms.

For each assignment mechanism we compare the mass of students who obtain a slot in a school preferred to that of their catchment areas, as a measure of students' real choice (since catchment school is the default option when no choice is available). We call this measure *Access to Better Schools*, denoted *ABS*.

Pareto-domination clearly implies having higher ABS, although intuitively the converse is not true. While an ordering of mechanisms regarding ABS is calculated, we point out that it does not necessarily imply existence of a Pareto-domination ranking. Indeed,

[11] In previous versions of this paper we use a more general model with an arbitrary number of schools. We also discuss the impact of adding additional priority criteria, and we discuss finite economies as well. This material is available upon request.

[12] Those who prefer their catchment area school obtain a sure slot there, so we can safely ignore them and their occupied slots. In case some school's capacity is less than the number of prioritized students who prefer it, the model becomes uninteresting in that this school gives all their slots to prioritized students only, in all the mechanisms we study.

mechanisms that may induce higher access to better schools may do so in exchange of placing prioritized students from good schools' catchment areas to bad schools. At the same time, mechanisms that perform poorly in terms of ABS may be protecting these students from being assigned to the bad school. Finally, we also observe that rank-domination (Featherstone, 2014) does not imply (or is implied by) ABS domination. A mechanism may be good in placing students from the worst school's catchment area in either one of the good schools, while being bad in placing them to her favorite good school. Another mechanism could rank-dominate the former and at the same time it could be dominated by it in terms of ABS.

## 4.1 Worst and Best Cases under the Boston Mechanism

It is well known that both DA and TTC are strategy-proof: there is a dominant strategy equilibrium in which all students submit the true ranking of schools according to their preferences. It is also known that agents have incentives to manipulate their rankings in BM, depending on their cardinal preferences– see for example Abdulkadiroğlu and Sönmez (2003), Miralles (2008) or Abdulkadiroğlu, Che and Yausda (2011). We illustrate two extreme cases that may constitute part of a Nash equilibrium (in undominated strategies) for limit preference structures.

The first extreme, which we call **Boston Mechanism (worst case)**, involves every student manipulating her preferences in order to minimize the probability of being assigned the bad school. More precisely, *for each underprioritized good school, all the prioritized students rank it first. And for every overprioritized good school, the number of prioritized students who rank it first exceeds the school's capacity.* This maximum manipulation arises as the unique Nash equilibrium outcome when the valuation of the bad school is sufficiently bad for every student. It is also one, yet not necessarily unique, Nash equilibrium prediction when both good schools are overprioritized, regardless of how worse the bad school is (compared to the second school). It is easy to envision that this maximal manipulation equilibrium leads to a dreadful level of access to better schools, since prioritized students use their safest options, resigning from achieving a slot in a better school while blocking others from getting access to good schools.

**Lemma 1:** There is $\tilde{v} \in (0, 1)$ such that if the range of $v$ lies strictly above $\tilde{v}$, every Nash equilibrium (undominated strategies) of the game induced by BM meets the worst case of the Boston Mechanism.

Appendix A contains the proofs to all lemmas and propositions.

On the other extreme we have the **Boston Mechanism (best case)**, where *agents submit a ranking according their true preferences.* This is a limit Nash equilibrium prediction when the valuation every agent has for the worst school is almost as good as the valuation for her second-best school. That is, there is almost no punishment for being rejected in the first round of the assignment algorithm, thus being sincere is optimal. In this case it is clear that access to better schools is dramatically increased. Prioritized students at good schools make no use of this privilege, aiming to better schools and thereby letting others get access to their preferred schools.[13]

**Lemma 2:** There exists $\bar{v} \in (0,1)$ such that if the range of $v$ lies strictly below $\bar{v}$, there is a Nash equilibrium (undominated strategies) of the game induced by BM that coincides with the best case of the Boston Mechanism.

The Nash equilibrium (in undominated strategies) outcome of the Boston Mechanism is neither unique nor predictable without knowing the distribution of cardinal von Neumann - Morgenstern utilities. Fortunately, we can state that the access to better schools measure would stay between the lower bound and the upper bound provided by the worst case and best case scenarios. Interestingly, this is also true if there is a proportion of nonstrategic students that compulsory rank the schools according to their true preferences.

We calculate ABS under BM (worst case and best case). In the worst-case scenario, agents aim to minimize the probability of ending assigned at the bad school. A school $s$ is *overdemanded* if the mass of students ranking it in first position exceeds its capacity. School $s$ is *underdemanded* otherwise. Notice that since ranking the bad school other than last is part of a dominated strategy, it cannot be the case that both two good schools are underdemanded. Here is our result. Let $\underline{ABS}= \sum_{s\in\{1,2\}} \max\{0, C_s - N_s\}$ be the lower bound for the access to better schools that can be attained by any mechanism. Essentially, every slot at each good school is given to a prioritized student, until either all students of its catchment area have been already assigned or the school capacity is filled.

**Proposition 1** $ABS_{BMworst} = \underline{ABS}$.

In the Boston Mechanism (best case), all the students truthfully rank their schools according to their preferences. As in the previous case, it cannot be that both two good schools are underdemanded in the first round of the assignment algorithm. Thus, either $N_1 + N_{w2} > C_2$ or $N_2 + N_{w1} > C_1$ (or both). Let $\overline{ABS} = C_1 + C_2$ be the obvious upper bound for the access to better schools that can be attained by any mechanism. If both schools have sufficient number of students who have it as its first best, then maximal access to better schools is attained, since both good schools are filled by students who desire it most. If one of the schools does not have a sufficient amount of students who prefer it, $N_o + N_{wo} < C_u$, then the remaining capacity after the first round, $C_u - N_o + N_{wo}$, will be filled by those who have priority at it that did not get their preferred school. If there is still capacity after that, then it will be filled by individuals from the bad neighborhood. Therefore, the only seats that will not be used to improve access to better schools will be the capacity available in the second round occupied by applicants to the overdemended school that did not get in and that live in the underdemanded school.

**Proposition 2** *1) If both schools are filled when all students who have it as their first best demand it, that is,* $N_1 + N_{w2} > C_2$ *and* $N_2 + N_{w1} > C_1$*, then* $ABS_{BMbest} = \overline{ABS}$.

[13]See Kojima and Ünver (2014) for a characterization of the Boston Mechanism under truthtelling.

*2) If one school is not filled when all students who have it as their first best demand it (label this school as u and the other school as o), that is, if $N_o + N_{wo} < C_u$, then* $ABS_{BMbest} = \overline{ABS} - \min\left\{N_u\left(1 - \frac{C_o}{N_u + N_{wo}}\right), C_u - N_o - N_{wu}\right\}$.

## 4.2 Access to better schools under Deferred Acceptance

The properties derived from DA in this paper are direct implications of it being stable. In other words, the results in DA would result from any stable mechanism. Stability in our framework limits ABS largely, because it leads to individuals in the bad neighborhoods blocking pareto improving exchanges between individuals in the good schools. Individuals in the good neighborhoods will apply for their first best and if they do not get in they will apply and get their neighborhood school, capacity permitting. Therefore no individual living in a good neighborhood will be placed in the bad school, unless her neighborhood school is overprioritized. If both schools are overprioritized, then clearly no individual from the bad neighborhood will have access to a good school. But by stability then, no individual from a good neighborhood can get better access, since, lacking priority to their preferred school requires that they get a higher lottery number than *any* student in the bad neighborhood. Therefore ABS in that case is 0.

On the other hand, if one school is underprioritized, $C_h - N_h > 0$, then some trade may arise. The intuition is as follows. Since $C_h > N_h$ some seats at $h$ are available that no individual living in $h$ will "claim" back. This implies that some individuals from the bad and from the other good neighborhood will access those seats and thereby achieve improved access. Additional access may be provided if the individuals from the other good school that access school $h$ do not fill their own school, creating some leftover capacity to be filled by non-prioritized students (this will depend on how overprioritized the other good school is). This process may lead to ABS being substantial in this case. The following proposition formally states these results.

**Proposition 3** *1) If both good schools are (weakly) overprioritized,* $ABS_{DA} = 0$.

*2) If one school (h) is underprioritized and the other one is (weakly) more prioritized than the bad school,* $ABS_{DA} = C_h - N_h$.

*3) If one school is underprioritized and the other one is less prioritized than the bad school,* $ABS_{DA} = \overline{ABS} - N_h(1 - c_2) - N_2(1 - c_h)$.

where $c_s \in [0, 1]$, are cut-offs such that no non-prioritized student with lottery number above the cut-off can obtain access to school $s$. The Appendix provides the exact analytical expressions for these cut-offs. Notice that in cases 1 and 2 DA reaches the lower bound of ABS, that is, $ABS_{DA} = \underline{ABS}$. More generally, this "trade blocking" systematically happens as long as $\min\{\rho_1, \rho_2\} \leq \rho_w$. Performance under DA improves when schools are less prioritized. As long as more slots at good schools can be given to students from the catchment area of the bad school, access to better schools is less blocked. Satisfying students with best lottery numbers from the bad school catchment area allows students from the catchment area of good schools to "exchange slots", as long as they also have sufficiently good lottery numbers. However, DA never reaches the upper bound $\overline{ABS}$.

## 4.3 Access under Boston versus Deferred Acceptance

We can now provide a comparison between BM and DA regarding access to better schools.

**Proposition 4** *Access to better schools under BM and DA:*
*1) If* $\min\{\rho_1, \rho_2\} \leq \rho_w$*, then* $\underline{ABS} = ABS_{DA} = ABS_{BMworst} < ABS_{BMbest}$.
*2) If* $\min\{\rho_1, \rho_2\} > \rho_w$*, then* $\underline{ABS} = ABS_{BMworst} < ABS_{DA} < ABS_{BMbest}$.

We first consider the case where at least one good school is more prioritized than the bad school. This case arises, for instance (though not exclusively), when the bad school is underprioritized. Deferred Acceptance provides the minimum possible level of access to better schools, coinciding with the worst case of the Boston Mechanism. Yet the best case scenario of the Boston Mechanism provides higher access to better schools. In some cases, it reaches the maximum attainable level of access. In such environments, switching from BM to DA can only worsen the level of access to better schools.

Secondly, we consider the case where both good schools are less prioritized than the bad school. This case arises, for instance (though not only), when both good schools are underprioritized. While the worst case scenario of BM still obtains the minimal level of access to better schools, DA improves upon that. Yet it does reach the access level of the BM (best case), which is maximal in case of overdemands. In such environments, the comparison between DA and BM is ambiguous, crucially depending on the level of preference manipulation under BM.

## 4.4 An Alternative: Top Trading Cycles

An alternative would be Top Trading Cycles (TTC), which has been proven to be strategy proof and efficient. With TTC the lower bound $\underline{ABS}$ is never reached. The reason is simple. In our model we have that all $i_1$ individuals prefer school 2, and viceversa. Therefore, at least $2\min\{C_1, N_1, C_2, N_2\}$, agents with sufficiently good lottery numbers and priority at a good school get access to their preferred school through exchanging their priorities for the good schools. Contrarily to what happens at DA, students from the worst school's catchment area cannot "block" this trade. We provide a full calculation of ABS given by TTC in the Appendix (Proposition 7). The next result summarizes comprehensible corollaries that help us characterize how good TTC is regarding ABS.

**Proposition 5** *1)* $\overline{ABS} - ABS_{TTC} \leq |\min\{C_1, N_1\} - \min\{C_2, N_2\}|$.
*2) This upper bound is not necessarily binding. There are cases in which* $|\min\{C_1, N_1\} - \min\{C_2, N_2\}| > 0$ *and* $ABS_{TTC} = \overline{ABS}$.
*3) In all cases we have* $ABS_{TTC} > \underline{ABS} = ABS_{BMworst}$.

TTC clearly outperforms DA when $\min\{\rho_1, \rho_2\} \leq \rho_w$, that is, when DA obtains minimal access to better schools. The following proposition shows that this also true in general.

**Proposition 6** *TTC dominates DA in terms of ABS, that is,* $ABS_{TTC} > ABS_{DA}$.

We are not able to provide a clear characterization of the ordering between TTC and BM with respect to ABS. There are environments in which TTC reaches maximal ABS whereas BM (best case) does not, and cases in which the opposite happens. We provide examples illustrating both possibilities.

**Example 1:** When $N_1 = N_2 < \min\{C_1, C_2\}$ and some good school is underdemanded under BM (best case), we have $\overline{ABS} = ABS_{TTC} > ABS_{BMbest}$. □

**Example 2:** When $\min\{N_1, N_2\} > \max\{C_1, C_2\}$ and $C_1 \neq C_2$, we have: $\overline{ABS} = ABS_{BMbest} > ABS_{TTC} = 2\min\{C_1, C_2\}$. □

ABS is an aggregate measure of choice that abstracts from potentially important aspects when allowing families to choose. For instance, BM (best case) may outperform TTC or vice versa, but qualitatively there are differences in the identity of the agents who obtain a better choice outside their catchment area. In TTC, no prioritized student from an underprioritized good school could ever be assigned to a bad school. Hence, access to better schools arises primarily because students with priority at good schools exchange their positions. Note that students from the bad school's catchment area get no indirect benefit from this. Conditional on not having traded a good school seat, the prioritized student's lottery number is worse than that of the students from the bad school's catchment area. This way the latter students may obtain access to better schools as well. In BM (best case), the first round works as if no priorities existed. Prioritized students at good schools apply for a different school, directly emptying a slot for other students. Consequently, chances are that a student from a good school's catchment area ends assigned at the bad school. The good side of this is that students from the bad school's catchment area have more chances to get access to a good school.

# 5 Discussion

## 5.1 Reducing the Number of Prioritized Seats

In cities such as Boston, it is well-known that for almost every school only half of available seats are prioritized according to the catchment area priority criterion. That is, the catchment area only has a bite in allocating half of the slots. The other half is assigned according to the lottery number only. The assignment process in Boston lies between our base model, where all the slots are prioritized, and a school assignment problem with no catchment area priorities. Not surprisingly, one would expect that access to better schools is enhanced by lowering the proportion of prioritized seats.

For each good school we can distinguish three cases. In the first case, the number of prioritized students ultimately applying for a slot at the school is lower than the number of prioritized seats. The outcome becomes identical to that of the base model. In the second case, the number of prioritized students applying for a slot at the school exceeds the number of prioritized seats, and the minimum lottery number among rejected students is too high to compete for non-prioritized seats. In that case, the number of prioritized seats acts as a cap to the loss with respect to $\overline{ABS}$. In the third case, the number of prioritized students applying for a slot at the school exceeds the number of prioritized seats, and the minimum lottery number among rejected students is sufficiently low to compete for non-prioritized seats. In that case, it is as if no catchment area priority existed.[14]

With respect to our base model, as the number of prioritized seats tend to zero, DA and TTC tend to be identical, both approaching the assignment of Random Serial Dictatorship. As for BM, its Nash equilibrium outcome tends to the competitive equilibrium outcome of a Pseudomarket with equal incomes, which is naturally ex-ante efficient (Miralles, 2008).

## 5.2 Access to Better Schools and other Measures

A fair objection to the concept of Access to Better Schools is that it ignores the assignment of those who did not improve upon their catchment area school. If one understands "choice" as a measure of how students stand with respect to their catchment areas, we should also take count of those students who actually end worse-off than when assigned to their catchment area schools. This count, that could be named Access to Worse Schools ($AWS$), would serve to calculate a "net" $ABS$, $NABS = ABS - AWS$.

Indeed one could imagine the following procedure in a scenario with $N_1 + N_2 < C_w$. First, send all students with priority at a good school to the bad school. Then, fill the remaining positions according to any prefixed algorithm. This procedure provides maximal ABS. This mechanism seems somewhat unintuitive and pervasive. Nevertheless, what lies behind this procedure is that students from the bad school catchment area have priority *at all schools* over students from good school catchment areas. For instance, the San Francisco Unified School District gives highest priority in all schools to families living in areas with "bad schools" (the lowest 20% percentile of average test scores).[15] According to the No Child Left Behind initiative, access to better schools is particularly important for these families. ABS accounts for it, yet NABS does not, due to a crowding-out effect. For each student from a "bad school" area that manages to enter a good overdemanded school, there is a student from that good school catchment area that is assigned to a worse school. Or else she opts for an outside option, namely a private school.

Going to out theoretical model, let us compare BM (best case) with TTC under the assumption that $\min\{N_1, N_2\} > C_1 = C_2$. Both mechanisms obtain maximal ABS. Nevertheless, TTC is ranked above BM (best case) in terms of NABS, since $AWS_{BMbest} > AWS_{TTC}$. More students with priority at a good school are assigned to the bad school under BM than under TTC. Ranking TTC over BM (best case) implies that placing students with priority at a good school better has more weight than placing students from the bad school catchment area in a better school. In some sense, condemning students from the bad school

[14]Dur, Kominers, Pathak and Sönmez (2014) show that the order in which the non-prioritized seats is distributed matters greatly in determining the number of neighborhood individuals who are accepted in their first choice.

[15]http://www.sfusd.edu/en/assets/sfusd-staff/enroll/files/2012-13/annual_report_march_5_2012_FINAL.pdf, page 81.

catchment area to stay there is rewarded, if NABS is used as an aggregate measure of choice.

Another alternative aggregate measure of school choice is the mass of students that obtain Access to their Favorite School ($AFS$). This approximation to students' satisfaction takes into account that ending in the actual top choice should not count equally as ending in the second-best school. Incidentally, in the context of the base model, AFS domination is equivalent to rank domination (since the mass of agents who obtain a slot in the least-preferred school is the same in every mechanism). However, we chose ABS because it counts students' chances to *escape from their catchment areas* towards a preferred school. The quantitative difference between AFS and ABS is the number of students from the bad school's catchment area that are assigned to their second-best schools.[16]

# 6 Conclusions

Since Abdulkadiroğlu and Sönmez (2003) the Boston Mechanism has been widely criticized in the school choice literature. Since then many cities around the world have substituted this mechanism by the Gale Shapley Deferred Acceptance mechanism.[17]Deferred Acceptance has been adapted from matching theory as a good alternative, since it is not manipulable, it protects nonstrategic parents and provides more efficient assignments in setups with strict priorities. The debate between these two mechanisms was based upon models that assumed extreme scenarios in the priority structure: either no priorities or strict priorities, and did not incorporate some important realities about the schools system, such as the vertical differentiation among schools. We solve a simple model of school choice with coarse residential priorities and vertical differentiation separating good from bad schools. We show that if school choice aims to improve *access to better schools* than the neighborhood school, then both mechanisms are likely to perform very poorly. We illustrate that the priority structure, under the presence of a stratified school system, can determine the final allocation to a great extent in both of these mechanisms.

The range of possible ABS outcomes of the Boston Mechanism is very wide, covering the lower and the upper bound of all possible ABS levels. The outcome depends on the degree of preference manipulation among participating parents. Manipulation harms access to better schools via the use of safe options, while sincerity is good because it makes improvements possible directly, but also because it empties a slot in a school (the neighborhood school) that is better preferred by other students. In districts where some schools are perceived as very bad, avoiding them becomes a primary objective, manipulation naturally arises. In school districts where vertical differentiation is rather low, sincere school application strategies are less punished, and ABS can be large.

We have seen that Deferred Acceptance unambiguously provides less access to better

[16]Incidentally, Table 1 in the preceding example would also report a measure of AFS. Both AFS and ABS may coincide numerically in several scenarios.

[17]See Pathak and Sönmez (2013) for evidence on the number of cities around the word where the Boston mechanims has been banned.

schools than the Boston Mechanism when some good school is relatively more prioritized (in terms of number of prioritized students divided by school capacity) than the bad school/s. ABS, in that case, reaches the lower bound of all possible outcomes. ABS improves under DA when all good schools are less prioritized than the bad school/s. The opportunity for students from a bad school's catchment area to occupy slots at good schools makes priority trading among prioritized students at good schools possible. However, access to better schools under Deferred Acceptance is always inferior to that of the Boston Mechanism if parents rank schools according to their true preferences.

We have also discussed a third, natural alternative in this debate, which is Top-Trading Cycles. TTC is more immune to the priority structure because prioritized students at good schools are allowed to trade their slots with no interferences from students of a bad school's catchment area. Top-Trading Cycles obtains higher access to better schools than Deferred Acceptance. It therefore constitutes a safe mechanism with respect to both the Boston Mechanism and Deferred Acceptance, in school choice problems where coarse zone priorities exist.

More generally this paper puts forth the extreme relevance that neighborhood priorities can have on the final allocation of students to schools, inhibiting the role that preferences may have in determining the final allocation. The literature has deemed these priorities as exogenous, but ultimately they constitute a key feature of the final assignment that the administration can a do change whenever needed.[18] Future work should incorporate the design of these priorities as a fundamental part of the mechanism design problem.

# 7 Appendix

## 7.1 Appendix A: ABS under DA in the finite economy example.

As said in the main text, students with priority at different good schools would like to "exchange" their guaranteed slots, yet then the students from the bad school catchment area may block this trade. We want to derive the chances of exactly a number $x$ of exchanges occurring. In order to gain more understanding we illustrate a simple case where $n = 2$ and $x = 1$. We calculate all the cases in which this event happens. It could be that the two top-ranked students in the tie-breaking lottery are one student of type $i_1$ and another one of type $i_2$, and the third-ranked student is $i_3$. We could have picked $\binom{2}{1} = 2$ students from each type, and the order between types $i_1$ and $i_2$ does not matter (there are $2! = 2$ ways to arrange them). There are also $(6-3)!$ ways to arrange the remaining students among themselves. Hence we find $2 \cdot 2 \cdot 2 \cdot 2! \cdot 3! = 96$ lottery outcomes satisfying this condition. But we have not covered all cases. It could also be that two students of type $i_1$ and another one of type $i_2$ occupy the first three positions in the lottery ranking, while the fourth position is occupied by an $i_3$ student. In this case There is only one way, or $\binom{2}{2}$, to pick two students out of the two existing $i_1$ students. We could still pick $\binom{2}{1} = 2$

[18]In cities such as Madrid, Barcelona, Boston, San Francisco and New Orleans, among others, priorities have changed over the last decade.

students from each of the other types. The way we arrange the two $i_1$ students and the $i_2$ student does not matter (there are 3! combinations). There are $(6-4)!$ ways to arrange the remaining students. We have found other $1 \cdot 2 \cdot 2 \cdot 3! \cdot 2! = 48$ such lottery outcomes. This number has to be multiplied by 2, to cover the final yet symmetric case in which two students of type $i_2$ and another one of type $i_1$ occupy the first three positions in the lottery ranking, while the fourth position is occupied by an $i_3$ student. We obtain a total of 192 favorable cases out of $6! = 720$ possible lottery outcomes. The probability of exactly one exchange with two students per school is $P(1,2) = \frac{4}{15}$. More generally

$$\begin{aligned}
P(x,n) &= \frac{1}{(3n)!}[\binom{n}{x}\binom{n}{x}n(2x)!(3n-2x-1)!+ \\
&\quad +2\sum_{i=x+1}^{n}\binom{n}{x}\binom{n}{i}n(x+i)!(3n-x-i-1)!] \\
&= \binom{n}{x}\left[\frac{n}{3n-2x}\frac{\binom{n}{x}}{\binom{3n}{2x}}+2\sum_{i=x+1}^{n}\frac{n}{3n-x-i}\frac{\binom{n}{i}}{\binom{3n}{x+i}}\right]
\end{aligned}$$

Let $X(n)$ denote the expected percentage of students that obtain a slot in a school better than their catchment area school under DA, when each school has $n$ slots and $n$ prioritized students. Then

$$X(n) = \frac{2}{3}\frac{1}{n}\sum_{x=1}^{n} xP(x,n)$$

The $\frac{2}{3}$ fraction appears because one third of students (those with priority at the bad school) have no chance to escape from the bad school. Values for $X(n)$ are reported in Table 1 (main text). It can be shown that $X(n) \to 0$, in fact quite fast (e.g. $X(20) = 0.0177$).[19]

## 7.2 Appendix B: proofs

**Proof Lemma1.**

Denote with $\tilde{N}_s$ the mass of students with priority at a good school $s$ who rank $s$ in first position. We want to find conditions under which $\tilde{N}_s < \min\{C_s, N_s\}$ cannot occur for any $s \in \{1,2\}$ in a Nash equilibrium. By way of contradiction, we suppose that $\tilde{N}_s < \min\{C_s, N_s\}$ for some $s \in \{1,2\}$.

We restrict attention to undominated strategies, which involve ranking the worst school in last position. Therefore, the mass of students applying for either school 1 or 2 in the first round of the assignment algorithm is 1, while the overall capacity of these schools is $C_1 + C_2$. One of these schools must be overdemanded in the first round: we call it $o$. The other good school is denoted as $u$.

[19] In a previous version of this paper we show that if we fix a proportion of agents wishing to exchange good school slots, the probability they *all* do so shrinks to zero at factorial speed as $n$ grows.

Suppose $u$ is underdemanded in the first round. Then, among those students applying for school $o$ in the first round without priority there, the chances of ending assigned at $w$ are at least $C_w = 1 - C_1 - C_2$, since the bad school is filled only with students that used this strategy. The payoff from this latter strategy for these agents is no more than $C_1 + C_2$. Instead, the payoff from applying for the underdemanded school is higher than $\tilde{v}$.

Setting $\tilde{v} \geq C_1 + C_2$ we force every Nash equilibrium to have both good schools overdemanded. Otherwise all students without priority at $o$ would best respond by ranking the underdemanded school $u$ first. Since all students with priority at $o$ would also best respond by ranking the preferred underdemanded school $u$ first, we would contradict the fact that $u$ is underdemanded.

Since both good schools are overdemanded, every rejected student at the first round is eventually assigned at the bad school. Use $q_s$ for the Nash equilibrium probability of being accepted at $s \in \{1, 2\}$ for a student without priority at $s$ that applies there in the first round of the BM algorithm. WLOG assume $q_u \geq q_o$ and notice that it must be the case that $q_o \leq C_1 + C_2$. Let $0 < \varepsilon < C_w$. We find conditions under which every Nash equilibrium meets $q_u \leq 1 - \varepsilon$. Suppose the contrary, thus applying for this school renders a payoff of more than $(1 - \varepsilon)\tilde{v}$. Applying for the other school renders a payoff bounded by $q_o \leq C_1 + C_2$. Set $\tilde{v} \geq \frac{C_1+C_2}{1-\varepsilon}$. Under this condition all non-prioritized students at $o$ (a mass $N_u + N_w$) would best respond by ranking $u$ first. But then, since $\tilde{N}_u = N_u$, our initial assumption implies $\tilde{N}_o < \min\{C_o, N_o\}$, yielding $o$ underdemanded, a contradiction.

Finally, we fix $\varepsilon \in (0, C_w)$ and we use $q_o \leq q_u \leq 1 - \varepsilon$. Let $\tilde{N}_s < \min\{C_s, N_s\}$ for some $s \in \{o, u\}$. For a prioritized student at school $s$ applying for the other good school, the payoff is not more than $1 - \varepsilon$, whereas applying for school $s$ gives payoff above $\tilde{v}$. Set $\tilde{v} = \max\{\frac{C_1+C_2}{1-\varepsilon}, 1 - \varepsilon\}$. With this, the best response for all students with priority at $s$ is to rank $s$ in first position, contradicting $\tilde{N}_s < \min\{C_s, N_s\}$ as part of a Nash equilibrium. $\square$

**Proof Lemma2.**

We restrict attention to undominated strategies, where all students rank the worst school in last position. Suppose that the profile of submitted rankings coincides with the profile of ordinal preferences. We show that the best response for all students is precisely to rank the schools sincerely, when valuations for second-best schools are sufficiently low (capped by a properly chosen $\bar{v}$).

First, consider the case where both good schools (labelled $s$ and $s'$) are overdemanded in the first round. For a student $i$ who prefers $s$ and abides by the sincere ranking strategy the payoff is $\frac{C_s}{N_{s'}+N_{ws}}$. Instead, the payoff from the best alternative, putting $s'$ first in the ranking, cannot exceed $v(i)$. Setting $\bar{v} \leq \bar{v}_1 = \min\{\frac{C_1}{N_2+N_{w1}}, \frac{C_2}{N_1+N_{w2}}\}$ we make sure that all such $i$ best respond by ranking schools truthfully.

Consider now the case in which one school, labelled $u$, is underdemanded in the first round (the other one, $o$, must be overdemanded). For a student who prefers $u$ the most, truthful ranking is obviously a best response. Consider a student $i$ who prefers $o$ the most. If she ranks $o$ in first position according to her true preferences, she obtains a payoff higher than $\frac{C_o}{N_u+N_{wo}}$. If she instead ranks $u$ first she obtains a payoff $v(i)$. Setting $\bar{v} \leq \bar{v}_2 = \frac{C_o}{N_u+N_{wo}}$ we make sure that all such $i$ best respond by ranking $o$ in first position. Notice that $\bar{v}_1 = \bar{v}_2$, thus $\bar{v} = \min\{\frac{C_1}{N_2+N_{w1}}, \frac{C_2}{N_1+N_{w2}}\}$ suffices to obtain the desired result. □

**Proof Proposition 1.**

In BM (worst case), if some good school $s$ is underprioritized, all of its prioritized students optimally rank the school of their catchment area first. If a good school $s$ is overprioritized, no Nash equilibrium (undominated strategies) exists where the number of prioritized students that rank $s$ first does not exceed its capacity. Altogether we obtain $ABS_{BMworst} = \sum_{s\in\{1,2\}} \max\{0, C_s - N_s\} = \underline{ABS}$. □

**Proof Proposition 2.**

Clear for overdemanded schools (all good slots are given to students that prefer them the most). When $u$ is underdemanded (hence $o$ is overdemanded), $C_u - N_o - N_{wu}$ slots of $u$ are still available after the first assignment round. This is the maximum number of slots that students with priority at $u$ who were rejected from $o$ in the first round, a total of $N_u\left(1 - \frac{C_o}{N_u+N_{wo}}\right)$, can occupy. All other slots are occupied by agents who had priority at a less-preferred school. □

**Proof Proposition 3.**

Following in Abdulkadiroğlu, Che and Yasuda (2014), the outcome of DA can be characterized via cutoffs $c_1$ and $c_2$. Provided a student $i$ with $\pi(i) \neq s$ applies at some point for school $s$, she would be *definitely* accepted if her lottery number meets $n(i) \leq c_s$. Obviously the cut-off for the worst school is 1. We easily adapt Abdulkadiroğlu, Che and Yasuda's method of calculus to the existence of a zone priority structure. Let $l$ be the good school with lowest cut-off, and $h$ the good school with highest cut-off. Then $c_l$ and $c_h$ meet

$$\begin{aligned} c_l(N_h + N_{wl}) &= \max\{0, C_l - (1 - c_h)N_l\} \\ c_h(N_l + N_{wh}) + (c_h - c_l)N_{wl} &= \max\{0, C_h - (1 - c_l)N_h\} \end{aligned}$$

1) $C_h - (1 - c_l)N_h \leq 0$. Then $c_l$ and $c_h$ are both zero. This case arises when both good schools are (weakly) overprioritized.

2) $C_l - (1 - c_h)N_l \leq 0$ and $C_h - (1 - c_l)N_h > 0$. In such a case we have $c_l = 0$ and $c_h = \frac{C_h - N_h}{N_l+N_w}$. This case arises when $C_h > N_h$ (that is, $\rho_h > 1$) and, after some algebra, $\rho_l \leq \rho_w$. This algebra goes as follows. $C_l - (1 - c_h)N_l \leq 0$ iff $C_l - (1 - \frac{C_h-N_h}{N_l+N_w})N_l \leq 0$, or $C_l - \frac{N_l+N_w+N_h-C_h}{N_l+N_w}N_l \leq 0$, or $C_l - \frac{1-C_h}{N_l+N_w}N_l \leq 0$, or $C_l - \frac{C_l+C_w}{N_l+N_w}N_l \leq 0$, or $\frac{C_l}{N_l} \leq \frac{C_l+C_w}{N_l+N_w}$, which happens if and only if $\frac{C_l}{N_l} \leq \frac{C_w}{N_w}$.

3): $C_l - (1 - c_h)N_l > 0$ and $C_h - (1 - c_l)N_h > 0$. Solving for the system of equations gives $c_l = \frac{N_l(\rho_l - \rho_w)}{N_h+N_{wl}}$ and $c_h = 1 - \rho_w$. We have, of course, that this is met when $\rho_h > 1$ and $\rho_l > \rho_w$. Notice that this implies $\rho_w < 1$.

In this case $ABS_{DA} = N_1c_2 + N_2c_1 + N_w \max\{c_1, c_2\} = N_1c_2 + N_2c_1 + N_w(1 - \rho_w) = N_1c_2 + N_2c_1 + N_w - C_w = N_1c_2 + N_2c_1 + C_1 + C_2 - N_1 - N_2$. $\square$

**Proof Proposition 4.**

We already saw that $ABS_{BMworst} = \underline{ABS}$. From the preceding Proposition it is clear that $ABS_{DA} = \underline{ABS}$ in its cases 1 and 2 (summarized as $\min\{\rho_1, \rho_2\} \leq \rho_w$), while $ABS_{DA} > \underline{ABS}$ if $\min\{\rho_1, \rho_2\} > \rho_w$. It is also clear that $ABS_{BMbest} = \overline{ABS} > ABS_{DA}$ if both good schools are overdemanded under BM (best case). It remains to check that $ABS_{BMbest} > ABS_{DA}$ also when there is one underdemanded good school under BM (best case). We use the labelling from previous propositions: $u$ for the underdemanded good school under BM (best case), $o$ for the overdemanded good school under BM (best case), $l$ for the good school with lowest cut-off under DA, and $h$ for the good school with highest cut-off under DA.

If $\min\{\rho_1, \rho_2\} \leq \rho_w$, we show that $ABS_{BMbest} > \max\{C_u - N_u, C_o - N_o\}$ $(\geq \underline{ABS})$. On the one hand, $ABS_{BMbest} \geq C_u + C_o - \{C_u - N_o - N_{wu}\} = C_o + N_o + N_{wu} > C_o - N_o$. On the other hand, $ABS_{BMbest} \geq C_u + C_o - N_u \left(1 - \frac{C_o}{N_u + N_{wo}}\right) > C_u - N_u$.

If $\min\{\rho_1, \rho_2\} > \rho_w$, we need to show that $\min\{N_u \left(1 - \frac{C_o}{N_u + N_{wo}}\right), C_u - N_o - N_{wu}\} < N_l(1 - c_h) + N_h(1 - c_l)$.

Case 1: $u = h$, $o = l$. It is enough to show $N_u \left(1 - \frac{C_o}{N_u + N_{wo}}\right) = N_h \left(1 - \frac{C_l}{N_h + N_{wl}}\right) < N_h(1 - c_l)$, or $c_l < \frac{C_l}{N_h + N_{wl}}$. This follows immediately since $c_l = \frac{C_l - N_l\rho_w}{N_h + N_{wl}}$.

Case 2: $u = l$, $o = h$. It suffices to show $C_u - N_o - N_{wu} = C_l - N_h - N_{wl} < N_l(1 - c_h) = N_l\rho_w$. We use the fact that $c_l = \frac{C_l - N_l\rho_w}{N_h + N_{wl}} < 1$ (since $c_l \leq c_h = 1 - \rho_w < 1$). This implies exactly $C_l - N_h - N_{wl} < N_l\rho_w$.

$\square$

**Proof Proposition 5.**

We use the following notation in order to shorten the exposition: $s$ and $s'$ denote generic good schools, $M_s = \min\{C_s, N_s\}$, and $\Delta_{ss'} = \frac{N_s N_w}{N_{s'} - N_{ws'}}$ if $N_{s'} > N_{ws'}$ and $+\infty$ otherwise. The calculations and the presentation of results include five cases.

**Proposition 7** *1) If* $C_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$, *then* $ABS_{TTC} = \overline{ABS} - [M_{s'} - C_s]$.

*2) If* $N_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$ *and* $C_s - N_s < \min\{\Delta_{ss'}, (M_{s'} - N_s)\frac{N_w}{N_{ws'}}\}$, *then* $ABS_{TTC} = \overline{ABS} - \left[M_{s'} - N_s - (C_s - N_s)\frac{N_{ws'}}{N_w}\right]$.

*3) If* $N_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$ *and* $\max\{C_s - N_s, \Delta_{ss'}\} \geq (M_{s'} - N_s)\frac{N_w}{N_{ws'}}$, *then* $ABS_{TTC} = \overline{ABS}$.

*4) If* $N_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$, $\Delta_{ss'} < \min\{C_s - N_s, (M_{s'} - N_s)\frac{N_w}{N_{ws'}}\}$ *and* $\frac{C_s - N_s - \Delta_{ss'}}{N_{ws} + N_s} < \frac{M_{s'} - N_s - \Delta_{ss'}\frac{N_{ws'}}{N_w}}{N_{ws'}}$, *then* $ABS_{TTC} = \overline{ABS} - [M_{s'} - N_s - \Delta_{ss'}\frac{N_{ws'}}{N_w} - (C_s - N_s - \Delta_{ss'})\frac{N_{ws'}}{N_{ws} + N_{s'}}]$.

*5) If* $N_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$, $\Delta_{ss'} < \min\{C_s - N_s, (M_{s'} - N_s)\frac{N_w}{N_{ws'}}\}$ *and* $\frac{C_s - N_s - \Delta_{ss'}}{N_{ws}+N_s} \geq \frac{M_{s'} - N_s - \Delta_{ss'}\frac{N_{ws'}}{N_w}}{N_{ws'}}$, *then* $ABS_{TTC} = \overline{ABS}$.

**Proof.** Case 1 is easy. Prioritized students at good schools trade their slots until school $s$ is filled. The assignment process continues with school $s'$ only, in order of priority, where $M_{s'} - C_s$ slots are assigned to students of school $s'$ catchment area. The waste with respect to maximal ABS, $\overline{ABS} - ABS_{TTC}$, is the mass of good school slots that are assigned to students of its catchment area.

For the rest of cases. Prioritized students at good schools trade their slots until every prioritized student at school $s$ obtains a slot at $s'$. The marginal prioritized student at school $s'$ that traded with a prioritized student at school $s$ has lottery number $n_0 = N_s/N_{s'}$. School $s$ now points to the non-prioritized student with best lottery number, who comes from the bad school catchment area and has number $\tilde{n}_0 = 0 < n_0$. With probability $\frac{N_{ws}}{N_w}$ she keeps this slot and with probability $\frac{N_{ws'}}{N_w}$ she points at $s'$, hence trading with the marginal prioritized student at school $s'$. Then, for every $\Delta$ slots that the assignment process continues giving at school $s$, it also assigns $\frac{N_{ws'}}{N_w}\Delta$ at school $s'$.

For prioritized students at school $s'$ the marginal lottery number for any given $\Delta$ evolves as $n_\Delta = \frac{N_s + \frac{N_{ws'}}{N_w}\Delta}{N_{s'}}$. For prioritized students at school $w$ the marginal lottery number evolves as $\tilde{n}_\Delta = \Delta/N_w$. $\Delta_{ss'}$ is the point at which both marginal lottery numbers coincide, $n_{\Delta_{ss'}} = \tilde{n}_{\Delta_{ss'}}$. This coincidence does not happen in cases 2 and 3. In case 2, all slots at school $s$ are filled before marginal lottery numbers equalize. In case 3, either all slots at school $s'$ are given before marginal lottery numbers equalize or all its prioritized students obtain a slot at $s$. In case 2, the assignment process continues with school $s'$ only, in order of priority. Finally, non-traded $M_{s'} - N_s - (C_s - N_s)\frac{N_{ws'}}{N_w}$ slots are assigned to students of school $s'$ catchment area (and this is $\overline{ABS} - ABS_{TTC}$). In case 3, students who remain to be placed can only be assigned to better schools than the catchment area school, thus there is no waste in terms of ABS.

In cases 4 and 5, marginal lottery numbers coincide, $n_{\Delta_{ss'}} = \tilde{n}_{\Delta_{ss'}}$, before either cases 2 or 3 arise. After lottery numbers coincide, school $s'$ continues pointing at the remaining student with best lottery number. Yet in this case, with probability $\frac{N_{ws}+N_{s'}}{N_w+N_{s'}}$ (notice the difference with respect to $\frac{N_{ws}}{N_w}$ because prioritized students at $s'$ do not longer have worse lottery numbers than other students) $s$ points at a student who directly keeps this slot and with probability $\frac{N_{ws'}}{N_w+N_{s'}}$ the pointed student points at $s'$, hence trading with the marginal prioritized student at school $s'$. In sum, for every $\delta$ slots given, $\delta\frac{N_{ws'}}{N_w+N_{s'}}$ are given at school $s'$ and $\delta\frac{N_{ws}+N_{s'}}{N_w+N_{s'}}$ are given at school $s$. As cases 2 and 3 do, cases 4 and 5 arise depending on whether all slots at school $s$ are filled first, or either all slots at school $s'$ are filled first or all its prioritized students obtain a slot at $s$. In case 4, the assignment process continues with school $s'$ only, in order of priority, where $M_{s'} - N_s - \Delta_{ss'}\frac{N_{ws'}}{N_w} - (C_s - N_s - \Delta_{ss'})\frac{N_{ws'}}{N_{ws}+N_{s'}}$ slots are finally assigned to students of school $s'$ catchment area (and this is $\overline{ABS} - ABS_{TTC}$).

In case 5, as in case 3, slots that remain to be placed belong to school $s$, that is better than the catchment area school for all unassigned students, thus there is no waste in terms of ABS. $\square$

**Proof Proposition 6.**

We omit cases where $\min\{\rho_1, \rho_2\} \leq \rho_w$, that is, when $ABS_{DA} = \underline{ABS}$. Consider the case where $C_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$. Then $\overline{ABS} - ABS_{DA} = N_s(1 - c_{s'}) + N_{s'}(1 - c_s) > N_{s'}(1 - c_s) \geq N_{s'} - C_s \geq \min\{C_{s'}, N_{s'}\} - C_s = \overline{ABS} - ABS_{TTC}$. The first inequality arises from the fact that no cutoff equals 1. The second inequality is a feasibility condition (at least $N_{s'} - C_s$ students with priority at school $s'$ cannot enter at school $s$).

We next consider the case where $N_s = \min\{C_s, N_s, C_{s'}, N_{s'}\}$, and we assume that $\overline{ABS} - ABS_{TTC} > 0$. Otherwise we would be done because DA never reaches this upper bound. Since all students with priority at school $s$ obtain a slot at $s'$, $\overline{ABS} - ABS_{TTC} > 0$ implies that a positive mass of students with priority at school $s'$ are assigned at school $s'$ under TTC (thus they cannot obtain access to the more preferred school $s$).

Focus then on TTC. Let $n_s^{s'}$ be the maximum lottery number among the students with priority at school $s'$ that are assigned at school $s$. Let $n_s^{ws}$ be the maximum lottery number among the students with priority at school $w$ who prefer $s$ among all schools and are assigned at school $s$. Notice that $n_s^{s'} \geq n_s^{ws}$. The reason is that the latter students can only gain access to school $s$ by directly being pointed by school $s$. They cannot be pointed by school $s'$ before $s$ fills all slots. Simply because not all students with priority at $s'$ are pointed by $s'$ as part of a trading cycle either (before $s$ fills all positions). On the contrary, students with priority at $s'$ can gain access to school $s$ either by directly being pointed by school $s$ or by being part of a trading cycle.

Moreover, notice that only students of the latter and the former types are assigned to school $s$. All students with priority at school $s$ obtain a slot at $s'$. No student with priority at school $w$ who prefers $s'$ among all schools is assigned to $s$. If they are pointed by school $s$, they must be part of a trading cycle (this is implied by the fact that a positive mass of students with priority at school $s'$ cannot obtain access to the more preferred school $s$). Therefore we have $n_s^{s'} N_{s'} + n_s^{ws} N_{ws} = C_s$.

But in DA where $\min\{\rho_1, \rho_2\} > \rho_w$, the cutoff for school $s$ meets $c_s(N_{s'} + N_{ws}) = C_s - (1 - c_{s'})N_s - \max\{0, (c_s - c_{s'})N_{ws'}\}$, implying $c_s < \frac{n_s^{s'} N_{s'} + n_s^{ws} N_{ws}}{N_{s'} + N_{ws}}$. Given that $n_s^{s'} \geq n_s^{ws}$ we obtain $n_s^{s'} > c_s$. But then $\overline{ABS} - ABS_{DA} = N_s(1 - c_{s'}) + N_{s'}(1 - c_s) > N_{s'}(1 - c_s) > N_{s'}(1 - n_s^{s'}) = \overline{ABS} - ABS_{TTC}$. $\square$

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