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On the Exhaustiveness of Truncation and Dropping Strategies in Many–to–Many Matching Markets*

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Abstract

We consider two–sided many–to–many matching markets in which each worker may work for multiple firms and each firm may hire multiple workers. We study individual and group manipulations in centralized markets that employ (pairwise) stable mechanisms and that require participants to submit rank order lists of agents on the other side of the market. We are interested in simple preference manipulations that have been reported and studied in empirical and theoretical work: *truncation strategies*, which are the lists obtained by removing a tail of least preferred partners from a preference list, and the more general *dropping strategies*, which are the lists obtained by only removing partners from a preference list (i.e., no reshuffling).

We study when truncation/dropping strategies are exhaustive for a group of agents on the same side of the market, i.e., when each match resulting from preference manipulations can be replicated or improved upon by some truncation/dropping strategies. We prove that for each stable mechanism, dropping strategies are exhaustive for each group of agents on the same side of the market (Theorem 1), i.e., independently of the quotas. Then, we show that for each stable mechanism, truncation strategies are exhaustive for each agent with quota 1 (Theorem 2). Finally, we show that this result cannot be extended neither to individual manipulations when the agent's quota is larger than 1 (even when all other agents' quotas equal 1 – Example 1), nor to group manipulations (even when all quotas equal 1 – Example 2).

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1 Introduction

In part–time labor markets and some professional entry–level labor markets a worker may be employed by a number of different firms. An important example of the latter are British entry–level medical labor markets which involve graduating medical students and teaching hospitals. Each student seeks two residency positions: one for a medical program and one for a surgical program. Roth (1991) modeled the British entry–level medical labor markets as many–to–two matching markets.

In this paper, we consider many–to–many matching markets in which each worker may work for multiple firms and each firm may hire multiple workers. Agents have preferences over subsets of potential partners.[1] An assignment between workers and firms is called a matching. A central concept in the matching literature is (pairwise) stability. A matching is called stable if all agents are matched to an acceptable subset of partners and there is no unmatched worker–firm pair who both would prefer to match (and possibly dismiss some current partners). Roth (1984a) studied a general many–to–many model and showed that if the agents' preferences satisfy substitutability then the set of stable matchings is non–empty.[2]

In many–to–many matching markets, the set of stable matchings might be different from the core (Blair, 1988) and also there might be stable matchings that can be blocked by coalitions of more than two agents (Roth and Sotomayor, 1990). Sotomayor (1999b) studied the stronger concept of setwise stability and showed that in the many–to–many model the set of stable matchings, the core, and the set of setwise stable matchings do not coincide. However, potential larger blocking coalitions in complex real–life settings might have more difficulties to organize themselves. In fact, Roth (1991, page 422) suggested that for many–to–many markets such as the British entry–level medical labor markets, stability is still of primary importance.

Many real–life matching markets employ a centralized mechanisms to match workers to firms and the only information that the matchmaker asks from the participating agents are their preferences over the other side of the market. In particular, we assume that the agents' quotas (i.e., the number of available slots) are commonly known by the agents (because, for instance, the quotas are determined by laws).[3] In practice, agents are only allowed to submit ordered lists of individual partners (potential partners that are not listed are assumed to be unacceptable). Presumably the agents' preferences over sets of potential partners are responsive

[1]Note that agents only have preferences over potential partners on the other side of the market and not over their colleagues.

[2]An agent has substitutable preferences if the agent continues to want to be partners with an agent even if other agents become unavailable. Note that substitutability excludes complementarities. Substitutability was introduced by Kelso and Crawford (1982) to show the existence of stable matchings in a many–to–one model with money.

[3]In particular, quotas cannot be manipulated (cf. Sönmez, 1997).

(Roth, 1985a): for each agent i, the convenience to match with an additional potential partner j by possibly replacing some partner k only depends on the individual characteristics of j and k (and whether the quota is reached). Throughout the current paper we focus on mechanisms that only demand ordered lists of potential individual and acceptable partners and keep the responsiveness assumption.[4] A mechanism is stable if for each reported profile of ordered lists it produces a matching that is stable with respect to the reported profile. Two important examples of such mechanisms are the so–called worker–optimal and firm–optimal stable mechanisms which are based on the deferred acceptance algorithm (introduced by Gale and Shapley, 1962, for the one–to–one case and adapted by Roth, 1984a, to the many–to–many case).

Even though there is evidence that clearinghouses that employ stable mechanisms often perform better than those that employ unstable mechanisms,[5] no stable mechanism is immune to preference manipulation (Dubins and Freedman, 1981, and Roth, 1982). This fact immediately triggers a question: What types of strategies should a strategic agent consider? In the present paper, we focus on two types of "simple" preference manipulations that have been reported and studied in empirical and theoretical work. The first class of preference manipulations is that of truncation strategies (Roth and Vande Vate, 1991). A truncation strategy is a list that is obtained from an agent's true preference list by removing a tail of its least preferred acceptable partners. Truncation strategies have been observed in practice, for instance, in the sorority rush (Mongell and Roth, 1991). The second class of preference manipulations consists of dropping strategies (Kojima and Pathak, 2009). A dropping strategy is a list that is obtained from an agent's true preference list by removing acceptable partners (i.e., no reshuffling). Obviously, each truncation strategy is also a dropping strategy. Roth and Rothblum (1999) studied the firm–optimal stable mechanism in the many–to–one model. They showed that if a worker's incomplete information is completely symmetric, then it might only gain by reporting a truncation strategy. Ehlers (2008) obtained a similar result for all so–called priority and linear programming mechanisms. Coles and Shorrer (2012) examined truncation strategies in the one–to–one model. They established that also in settings with asymmetric incomplete information about the strategies submitted by the other agents, workers can truncate lists with little risk of ending up unmatched, but with the potential to see large gains. Ma (2010) studied truncation strategies and the equilibrium outcomes induced by the worker–optimal stable mechanism in one–to–one and many–to–one matching markets. For one–to–one, he found that if in equilibrium each firm uses a truncation strategy, then the equilibrium outcome is the firm–optimal matching. For many–to–one, he found that if in equilibrium each firm uses a truncation strategy, then the equilibrium outcome is either the firm–optimal matching or an unstable matching with respect to the true preferences. Ashlagi and Klijn (2012) studied effects of manipulations in the direct-revelation game based on the worker–optimal stable mechanism in one–to–one and many–to–one matching markets. For one–to–one, they showed that under the worker–optimal stable mechanism, any weakly successful group manipulation by firms is weakly beneficial to all other firms

[4]Responsiveness implies substitutability, and hence the existence of a stable matching.

[5]See, for instance, Roth (1991).

and weakly harmful to all workers and any truncation strategy of a firm is weakly beneficial to all other firms and weakly harmful to all workers. They showed that neither of the results above extends in an appropriate way to many–to–one: a firm can have dropping strategies and successful manipulations that strictly harm some other firm and strictly benefit some worker.

Taking the stability requirement for a mechanism to perform well as granted, we study stable mechanisms, but do not restrict ourselves to the firm–optimal stable mechanism (as in Roth and Rothblum, 1999, and Coles and Shorrer, 2012). On the other hand, we assume a complete information environment. We consider the point of view of an individual worker while keeping the other agents' strategies fixed. In view of our analysis it is convenient to introduce the truncation/dropping correspondence that assigns to each preference relation the set of truncation/dropping strategies obtained from the induced list over individual agents. In one–to–one markets, the truncation correspondence is exhaustive (Roth and Vande Vate, 1991, Theorem 2) in the sense that for each strategy, the induced match can be replicated or improved upon by some truncation of the list induced by the agent's true preference relation.[6] Kojima and Pathak (2009, Lemma 1) proved that the dropping correspondence is exhaustive for a firm in the many–to–one model (where workers' quotas equal one).[7] However, their result does not say anything about possible joint manipulations by a group of workers or a group of firms, nor deals with the possibility of workers having a quota larger than one.[8] We show that for each stable mechanism, the dropping correspondence is exhaustive for each group of agents on the same side of the market (Theorem 1).

Since Roth and Vande Vate's (1991) model is one–to–one, their result would not apply to most real–life matching markets.[9] We extend Roth and Vande Vate's (1991) result by showing that for each stable mechanism, the truncation correspondence is exhaustive for each agent with quota 1 (Theorem 2). the truncation/dropping correspondence is exhaustive for a group of agents on the same side of the market. We complement our second result with two examples to show that it cannot be generalized in the following two ways. The truncation correspondence is

- neither necessarily exhaustive for an agent with quota larger than 1 even when all other agents' quotas equal 1 (Example 1);

[6] Roth and Vande Vate (1991) studied random stable mechanisms. We rephrase their Theorem 2 to fit it for our framework.

[7] In fact, Kojima and Pathak (2009) also considered strategic manipulation by underreporting quotas. We focus on manipulation via preference lists, and aim to establish "exhaustiveness results" (of truncation and dropping strategies) for different classes of quota vectors.

[8] Note that we only focus on (pairwise) stability and do not consider larger blocking coalitions than worker–firm pairs. This is not a conceptual contradiction to our study of joint manipulations, since larger blocking coalitions would involve agents from both sides of the market, while the joint manipulations we study only deal with groups of agents on the same side of the market. It seems more likely that a group of agents on the *same* side of the market can carry out a group manipulation that is actually binding on its members.

[9] For each many–to–one market, there is a one–to–one correspondence between its stable matchings and those of a related one–to–one market. Hence, many properties of the set of stable matchings in the one–to–one model carry over to the many–to–one model. Yet, with respect to strategic issues, Roth (1985a) showed that the two models are not equivalent.

- nor necessarily exhaustive for a group of agents on the same side of the market even when all quotas equal 1 (Example 2).

Our results suggest that if workers and firms are aware of the exhaustiveness of truncation or dropping correspondences, we can expect them to reveal truthful information regarding the relative rank order of the listed potential partners. To put our paper in perspective, we briefly mention some of the most closely related papers on many–to–many matching markets (apart from the already mentioned work by Roth, 1984a, and Sotomayor, 1999b). Alkan (1999,2001,2002), Baïou and Balinski (2000), Blair (1988), Fleiner (2003), Roth (1985b), and Sotomayor (1999a) provided important insights into the lattice structure of the set of stable matchings in different (many–to–many) models. Martínez et al. (2004) presented an algorithm to compute the full set of stable matchings when preferences are substitutable. Sotomayor (2004) provided a mechanism that implements the set of stable matchings when preferences are responsive. Klijn and Yazıcı (2012) studied the number and the set of filled slots in stable matchings when preferences are substitutable and weakly separable. Finally, Echenique and Oviedo (2006), Klaus and Walzl (2009), Konishi and Ünver (2006), and Sotomayor (1999b) analyzed the relation between various solution concepts different from (pairwise) stability on several domains of preferences.

The remainder of the paper is organized as follows. In Section 2, we introduce the model. In Section 3, we present and prove our results. Section 4 concludes.

2 Model

There are two finite and disjoint sets of agents: a set of *workers* W and a set of *firms* F. Let $I = W \cup F$ be the set of agents. We denote a generic worker, firm, and agent by w, f, and i, respectively. For each agent i, there is an integer *quota* $q_i \geq 1$. Worker w can work for at most q_w firms and firm f can hire at most q_f workers. Let $q = (q_i)_{i \in I}$.

Let $i \in I$. The set of *potential partners* of agent i is denoted by N_i. If $i \in W$, $N_i = F$ and if $i \in F$, $N_i = W$. A subset of potential partners $N \subseteq N_i$ is *feasible* (for agent i) if $|N| \leq q_i$. Let $\mathcal{N}(N_i, q_i) = \{N \subseteq N_i : |N| \leq q_i\}$ denote the collection of feasible subsets of potential partners. The element $\varnothing \in \mathcal{N}(N_i, q_i)$ denotes "being unmatched" or some outside option. Agent i has a complete, transitive, and strict *preference relation* $\succ_i$ over $\mathcal{N}(N_i, q_i)$. For each $N, N' \in \mathcal{N}(N_i, q_i)$, we write $N \succeq_i N'$ if agent i finds N at least as good as N', i.e., $N \succ_i N'$ or $N = N'$. Let $\mathcal{P}_i^{\succ}$ be the set of all preference relations for agent i. Let $\succ = (\succ_i)_{i \in I}$. For $A \subseteq I$, let $\succ_A = (\succ_i)_{i \in A}$ and $\succ_{-A} = (\succ_i)_{i \in I \setminus A}$.

Let P_i be the restriction of $\succ_i$ to $\{\{j\} : j \in N_i\} \cup \{\varnothing\}$, i.e., individual partners in N_i and being unmatched. For $j, j' \in N_i \cup \{\varnothing\}$, we write $j \, P_i \, j'$ if $j \succ_i j'$, and $j \, R_i \, j'$ if $j \succeq_i j'$.[10] Let $\mathcal{P}_i$ be the set of all such restrictions for agent i. Agent $j \in N_i$ is an *acceptable partner* for agent i if $j \, P_i \, \varnothing$. Let $P = (P_i)_{i \in I}$. For $A \subseteq I$, let $P_A = (P_i)_{i \in A}$ and $P_{-A} = (P_i)_{i \in I \setminus A}$.

We also represent an agent i's preferences P_i as an ordered list of the elements in $N_i \cup \{\varnothing\}$.

[10] With some abuse of notation we often write x for a singleton $\{x\}$.

For instance, $P_w = f_3\, f_2\, \varnothing\, f_1\, \ldots\, f_4$ indicates that w prefers f_3 to f_2, f_2 to being unmatched, and being unmatched to any other firm.

We assume that for each agent i, $\succ_i$ is a *responsive extension* of P_i (or responsive for short)[11] such that (r1) as long as an agent's quota is not reached, it prefers to fill a position with an acceptable partner rather than leaving it unfilled and (r2) an agent if faced with two sets of potential partners that differ only in one partner, it prefers the set of partners containing the more preferred partner, i.e., for all $N \in \mathcal{N}(N_i, q_i)$,

(r1) if $j \in N_i \backslash N$ and $|N| < q_i$, then $N \cup j \succ_i N$ if and only if $j\, P_i\, \varnothing$; and

(r2) if $j \in N_i \backslash N$ and $k \in N$, then $(N \backslash k) \cup j \succ_i N$ if and only if $j\, P_i\, k$.

A *(many–to–many matching) market* is given by $(W, F, \succ, q)$ or, when no confusion is possible, $(\succ, q)$ for short.[12]

Let $(W, F, \succ, q)$ be a market. A *matching* is a function $\mu : I \to 2^I$ such that (m1) each agent is matched to a feasible subset of potential partners and (m2) an agent is matched to a partner if and only if the partner is matched to the agent, i.e.,

(m1) for all $i \in I$, $\mu(i) \in \mathcal{N}(N_i, q_i)$; and

(m2) for all $w \in W$ and $f \in F$, $f \in \mu(w)$ if and only if $w \in \mu(f)$.

Let μ be a matching. Let $i, j \in I$. If $j \in \mu(i)$ then we say that i and j are *matched* to one another and that they are *mates* in μ. The set $\mu(i)$ is agent i's *match*.

Next, we describe desirable properties of matchings. First, we are interested in a voluntary participation condition over the matchings. Formally, a matching μ is *individually rational* if for each $i \in I$ and each $j \in \mu(i)$, $j\ P_i\ \varnothing$.[13]

Second, we aim to avoid particular blocking pairs that would render a matching unstable. A worker–firm pair (w, f) is a *blocking pair* for μ if (b1) a worker w and a firm f are not mates in μ, (b2) w would prefer to add f or replace another firm by f, and (b3) f would prefer to add w or replace another worker by w, i.e.,

(b1) $w \notin \mu(f)$;

(b2) $[\,|\mu(w)| < q_w$ and $f\ P_w\ \varnothing\,]$ or [there is $f' \in \mu(w)$ such that $f\ P_w\ f'$]; and

(b3) $[\,|\mu(f)| < q_f$ and $w\ P_f\ \varnothing\,]$ or [there is $w' \in \mu(f)$ such that $w\ P_f\ w'$].[14]

A matching is (pairwise) *stable* if it is individually rational and there are no blocking pairs. Let $S(\succ, q)$ be the set of stable matchings for market $(\succ, q)$. Roth (1984a) showed that the set of stable matchings is always non–empty. In fact, he showed that for each market $(\succ, q)$, there is a (worker–optimal) stable matching μ^W that is weakly preferred by all workers to any other stable matching in $S(\succ, q)$. Formally, for each $w \in W$ and each $\mu \in S(\succ, q)$, $\mu^W(w) \succeq_w \mu(w)$.

[11] See Roth (1985a) and Roth and Sotomayor (1989) for a discussion of this assumption.

[12] A many–to–one matching market is a market where each agent on one given side of the market has quota 1. A one–to–one or marriage market is a market where each agent has quota 1.

[13] Alternatively, by responsiveness condition (r1), a matching μ is individually rational if no agent would be better off by breaking a match, i.e., for each $i \in I$ and each $j \in \mu(i)$, $\mu(i) \succ_i \mu(i) \backslash j$.

[14] By responsiveness conditions (r1) and (r2), (b2) is equivalent to [$[|\mu(w)| < q_w$ and $\mu(w) \cup f \succ_w \mu(w)\,]$ or [there is $f' \in \mu(w)$ such that $(\mu(w) \smallsetminus f') \cup f \succ_w \mu(w)\,]$]. A similar equivalent statement holds for (b3).

Similarly, there is a (firm–optimal) stable matching μ^F that is weakly preferred by all firms to any other stable matching in $S(\succ, q)$. Note that stability does not depend on the particular responsive extensions of the agents' preferences over individual acceptable partners.[15] Hence, we can denote the set of stable matchings for $(\succ, q)$ by $S(P, q)$.

In many–to–one matching markets, the set of stable matchings coincides with the core defined by weak domination. In addition, ruling out blocking pairs is sufficient for ruling out blocking coalitions that involve more than two agents. This is not true in many–to–many matching markets. Not only might the set of stable matchings be different from the core, but also there might be stable matchings that can be blocked by coalitions of more than two agents (see Sotomayor, 1999b). However, Roth (1991, page 422) suggested that for certain many–to–many markets, stability is still of primary importance.

A mechanism assigns a matching to each market. We assume that quotas are commonly known by the agents (because, for instance, the quotas are determined by law).[16] Therefore, the only information that the mechanism asks from the agents are their preferences over the other side of the market. Many real–life centralized matching markets employ mechanisms that only ask for the ordered lists $P = (P_i)_{i \in I}$ of individual partners, i.e., they do not depend on the particular responsive extensions. Throughout the paper we focus on this class of mechanisms. Hence, a *mechanism* φ assigns a matching $\varphi(P, q)$ to each pair (P, q).[17] We often denote agent i's match $\varphi(P, q)(i)$ by $\varphi_i(P, q)$. A mechanism φ is *stable* if for each (P, q), $\varphi(P, q) \in S(P, q)$. Two important examples of such mechanisms are the worker–optimal stable mechanism φ^W and the firm–optimal stable mechanism φ^F which assign to each market its worker–optimal stable matching and firm–optimal stable matching, respectively.

An important question is whether stable mechanisms are immune to preference manipulations by strategic agents. A *strategy* is an (ordered) preference list of a subset of potential partners.[18] More precisely, for each agent i, $\mathcal{P}_i$ is the set of strategies. Dubins and Freedman (1981) and Roth (1982) showed that there is no stable mechanism that is strategy–proof.[19] Formally, for each stable mechanism, φ, there is a market $(\succ, q)$ in which some agent i can submit a preference list P'_i different from its true preference list P_i and obtain a better match, i.e., $\varphi_i(P'_i, P_{-i}, q) \succ_i \varphi_i(P, q)$.

Next, we provide the formal definition of two important classes of strategies that have been studied in the literature. A truncation strategy of a worker w is an ordered list P'_w obtained from P_w by making a tail of acceptable firms unacceptable (Roth and Vande Vate, 1991). Formally, for a worker w with preferences P_w over individual firms, P'_w is a *truncation strategy* if for

[15] In fact, the set of stable matchings does not depend on the agents' orderings of the (individual) unacceptable partners either.

[16] In particular, quotas cannot be manipulated (cf. Sönmez, 1997).

[17] We do not suppress the notation q since the quotas play a role in the definition of stability. Moreover, our results are also conditional on the values of the quotas.

[18] The listed potential partners are interpreted as the acceptable potential partners. The other potential partners are unacceptable and, since we focus on stable mechanisms, their relative ordering is irrelevant.

[19] However, some stable mechanisms are strategy–proof for one side of the market if each agent on that side of the market has quota 1 (Roth, 1982, Theorem 5).

any firms $f, f' \in F$, (a) [if $f\, R'_w\, f' R'_w\, \varnothing$ then $f\, R_w\, f' R_w\, \varnothing$], and (b) [if $f\, P'_w\, \varnothing$ and $f'\, P_w\, f$ then $f'\, P'_w\, \varnothing$]. We define a truncation strategy of a firm similarly.

A dropping strategy of a worker w is an ordered list P'_w obtained from P_w by removing some acceptable firms, i.e., not necessarily a tail of least preferred firms (Kojima and Pathak, 2009). Formally, for a worker w with preferences P_w over individual firms, P'_w is a *dropping strategy* if for any firms $f, f' \in F$, [$f\, R'_w\, f' R'_w\, \varnothing$ implies $f\, R_w\, f' R_w\, \varnothing$]. We define a dropping strategy of a firm similarly.

A *strategy space reductor* for i is a correspondence Σ that maps each preference relation $\succ_i$ to a subset of the set of strategies. Formally, a strategy space reductor is a correspondence $\Sigma : \mathcal{P}_i^{\succ} \rightrightarrows \mathcal{P}_i$ such that for each $\succ_i \in \mathcal{P}_i^{\succ}$, the (non–empty) *reduced strategy space* $\Sigma(\succ_i)$ is a subset of $\mathcal{P}_i$. We focus on two strategy space reductors: the truncation correspondence and the dropping correspondence. The *truncation correspondence* $\boldsymbol{\tau}$ associates each preference relation $\succ_i$ with the set of truncation strategies obtained from the corresponding restriction P_i. Similarly, the *dropping correspondence* δ associates each preference relation $\succ_i$ with the set of dropping strategies obtained from the corresponding restriction P_i.

We next define the exhaustiveness of a strategy space reductor for an individual agent, i.e., when a strategy space reductor is rich enough to replicate or improve upon any possible match. Let q be a quota vector, φ be a mechanism and Σ be a strategy space reductor. The strategy space reductor Σ is *φ–exhaustive for agent i* if for each $\succ_i$, each P'_i, and each P_{-i}, there exists $Q_i \in \Sigma(\succ_i)$ such that $\varphi_i(Q_i, P_{-i}, q) \succeq_i \varphi_i(P'_i, P_{-i}, q)$.

When groups of agents on the same side of the market can jointly carry out strategic manipulations, we extend the previous definition as follows. Let q be a quota vector, φ be a mechanism, and $A' \subseteq A$ be a group of agents on the same side of the market $A \in \{W, F\}$. A (common) strategy space reductor Σ is *φ–exhaustive for group A'* if for each $\succ_{A'}$, each $P'_{A'}$, and each $P_{-A'}$, there exists $Q_{A'} \in \prod_{i \in A'} \Sigma(\succ_i)$ such that for each $i \in A'$, $\varphi_i(Q_{A'}, P_{-A'}, q) \succeq_i \varphi_i(P'_{A'}, P_{-A'}, q)$.

Note that φ–exhaustiveness for a group of agents implies φ–exhaustiveness for an agent, but the reverse is not true (see, for instance, Theorem 2 and Example 2).

3 Results

In this section, we present and prove our results. Recall that the quotas $(q_i)_{i \in I}$ are fixed and cannot be manipulated. We first consider the dropping correspondence and seek to determine when it is exhaustive.

Kojima and Pathak (2009) considered a many–to–one matching model where for each $w \in W$, $q_w = 1$. Their Lemma 1 implies that for each stable mechanism φ, the dropping correspondence is φ–exhaustive for each firm $f \in F$. We extend this result by showing that for each stable mechanism φ, the dropping correspondence is φ–exhaustive for a group of agents on the same side of the market, independently of the vector of the quotas. The proof parallels that of Kojima and Pathak (2009, Lemma 1). The main difference with their proof is that we need to show that during the procedure to get a stable matching only firms with vacant positions can be part of blocking pairs.

The constructive proof of Theorem 1 works as follows: For each stable mechanism φ, each group of workers W', and each worker $w \in W'$, (1) take any strategy P'_w, (2) find the matching under this strategy for φ, (3) suppose that w reported the acceptable firms that he was matched to under P'_w in the same relative order, (4) prove that w is matched to these firms in some stable matching and that stable mechanism matches him to the same mates.

Theorem 1. *Let φ be a stable mechanism. The dropping correspondence δ is φ–exhaustive for a group of agents on the same side of the market.*[20]

Proof. Let φ be a stable mechanism. Let $(\succ, q)$ be a market. Let P be the restriction of $\succ$ to individual partners and being unmatched. Without loss of generality, let $A = W$. Let $W' \subseteq W$.

Let $P'_{W'} = (P'_i)_{i \in W'}$ be a strategy–profile for W'. Let $\mu = \varphi(P'_{W'}, P_{-W'}, q)$. For each $w \in W'$, let $I^{\mu}(w) = \{f : f \in \mu(w) \text{ and } f\, P_w\, \varnothing\}$ be the set of firms matched to w at μ and that are acceptable for w with respect to P_w. For each $w \in W'$, let $Q_w \in \delta(\succ_w)$ be the dropping strategy obtained from P_w by ranking the firms in $I^{\mu}(w)$ according to the true relative ordering and making all other firms unacceptable. We need to show that for all $w \in W'$, $\varphi_w(Q_{W'}, P_{-W'}, q) \succeq_w \varphi_w(P'_{W'}, P_{-W'}, q)$. Note that by (r1) in the definition of responsiveness it is sufficient to show that for each $w \in W'$, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^{\mu}(w)$.

For each $w \in W$, let

$$\mu'_0(w) = \begin{cases} I^{\mu}(w) & \text{if } w \in W'; \\ \mu(w) & \text{if } w \notin W'. \end{cases}$$

Suppose μ'_0 is stable with respect to $(Q_{W'}, P_{-W'}, q)$. Let $w \in W'$. Note that in μ'_0 agent w is assigned to all its acceptable partners (with respect to Q_w). Hence, by Alkan (2002, Proposition 6), for each stable matching $\nu \in S(Q_{W'}, P_{-W'}, q)$, $\nu(w) = \mu'_0(w) = I^{\mu}(w)$.[21] By stability of φ, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^{\mu}(w)$, which we needed to establish.

Suppose μ'_0 is *not* stable with respect to $(Q_{W'}, P_{-W'}, q)$. Before we apply an iterative procedure to transform μ'_0 into a stable matching, we first establish a few properties of μ'_0:

P1($\boldsymbol{\mu'_0}$) μ'_0 is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$.

P2($\boldsymbol{\mu'_0}$) If (w, f) is a blocking pair for μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, then $w \notin W'$.

P3($\boldsymbol{\mu'_0}$) If (w, f) is a blocking pair for μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, then $|\mu'_0(f)| < q_f$.

Proof. P1(μ'_0) and P2(μ'_0) are immediate. Next, we show P3(μ'_0). Suppose it is not the case. Then, $|\mu'_0(f)| = q_f$. Since $\mu'_0(f) \subseteq \mu(f)$ and $|\mu(f)| \leq q_f$, $\mu'_0(f) = \mu(f)$. By P2(μ'_0), $w \notin W'$.

[20] The proof of Theorem 1 shows that any group of agents on the same side of the market only needs to consider dropping strategies in which the number of acceptable firms they report is at most their quota. However, they cannot only focus on dropping strategies in which the number of acceptable firms is equal to their quota (Example 3). We would like to thank the associate editor for pointing out this fact. We refer to Section 4 for further details.

[21] Proposition 6 in Alkan (2002) is an extension of part of a result that is known as the Rural Hospital Theorem (Roth, 1984b) which states that each agent is matched to the same number of partners in every stable matching.

Hence, $\mu_0'(w) = \mu(w)$. So, (w, f) also blocks μ with respect to $(P'_{W'}, P_{-W'}, q)$, which contradicts the stability of μ with respect to $(P'_{W'}, P_{-W'}, q)$. $\diamond$

Set $\mu' := \mu_0'$. As long as μ' is not stable with respect to $(Q_{W'}, P_{-W'}, q)$, apply the following procedure.

Begin Procedure.

By P1(μ'), there is at least one blocking pair for μ' with respect to $(Q_{W'}, P_{-W'}, q)$. Let f' be a firm that is a

member of one such blocking pair. Among all workers w involved in blocking pairs (w, f') for μ' with respect to $(Q_{W'}, P_{-W'}, q)$, let w' be the most preferred worker with respect to $P_{f'}$. By P2(μ'), $w' \notin W'$. By P3(μ'), $|\mu'(f')| < q_{f'}$. Define

$$\mu''(w) = \begin{cases} \mu'(w') \cup f' & \text{if } w = w' \text{ and } |\mu'(w')| < q_{w'}; \\ (\mu'(w') \cup f') \backslash \arg\min_{P_{w'}} \{f : f \in \mu'(w')\} & \text{if } w = w' \text{ and } |\mu'(w')| = q_{w'}; \\ \mu'(w) & \text{if } w \in W \backslash \{w'\}. \end{cases}$$

Then,

P1(μ'') μ'' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$;

P2(μ'') If (w, f) is a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, then $w \notin W'$; and

P3(μ'') If (w, f) is a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, then $|\mu''(f)| < q_f$.

Set $\mu' := \mu''$.

End Procedure.

In each iteration, one worker $w' \notin W'$ gets a strictly better match (with respect to $P_{w'}$) and all other workers keep their match. (This follows from the fact that firm f' has a vacant position in μ'.) Therefore, the iterative procedure terminates after a finite number of steps. The resulting matching μ^* is stable with respect to $(Q_{W'}, P_{-W'}, q)$. Let $w \in W'$. Since in each iteration of the procedure w keeps it match, $\mu^*(w) = \mu_0'(w) = I^\mu(w)$. Note that in μ^* agent w is assigned to all its acceptable partners (with respect to Q_w). Hence, by Alkan (2002, Proposition 6), for each stable matching $\nu \in S(Q_{W'}, P_{-W'}, q)$, $\nu(w) = \mu^*(w) = I^\mu(w)$. By stability of φ, $\varphi_w(Q_{W'}, P_{-W'}, q) = I^\mu(w)$, which we needed to establish.

It only remains to show that in each iteration, μ'' is a matching that satisfies P1(μ''), P2(μ''), and P3(μ''). We do this by induction. Suppose that in iteration 1 up to $k-1$ the resulting matching satisfies P1(.), P2(.), and P3(.). Let μ' be the matching at the beginning of iteration k (and suppose it is not stable with respect to $(Q_{W'}, P_{-W'}, q)$. Hence, P1(μ'), P2(μ'), and P3(μ') hold.) We will show that the matching μ'' that is obtained in iteration k satisfies P1(μ''), P2(μ''), and P3(μ'').

Proof of P1(μ''). By the induction hypothesis, μ' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$. The only new mates in μ'' with respect to μ' are the pair $\{w', f'\}$. Since (w', f') is a blocking pair for μ' and since μ' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$, it immediately follows that w' and f' are mutually acceptable with respect to $(Q_{W'}, P_{-W'}, q)$. Therefore, μ'' is individually rational with respect to $(Q_{W'}, P_{-W'}, q)$. $\diamond$

Proof of P2(μ''). Suppose $w \in W'$. Since (w,f) blocks μ'' with respect to $(Q_{W'}, P_{-W'}, q)$, $f \notin \mu''(w)$. By the induction hypothesis, in iterations 1 up to k, agent w has kept it original match, i.e., $\mu''(w) = \mu'_0(w)$. Hence, w blocks μ'' together with $f \notin \mu'_0(w)$. Recall that Q_w is a dropping strategy for which the firms in $\mu'_0(w)$ are the only acceptable ones for w. This gives a contradiction to (b2) in the definition of blocking pair and the individual rationality of μ'_0 with respect to $(Q_{W'}, P_{-W'}, q)$, which was established in P1(μ'_0). Hence, $w \notin W'$. ⋄

Proof of P3(μ''). Let (w,f) be a blocking pair for μ'' with respect to $(Q_{W'}, P_{-W'}, q)$. By P2(μ''), $w \notin W'$. Suppose $|\mu''(f)| = q_f$. Then, by (b3) in the definition of blocking pair, $w\, P_f\, \tilde{w}$ for some $\tilde{w} \in \mu''(f)$. We distinguish between two cases.

Case I. $(\tilde{w}, f)$ was a blocking pair matched in some iteration l, $l \leq k$.
By the induction hypothesis, in iterations $l+1$ up to k, worker $w \notin W'$ either keeps its match from iteration l or obtains a strictly better match by (possibly repeatedly) adding an acceptable firm and/or replacing its least preferred mate by a more preferred firm (if its quota is reached). Therefore, since (w,f) is a blocking pair for μ'' at the end of iteration k, w is also willing to block (with f) the initial matching in iteration l and w and f are not mates at the initial matching in iteration l. Since $w\, P_f\, \tilde{w}$, firm f did not block with the best possible worker in iteration l, which contradicts the definition of the procedure.

Case II. $\tilde{w}$ is matched to f in all matchings of iterations $1, \ldots, k$.
By the induction hypothesis, in iterations 1 up to k, worker $w \notin W'$ either keeps its match $\mu'_0(w)$ or obtains a strictly better match by (possibly repeatedly) adding an acceptable firm and/or replacing its least preferred mate by a more preferred firm (if its quota is reached). Therefore, since (w,f) is a blocking pair for μ'' at the end of iteration k, w is also willing to block (with f) matching μ'_0 (with respect to P_w) and $w \notin \mu'_0(f)$. Since $w\, P_f\, \tilde{w}$ and (by assumption) $\tilde{w} \in \mu'_0(f)$, (w,f) is a blocking pair for μ'_0 with respect to $(P'_{W'}, P_{-W'}, q)$. Since $w \notin W'$, it follows from the definition of μ'_0 that $\mu(\tilde{w}) = \mu'_0(\tilde{w})$. Hence, (w,f) is a blocking pair for μ with respect to $(P'_{W'}, P_{-W'}, q)$, which contradicts the stability of $\mu = \varphi(P'_{W'}, P_{-W'}, q)$. ⋄ □

Next, we consider the truncation correspondence and seek to determine when it is exhaustive.[22] Roth and Vande Vate (1991) studied a matching model making the following assumptions: (1) $|W| = |F|$, (2) each agent is acceptable to all agents on the other side of the market, and (3) for each $i \in I$, $q_i = 1$. Their Theorem 2 says that for each stable mechanism φ, the truncation correspondence τ is φ–exhaustive for each agent. It can easily be seen that the first two assumptions can be disposed of. Below, we further extend the result by relaxing the third assumption as well.

Theorem 2. *Let $A \in \{W, F\}$. Let φ be a stable mechanism. Suppose for some $a \in A$, $q_a = 1$. Then, the truncation correspondence τ is φ–exhaustive for agent a.*

[22]Truncation strategies have been extensively studied in the matching literature (Roth and Vande Vate, 1991, Ehlers, 2008, Romm, 2011, Ashlagi and Klijn, 2012, Coles and Shorrer, 2012, among others). Moreover, they have been used in practice, for instance, in the sorority rush (Mongell and Roth, 1991).

Proof. Let φ be a stable mechanism. Let $(\succ, q)$ be a market. Let P be the restriction of $\succ$ to individual partners and being unmatched. Without loss of generality, let $A = W$. Let $w \in W$ be such that $q_w = 1$.

Let P'_w be a strategy for w. We identify a truncation strategy $Q_w \in \tau(\succ_w)$ with $\varphi_w(Q_w, P_{-w}, q) \; R_w \, \varphi_w(P'_w, P_{-w}, q)$. By Theorem 1, there is a dropping strategy $P^*_w = \varphi_w(P'_w, P_{-w}, q)$ with $\varphi_w(P^*_w, P_{-w}, q) \, R_w \, \varphi_w(P'_w, P_{-w}, q)$. Then, it is enough to identify a truncation strategy Q_w with $\varphi_w(Q_w, P_{-w}, q) \; R_w \, \varphi_w(P^*_w, P_{-w}, q)$. We distinguish between two cases.

Case I. $\varnothing \, R_w \, \varphi_w(P^*_w, P_{-w}, q)$.
Let $Q_w = \varnothing$ be the empty truncation strategy. Then, by the stability of φ, $\varphi_w(Q_w, P_{-w}, q) = \varnothing$. Hence, $\varphi_w(Q_w, P_{-w}, q) \, R_w \, \varphi_w(P^*_w, P_{-w}, q)$.

Case II. $\varphi_w(P^*_w, P_{-w}, q) \, P_w \, \varnothing$.
Note that $\varphi_w(P^*_w, P_{-w}, q) \in F$. Let $f^* = \varphi_w(P^*_w, P_{-w}, q)$. Let Q_w be the truncation of P_w such that f^* is the last acceptable firm. Let $Q = (Q_w, P_{-w})$. We first show that for all $\mu \in S(Q, q)$, $\mu(w) \, R_w \, f^*$.

Suppose, to the contrary, that there is some $\tilde{\mu} \in S(Q, q)$ with $f^* \, P_w \, \tilde{\mu}(w)$. Then, since each firm f with $f^* P_w f$ is not listed (i.e., acceptable) in Q_w and since $\tilde{\mu}$ is individually rational with respect to Q, $\tilde{\mu}(w) = \varnothing$. By Alkan (2002, Proposition 6), for all $\mu \in S(Q, q)$, $\mu(w) = \varnothing$. In particular, $\varphi^W_w(Q, q) = \varnothing$.

We need to show that $\varphi^W(Q, q)$ is stable under (P^*_w, P_{-w}, q). Suppose, to the contrary, that there is a blocking pair for $\varphi^W(Q, q)$ under (P^*_w, P_{-w}, q). Then, the same pair blocks $\varphi^W(Q, q)$ under (Q, q). Hence, $\varphi^W(Q, q)$ is not stable under (Q, q), contradicting the stability of φ^W. Since $\varphi^W_w(Q, q) = \varnothing$, by Alkan (2002, Proposition 6) and the stability of φ, $\varphi_w(P^*_w, P_{-w}, q) = \varnothing$, contradicting $\varphi_w(P^*_w, P_{-w}, q) = f^*$. Hence, for all $\mu \in S(Q, q)$, $\mu(w) \, R_w \, f^*$. Since $\varphi(Q, q) \in S(Q, q)$, $\varphi_w(Q, q) \, R_w \, f^* = \varphi_w(P^*_w, P_{-w}, q)$. □

We complement Theorem 2 with two examples to show that it cannot be extended in the following two ways. The truncation correspondence is

- neither necessarily φ–exhaustive for an agent a if $q_a > 1$ and for all $i \in I \backslash \{a\}$, $q_i = 1$ (Example 1);
- nor necessarily φ–exhaustive for a group of agents on the same side of the market if for all $i \in I$, $q_i = 1$ (Example 2).

Example 1. (The truncation correspondence τ is not necessarily φ–exhaustive for an agent $a \in A$ if $q_a > 1$.)[23]

Consider a many–to–one matching market $(W, F, \succ, q)$ with 3 workers, 4 firms, and preferences over individual partners P given by the columns in Table 1. All potential partners are

[23]The preferences P are adapted from Roth (1985a, p. 283, Table I) and Roth and Sotomayor (1990, p. 146).

acceptable. For instance, $P_{f_1} = w_3\, w_1\, w_2\, \varnothing$. Worker w_1 has quota $q_{w_1} = 2$. Any other agent i has quota $q_i = 1$.

Table 1: Preferences P in Example 1

Workers			Firms			
w_1	w_2	w_3	f_1	f_2	f_3	f_4
$\boldsymbol{f_1}$	f_1	$\boldsymbol{f_3}$	$\boxed{w_3}$	$\boxed{\boldsymbol{w_2}}$	$\boxed{w_1}$	$\boxed{\boldsymbol{w_1}}$
f_2	$\boxed{\boldsymbol{f_2}}$	$\boxed{f_1}$	$\boldsymbol{w_1}$	w_1	$\boldsymbol{w_3}$	w_2
$\boxed{f_3}$	f_3	f_2	w_2	w_3	w_2	w_3
$\boxed{\boldsymbol{f_4}}$	f_4	f_4				

One easily verifies that the unique stable matching μ for (P, q) is given by

$$\mu: \begin{array}{ccc} w_1 & w_2 & w_3 \\ | & | & | \\ \{f_3, f_4\} & f_2 & f_1 \end{array}$$

which is the boxed matching in Table 1.

Consider the (dropping) strategy $P'_{w_1} = f_1\, f_4$ for worker w_1. Let $P' = (P'_{w_1}, P_{-w_1})$. The unique stable matching for (P', q) is given by

$$\mu': \begin{array}{ccc} w_1 & w_2 & w_3 \\ | & | & | \\ \{f_1, f_4\} & f_2 & f_3 \end{array}$$

which is the boldfaced matching in Table 1. Note that $\mu'(w_1) = \{f_1, f_4\} \succ_{w_1} \{f_3, f_4\} = \mu(w_1)$ for each responsive extension $\succ_{w_1}$ of P_{w_1}. Since μ and μ' are the unique stable matchings for (P, q) and (P', q), respectively, it follows that under each stable mechanism, in market $(\succ, q)$ firm w_1 can strictly improve its match by misreporting its preferences.

Table 2: Truncations and matches of w_1 in Example 1

Q_{w_1}	$\varphi_{w_1}(Q_{w_1}, P_{-w_1})$
$f_1\, f_2\, f_3\, f_4$	$\{f_3, f_4\}$
$f_1\, f_2\, f_3$	f_3
$f_1\, f_2$	f_1
f_1	f_1

In Table 2, we indicate the match of worker w_1 under each stable mechanism φ and for each profile (Q_{w_1}, P_{-w_1}) where Q_{w_1} is a truncation strategy. One immediately verifies that no individual truncation strategy for w_1 replicates or improves upon the match $\{f_1, f_4\} = \varphi_{w_1}(P'_{w_1}, P_{-w_1})$.

Following the proof of Theorem 1, for each truncation strategy Q_{w_1} of w_1, we provide a dropping strategy that consists of the acceptable firms that are matched to w_1 at $\varphi(Q_{w_1}, P_{-\{w_1\}})$ in the true relative order. Instead of truncation strategy $Q_{w_1} = f_1\, f_2\, f_3\, f_4$, w_1 can use dropping strategy $Q'_{w_1} = f_3\, f_4$ to be matched to $\{f_3, f_4\}$. Instead of $Q_{w_1} = f_1\, f_2\, f_3$, w_1 can use dropping strategy $Q'_{w_1} = f_3$ to be matched to $\{f_3\}$. Instead of truncation strategies $Q_{w_1} = f_1\, f_2$ and $Q_{w_1} = f_1$, w_1 can use dropping strategy $Q'_{w_1} = f_1$ to be matched to $\{f_1\}$.

Also, note that by introducing additional workers and firms, the negative result here can be extended in a straightforward way to situations in which for all $i \in I \backslash \{a\}$, $q_i \geq 1$. $\diamond$

Example 2. (The truncation correspondence $\boldsymbol{\tau}$ is not necessarily $\boldsymbol{\varphi}$–exhaustive for a group of agents on the same side of the market if for all $\boldsymbol{i \in I}$, $\boldsymbol{q_i = 1}$.)

Consider the one–to–one matching market $(W, F, \succ, q)$ with 4 workers, 4 firms, and preferences P given by the columns in Table 3. Only acceptable partners are depicted in Table 3. For instance, $P_{w_1} = f_4\, f_2\, f_3\, \varnothing\, f_1$. For each agent $i \in I$, $q_i = 1$.

Table 3: Preferences P in Example 2

Workers				Firms			
w_1	w_2	w_3	w_4	f_1	f_2	f_3	f_4
f_4	$\boldsymbol{f_1}$	$\boldsymbol{f_3}$	$\boldsymbol{f_4}$	$\boxed{w_3}$	$\boxed{w_4}$	$\boxed{w_1}$	$\boxed{w_2}$
$\boldsymbol{f_2}$	$\boxed{f_4}$	$\boxed{f_1}$	f_3	w_4	$\boldsymbol{w_1}$	w_4	w_1
$\boxed{f_3}$			f_1	$\boldsymbol{w_2}$		$\boldsymbol{w_3}$	$\boldsymbol{w_4}$
			$\boxed{f_2}$				

One easily verifies that the firm–optimal stable matching $\mu = \varphi^F(P, q)$ is given by

$$\mu : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ f_3 & f_4 & f_1 & f_2 \end{array}$$

which is the boxed matching in Table 3.

Consider the profile of (dropping) strategies (P'_{w_1}, P'_{w_2}) where $P'_{w_1} = f_2$ and $P'_{w_2} = f_1$. Let $P' = (P'_{w_1}, P'_{w_2}, P_{-\{w_1, w_2\}})$. The firm–optimal stable matching $\mu' = \varphi^F(P', q)$ now equals

$$\mu' : \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \\ | & | & | & | \\ f_2 & f_1 & f_3 & f_4 \end{array}$$

which is the boldfaced matching in Table 3. Note that $\mu'(w_1) = f_2\, P_{w_1}\, f_3 = \mu(w_1)$ and $\mu'(w_2) = f_1\, P_{w_2}\, f_4 = \mu(w_2)$. It follows that under the firm–optimal stable mechanism, in market $(\succ, q)$ workers $\{w_1, w_2\}$ can strictly improve their matches by jointly misreporting their preferences.

Table 4: Truncations of w_1, w_2 and matches of w_2 in Example 2

Q_{w_1}	Q_{w_2}	$\varphi^F_{w_2}(Q_{w_1}, Q_{w_2} P_{-\{w_1,w_2\}})$
$f_4\, f_2\, f_3$	$f_1\, f_4$	f_4
$f_4\, f_2$	$f_1\, f_4$	f_4
f_4	$f_1 f_4$	f_4
$f_4\, f_2\, f_3$	f_1	$\varnothing$
$f_4\, f_2$	f_1	$\varnothing$
f_4	f_1	$\varnothing$

In Table 4, we indicate the match of worker w_2 under the firm–optimal stable mechanism φ^F for each profile $(Q_{w_1}, Q_{w_2} P_{-\{w_1,w_2\}})$ where Q_{w_1} and Q_{w_2} are truncation strategies.[24] One immediately verifies that no pair of truncation strategies for w_1 and w_2 leads to a match for w_2 that is weakly preferred to $f_1 = \varphi^F_{w_2}(P'_{w_1}, P'_{w_2}, P_{-\{w_1,w_2\}})$. $\diamond$

We conclude with Table 5, which summarizes all our (positive and negative) findings.

Table 5: Summary of results. Given the quotas of the workers and firms, + (−) means that the correspondence is (not necessarily) exhaustive.

<table>
<tr><th rowspan="2"></th><th colspan="3">Quotas</th><th rowspan="2">φ–exhaustive for worker w</th><th colspan="2">Quotas</th><th rowspan="2">φ–exhaustive for a group of workers</th></tr>
<tr><th>Worker w</th><th>Other workers</th><th>Firms</th><th>Workers</th><th>Firms</th></tr>
<tr><td>Dropping correspondence</td><td colspan="2">≥ 1</td><td>≥ 1</td><td>+ (Theorem 1)</td><td>≥ 1</td><td>≥ 1</td><td>+ (Theorem 1)</td></tr>
<tr><td rowspan="2">Truncation correspondence</td><td>= 1</td><td>≥ 1</td><td>≥ 1</td><td>+ (Theorem 2)</td><td rowspan="2">= 1</td><td rowspan="2">= 1</td><td rowspan="2">− (Example 2)</td></tr>
<tr><td>> 1</td><td>= 1</td><td>= 1</td><td>− (Example 1)</td></tr>
</table>

4 Concluding Remarks

In this section, we discuss three important issues. First, we briefly comment on setwise stable mechanisms. Second, we show that there is a subcorrespondence of the dropping correspondence that is exhaustive. We also show this subcorrespondence cannot be "reduced" further. Finally, we explore the number of truncation strategies an agent with quota 1 has to consider in any stable mechanism.

Our results also hold for setwise stable mechanisms.[25] Theorem 1 and Theorem 2 still hold

[24] For each pair of truncation strategies of w_1 and w_2, we can construct a pair of dropping strategies that consist of the acceptable firms that they are matched to at $\varphi(Q_{w_1}, Q_{w_2} P_{-\{w_1,w_2\}})$ in the true relative order (as described in the proof of Theorem 1). Then, these dropping strategies yield the same matches for w_1 and w_2.

[25] A setwise stable matching is an individually rational matching that cannot be blocked by a coalition that forms new matches only among its members, but may preserve some of its matches outside of the coalition. See Roth (1984a), Sotomayor (1999b), and Echenique and Oviedo (2006) for a discussion on setwise stability.

under setwise stability since whenever the set of setwise stable matchings is non-empty, it is a subset of the set of pairwise stable matchings. Moreover, the conclusions in Examples 1 and 2 are still valid since the matchings in the examples are setwise stable.

An exhaustive correspondence is minimal if there is no proper subcorrespondence that is exhaustive as well. The dropping correspondence δ is not minimal. To see this, for an agent i with quota q_i, consider the subcorrespondence $\delta_{\leq}$ that associates each $\succ_i$ with the subset of dropping strategies with at most q_i acceptable partners. Formally, let $A(P_i)$ be the set of acceptable partners under P_i and $\delta_{\leq}(\succ_i) = \{P_i'$ is a dropping strategy of P_i and $0 \leq |A(P_i')| \leq q_i\}$. The proof of Theorem 1 shows that for any stable mechanism φ, $\delta_{\leq}$ is φ–exhaustive for a group of agents on the same side of the market. However, agents cannot exclusively focus on dropping strategies in which the number of acceptable partners is equal to their quota. Formally, for each agent i, let $\delta_{=}(\succ_i) = \{P_i'$ is a dropping strategy of P_i and $|A(P_i')| = q_i\}$. In the next example, we show that $\delta_{=}$ is not necessarily φ^F–exhaustive for a worker.

Example 3. ($\delta_{=}$ is not necessarily φ^F–exhaustive for a worker.)

Consider the many–to–one matching market $(W, F, \succ, q)$ with 2 workers, 2 firms, and preferences over individual partners P given by the columns in Table 6. All potential partners are acceptable. Worker w_1 has quota $q_{w_1} = 2$. Any other agent i has quota $q_i = 1$.

Table 6: Preferences P in Example 3

Workers		Firms	
w_1	w_2	f_1	f_2
$\boldsymbol{f_1}$	$\boldsymbol{f_2}$	$\boxed{w_2}$	$\boxed{w_1}$
$\boxed{f_2}$	$\boxed{f_1}$	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$

One easily verifies that the firm–optimal stable matching $\mu = \varphi^F(P, q)$ is given by

$$\mu : \begin{array}{cc} w_1 & w_2 \\ | & | \\ f_2 & f_1 \end{array}$$

which is the boxed matching in Table 6.

Note that $\delta_{=}(\succ_{w_1}) = \{P_{w_1}\}$. Now, consider the strategy $P'_{w_1} = f_1$ for worker w_1. Let $P' = (P'_{w_1}, P_{-w_1})$. The firm–optimal stable matching $\mu' = \varphi^F(P', q)$ is given by

$$\mu' : \begin{array}{cc} w_1 & w_2 \\ | & | \\ f_1 & f_2 \end{array}$$

which is the boldfaced matching in Table 6.

Then, $\varphi^F_{w_1}(P'_{w_1}, P_{-w_1}, q) \succ_{w_1} \varphi^F_{w_1}(P_{w_1}, P_{-w_1}, q)$. Hence, the correspondence $\delta_=$ is not φ^F–exhaustive for w_1. ◇

The minimum number of strategies that an agent should consider depends on the mechanism at hand. For instance, in the many–to–one matching model (where workers' quotas equal one), the worker–optimal stable mechanism is strategy–proof for the workers. Hence, in that case each worker w only needs 1 truncation strategy, namely P_w. Formally, let the identity correspondence Ψ be defined by $\Psi(\succ_w) = \{P_w\}$ for all preference relations $\succ_w$. Then, under the worker–optimal stable mechanism φ^W, the identity correspondence is exhaustive and trivially minimal for each worker.

However, the next example shows that under the firm–optimal stable mechanism φ^F, the truncation correspondence is (exhaustive and) minimal for each worker. The two observations about φ^W and φ^F imply that for any stable mechanism the number of truncation strategies that a worker w has to consider is between 1 and $\max\{1, |A(P_w)|\}$. (The worker–optimal stable mechanism and the firm–optimal stable mechanism show that the bounds are tight.)

Example 4. Consider the one–to–one matching market (W, F, P, q) with k workers, k firms, and preferences over individual partners P given by the columns in Table 7. Only acceptable partners are depicted in Table 7. For each $i \in I$, $q_i = 1$.

Table 7: Preferences P in Example 4

Workers							Firms						
w_1	…	w_{p-1}	w_p	…	w_{k-2}	w_{k-1}	w_k	f_1	…	f_{p-1}	f_p	…	f_{k-2}	f_{k-1}	f_k
f_1	$\vdots$	f_{p-1}	f_{p+1}	$\vdots$	f_{k-1}	f_k	f_1	w_1	$\vdots$	w_{p-1}	w_p	$\vdots$	w_{k-2}	w_{k-1}	w_k
			f_p	$\vdots$	f_{k-2}	f_{k-1}	f_2				w_k	$\vdots$	w_k	w_k	w_{k-1}
							$\vdots$				w_{p-1}	$\vdots$	w_{k-3}	w_{k-2}	
							f_p								
							$\vdots$								
							f_k								

One easily verifies that the firm–optimal stable matching $\mu = \varphi^F(P, q)$ is given by

$$\mu : \begin{array}{cccccccc} w_1 & \dots & w_{p-1} & w_p & \dots & w_{k-2} & w_{k-1} & w_k \\ | & \dots & | & | & \dots & | & | & | \\ f_1 & \dots & f_{p-1} & f_p & \dots & f_{k-2} & f_{k-1} & f_k. \end{array}$$

Now, consider the truncation strategy $P'_{w_k} = f_1 f_2 \dots f_p$ for worker w_k. Let $P' = (P'_{w_k}, P_{-w_k})$. The firm–optimal stable matching $\mu' = \varphi^F(P', q)$ is given by

$$
\mu' : \begin{array}{cccccccc} w_1 & \ldots & w_{p-1} & w_p & \ldots & w_{k-2} & w_{k-1} & w_k \\ | & \ldots & | & | & \ldots & | & | & | \\ f_1 & \ldots & f_{p-1} & f_{p+1} & \ldots & f_{k-1} & f_k & f_p. \end{array}
$$

One immediately verifies that no other truncation strategy leads to a match for w_k that is preferred to $f_p = \varphi^F_{w_k}(P', q)$. (Under any truncation strategy in which the number of acceptable firms l is such that $p < l \leq k$, w_k is matched to f_l and $f_p P_{w_k} f_l$. Under any truncation strategy in which the number of acceptable firms is such that $l < p$, w_k remains unmatched and $f_p P_{w_k} \emptyset$.)

Note that in this example, by varying p between 1 and k, we obtain a problem in which P'_{w_k} is the unique optimal truncation strategy of w_k (which matches him to f_p). Hence, for each truncation strategy with at least one acceptable firm, there is a problem in which the worker has to use this truncation strategy. For the truncation strategy with no acceptable firms, consider a problem in which a worker has a preference relation with no acceptable firms. Then, the worker uses his unique truncation strategy, namely the empty truncation strategy. Hence, under φ^F, worker w has to consider $\max\{1, |A(P_w)|\}$ truncation strategies.[26] ⋄

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[26] Note that by introducing additional workers and firms, the result here can be extended in a straightforward way to many–to–one matching markets.

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