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**Salvador Barberà  
Anke Gerber**

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# Sequential Voting and Agenda Manipulation: The Case of Forward Looking Tie-Breaking\*

Salvador Barberà<sup>†</sup>      Anke Gerber<sup>‡</sup>

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## Abstract

We study the possibilities for agenda manipulation under strategic voting for two prominent sequential voting procedures, the amendment and the successive procedure. We show that a well-known result for tournaments, namely that the successive procedure is (weakly) more manipulable than the amendment procedure at any given preference profile, extends to arbitrary majority quotas. Moreover, our characterizations of the attainable outcomes for arbitrary quotas allow us to compare the possibilities for manipulation across different quotas. It turns out that the simple majority quota maximizes the domain of preference profiles for which neither procedure is manipulable, but at the same time neither the simple majority quota nor any other quota uniformly minimize the scope of manipulation, once this becomes possible. Hence, quite surprisingly, simple majority voting is not necessarily the optimal choice of a society that is concerned about agenda manipulation.

*Keywords:* Sequential voting, agendas, manipulation

*JEL-Classification:* C72, D02, D71, D72.

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<sup>†</sup>MOVE, Universitat Autònoma de Barcelona and Barcelona GSE, Facultat d’Economia i Empresa, Edifici B, 08193 Bellaterra (Barcelona), Spain; e-mail: salvador.barbera@uab.cat.

<sup>‡</sup>Department of Economics, University of Hamburg, Von-Melle-Park 5, D-20146 Hamburg, Germany; e-mail: anke.gerber@wiso.uni-hamburg.de.

# 1 Introduction

Many societies and institutions, when choosing among alternatives, resort to sequential (multi stage) decision procedures, whereby different voters can determine, in a sequence of different steps, which alternatives are definitely out and which ones retain a chance to be considered again, until one of them is definitely selected. In this paper we study two families of classical methods of that sort, the amendment and the successive elimination procedures, both of which are used extensively in many parts of the world.<sup>1</sup>

It is known since ancient times<sup>2</sup> that the order in which different alternatives are considered along a sequential decision procedure can affect the final choice that a given society may reach, even if the preferences of its members stay the same. Therefore, setting the agenda is a very influential decision, and whoever controls the order of vote often has the possibility to engage in agenda manipulation, that is, of determining what will be the outcome of the choice process.<sup>3</sup> That power is not absolute, however, since there may be cases where any agenda would lead to the same outcome, as long as the rest of features defining a rule remains unchanged, and others where the range of choices that may be obtained is limited to some subset of all possible alternatives. In this paper we analyze the extent to which, given the preferences of voters and assuming strategic voting, an agenda setter could choose among several outcomes, and exactly what these outcomes may be in each voting situation. This is well-known for the special case of tournaments, for which Miller (1977) showed that the set of alternatives that are

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<sup>1</sup>Precise definitions of these rules are provided in section 2. These rules were named by Farquharson (1969) and then studied by Miller (1977, 1980) in the special but important case where decisions are made by simple majority. A recent axiomatic characterization is in Apesteguia et al. (2014). The relevance of these methods in parliamentary practice, and their use in different countries is discussed in Rasch (2000).

<sup>2</sup>See the Letter to Titus Aristo by Pliny the Young (A.D. 105) reproduced in McLean and Urken (1995). Farquharson's path-breaking book (1969) uses that letter extensively for motivation and analysis.

<sup>3</sup>We concentrate on manipulations that involve changes in the order of vote, while keeping the same set of alternatives. Other forms of agenda manipulation involve the addition of new items to the agenda, or the removal of some alternatives. This has been studied, among others, by Dutta et al. (2004) and Duggan (2006).

attainable by the successive procedure coincides with the *top cycle* and for which Banks (1985) provided a characterization of the attainable set for the amendment procedure which became to be known as the *Banks set*. However, to the best of our knowledge we are the first to provide characterizations of the sets of attainable alternatives for all possible majority quotas applied to the amendment and the successive voting procedure. Our characterizations differ from those of Miller (1977) and Banks (1985) and hence are no straightforward extensions from the case of tournaments to arbitrary quotas. Moreover, our general characterization results allow us to compare the power of the agenda setter across different quotas. This is not only relevant for institutional design but it is also the basis for a potential extension of the analysis of self-stable majority rules and constitutions from binary to arbitrary finite choice sets (Barberà and Jackson, 2004).

The exact characteristics of a sequential voting rule are determined by combining several ingredients. The first one is what we can call a *tree form*, which determines two aspects of the sequential process. One is the number and the nature of actions that agents can take at any node, starting from an initial one, until a terminal node is reached at the end of each path. But since one and only one alternative will eventually be attached to each terminal node, in order to define trees, a tree form is also defined by any restriction that may be imposed on the possible assignment of the same alternative to different terminal nodes. The two families of procedures we study here are based on binary tree forms, where each non-terminal node has two successors. The second ingredient defining a sequential rule is the *agenda*, that is, the specific assignment of alternatives to terminal nodes, respecting the restrictions imposed by the tree form. That assignment determines what choices will be made by society after following the possible path that leads to each terminal node. In all the cases we study, an agenda is just an order over the alternatives, because we provide specific and unique rules that translate each possible order into a unique admissible assignment of alternatives to the terminal nodes of the tree forms that we consider. A *tree* is then given by a tree form and by an agenda. Now, in order to turn a tree into a *sequential voting rule*, we must specify how will the different members of a voting body influence the choice of paths along the tree. Since we are working with binary trees, and we want to consider methods that treat all agents on the same foot, we consider

as possible methods all those that are defined by a quota  $q$ , with  $q$  between 1 and the number of voters. When confronted with two choices at any node, society will move to a pre-specified follower of that node if at least  $q$  people vote for it, and will otherwise take the opposite path.<sup>4</sup>

A sequential voting rule will thus be fully specified once we have a tree and a quota. Of course, a voting rule is defined independently of the preferences that may be held by different agents regarding the alternatives. It sets the rules through which agents will be able to contribute to the social decision. But in order to study the behavior of different agents under these rules, we need to know what their preferences will be. And then, given a profile of preferences, we'll have all the elements to study the strategic behavior of those agents. A tree and a quota then provide a game form, and when we add to them a preference profile we have a game.

Although our motivation is to study the strategic behavior of voters under these sequential rules, it turns out that most of our analysis can be carried out by just knowing a dominance relation among alternatives that generalizes the notion of a tournament, and that can be used to represent the preferences of society. Whereas a tournament is any complete and asymmetric relation over alternatives, the binary relations generated by comparing alternatives according to quotas different than simple majority give rise to relations that may fail one of these two properties. Moreover, some relations that are either complete, but not asymmetric, or asymmetric and not complete, may never be obtained as the dominance relation induced by a quota and a preference profile. Yet, our main characterization results still hold for this larger class of social preferences. Because of that, our work can also be understood as a natural extension of tournament theory, and the sets we identify can be compared to the different solution sets proposed for tournaments and for their extensions (Miller, 1977 and 1980; Shepsle and Weingast, 1984; Banks, 1985; Moulin, 1986; Banks and Bordes, 1988; Laslier, 1997).

We first provide characterizations of the unique equilibrium outcomes obtained by iterative elimination of weakly dominated strategies for each of the two

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<sup>4</sup>As we shall see later, this description implicitly implies the choice of a criterion to break ties, when these arise.

families of games we consider. We then use these characterizations to identify the sets of alternatives that could be the outcome of games that share the same tree form and the same rule to choose among nodes, but differ on the agenda. Comparisons among these sets allow us to discuss the degree of agenda manipulability of different rules in our classes. It turns out that both procedures are non-manipulable on the same set of preference profiles, namely those profiles for which there exists a (generalized) Condorcet winner, i.e. an alternative that dominates all others and in turn is not dominated. Moreover, if there is no Condorcet winner, then the successive procedure is more vulnerable towards agenda manipulation than the amendment procedure in the following sense: at any preference profile (or more generally, for any dominance relation), any outcome that can be achieved for some agenda under the amendment procedure can also be achieved by some agenda under the successive procedure, while the reverse is not true in general. While this result was already known for tournaments given the characterizations of Miller (1977) and Banks (1985), we are able to show that it holds for all quotas.

Comparing different quotas under the same sequential voting procedure we find that the set of preference profiles which do not allow for manipulation is maximized at simple majority voting and is otherwise weakly decreasing (increasing) in the quota for supermajority (submajority) quotas. This gives some support for simple majority voting if the possibility of agenda manipulation is a concern. On the other hand, if at a given preference profile there are opportunities for agenda manipulation under simple majority voting, then there is no quota that uniformly minimizes the degree of manipulability, neither for the successive nor for the amendment procedure. There are even cases where a submajority quota minimizes the possibilities for manipulation.

The outline of the paper is the following. In section 2 we introduce general binary voting games and derive the equilibrium outcome of the voting game for the amendment and sequential procedure at a given agenda. In sections 3 and 4 we characterize the set of outcomes that can be obtained by agenda manipulation for the amendment and sequential procedures. In section 5 we compare the scope of manipulation under the amendment and successive procedures for different quotas. Section 6 concludes.

## 2 Sequential Binary Voting Games

Let there be a finite set of alternatives  $X$  with  $\#X \geq 2$ .<sup>5</sup> A *binary voting tree on  $X$*  is a tree in which every non-terminal node has exactly two successors, left and right, and to every terminal node an alternative in  $X$  is assigned, so that this mapping is onto.<sup>6</sup> Formally, we define a binary voting tree on  $X$  to be a quadruple  $(X, N, \triangleright, \phi)$ , such that the following conditions are satisfied.

1.  $N$  is a finite set of *nodes*,
2.  $\triangleright$  is a binary relation on  $N$  which satisfies the following conditions.
  - (i) there exists a unique  $\nu_0 \in N$  (the *initial node*) such that
$$\{\nu \mid \nu \in N \text{ and } \nu_0 \triangleright \nu\} = \emptyset,$$
  - (ii) for all  $\nu \in N \setminus \{\nu_0\}$ , there exists a unique  $\nu' \in N$  with  $\nu \triangleright \nu'$ ,
  - (iii) there exists a nonempty subset  $T \subset N$  of *terminal nodes* such that for all  $\nu \in T$ ,
$$\{\nu' \mid \nu' \in N \text{ and } \nu' \triangleright \nu\} = \emptyset,$$
  - (iv) for all  $\nu \in N \setminus T$ ,  $\{\nu' \mid \nu' \triangleright \nu\} = \{l(\nu), r(\nu)\}$ ,<sup>7</sup>
3.  $\phi : T \rightarrow X$  is an onto function assigning to each terminal node a unique alternative in  $X$ .

If  $\nu \triangleright \nu'$  for  $\nu, \nu' \in N$ , then we call  $\nu$  a *successor* of  $\nu'$  and  $\nu'$  a *predecessor* of  $\nu$ . A non-terminal node of a binary voting tree on  $X$  is called a *decision node*.

Let there be  $n$  voters. Every voter  $i$  has a strict preference ordering  $\mathcal{P}_i$  over  $X$ , i.e.  $\mathcal{P}_i$  is complete (for all  $x, y \in X$  with  $x \neq y$ , it is true that  $x\mathcal{P}_iy$  or  $y\mathcal{P}_ix$ ),

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<sup>5</sup> $\#A$  denotes the number of elements in a finite set  $A$ .

<sup>6</sup>For purposes of expediency we define trees directly, rather than starting with tree forms as introduced in section 1. Thus, at this stage the role of agendas is implicit, and the one suggested in section 1. It will become more explicit when we introduce the binary voting games for the amendment and successive procedure below.

<sup>7</sup>Since every decision node is assumed to have exactly two successors, we follow Austen-Smith and Banks (2005) and label the successors of every  $\nu \in N \setminus T$  as  $l(\nu)$  (left successor), and  $r(\nu)$  (right successor).

transitive (for all  $x, y, z \in X$ , if  $x\mathcal{P}_iy$  and  $y\mathcal{P}_iz$ , then  $x\mathcal{P}_iz$ ) and asymmetric (for all  $x, y \in X$ , if  $x\mathcal{P}_iy$ , then  $\neg y\mathcal{P}_ix$ ). Let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_n)$  be the profile of voters' preferences. Then, for every binary voting tree  $(X, N, \triangleright, \phi)$  on  $X$  and any quota  $q \in \{1, \dots, n\}$  we can define a *sequential binary voting game on  $X$* ,  $(X, N, \triangleright, \phi, \mathcal{P}, q)$ , as follows: at every non-terminal node  $\nu$  there is a  $q$ -majority vote over the successors  $l(\nu)$  and  $r(\nu)$ , such that  $r(\nu)$  wins, if at least  $q$  voters vote in favor of  $r(\nu)$ , and  $l(\nu)$  wins otherwise. Obviously, unless  $n$  is odd and  $q = (n + 1)/2$ , the outcome of the vote may depend on the labeling of the successors of a decision node. If  $r(\nu)$  wins, the next  $q$ -majority vote is over the successors  $l(r(\nu))$  and  $r(r(\nu))$  of  $r(\nu)$ , while if  $l(\nu)$  wins, the next  $q$ -majority vote is over the successors  $l(l(\nu))$  and  $r(l(\nu))$  of  $l(\nu)$ . Voter  $i$ 's strategy then is a function  $\sigma_i : N \setminus T \rightarrow N$  such that  $\sigma_i(\nu) \in \{l(\nu), r(\nu)\}$  for all  $\nu \in N \setminus T$ . By following the winning successors through the tree every strategy profile  $\sigma = (\sigma_1, \dots, \sigma_n)$  determines a unique path from the initial node  $\nu_0$  to a terminal node  $\nu(\sigma) \in T$  which is associated with a unique alternative  $\phi(\nu(\sigma)) \in X$ .

Since the sequential binary voting games defined above can have very implausible Nash equilibria, where all voters coordinate on the same strategy irrespective of their preferences, we restrict to the class of Nash equilibria in undominated strategies as it is common in the literature on voting games. Recall that a normal form game is *dominance solvable*, if all players are indifferent between all strategy profiles that survive the iterative procedure where all weakly dominated strategies of all players are simultaneously eliminated at each stage. An extensive form game (like the sequential binary voting game defined above) is dominance solvable, if the corresponding normal form game is dominance solvable.<sup>8</sup> We will now argue that the sequential binary voting game  $(X, N, \triangleright, \phi, \mathcal{P}, q)$  is dominance solvable for all quotas  $q$ : first, for every voter  $i$  we can eliminate all strategies, where  $i$  does not vote for his preferred terminal node at every last decision node, i.e. at every decision node whose successors are two terminal nodes. Observe that given the strict preference ordering  $\mathcal{P}_i$ , voter  $i$  is indifferent between two terminal nodes  $\nu$  and  $\nu'$  if and only if  $\phi(\nu) = \phi(\nu')$ . Hence, voter  $i$  is indifferent between two terminal nodes if and only if all voters  $j \neq i$  are indifferent between these

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<sup>8</sup>In voting theory dominance solvability is also known as “sophisticated voting” (see Farquharson, 1969).



nodes. Thus, conditional on reaching a specific terminal decision node, all strategy profiles surviving the first elimination round are outcome equivalent. Hence, after the first elimination round every voter has well defined preferences over all last decision nodes since all these nodes are associated with a unique outcome (alternative) under the surviving strategy profiles. We continue by eliminating for every voter  $i$  all strategies where  $i$  does not vote for his preferred successor node at every penultimate decision node, i.e. at every decision node that is a predecessor of two last decision nodes. Again, after this second elimination round every voter has well defined preferences over all penultimate decision nodes and if one voter is indifferent between two penultimate decision nodes, all voters are indifferent. Continuing in this way we finally arrive at the initial node and we eliminate for every voter  $i$  all strategies where  $i$  does not vote for his preferred successor node. Then, all voters are indifferent between all remaining strategy profiles and all these surviving profiles  $\sigma$  lead to the same alternative  $\phi(\nu(\sigma)) \in X$  which we call the *outcome*,  $o(X, N, \triangleright, \phi, \mathcal{P}, q)$ , of the sequential binary voting game. Hence, we have the following result (cf. McKelvey and Niemi, 1978; Moulin, 1979; Gretlein, 1982; Austen-Smith and Banks, 2005).

**Theorem 2.1** *Every sequential binary voting game  $(X, N, \triangleright, \phi, \mathcal{P}, q)$  is dominance solvable.*

In this paper we will focus on two specific binary voting trees on  $X$  which represent two prominent sequential voting procedures: the *amendment procedure* and the *successive procedure*. Both procedures start with an *agenda*, i.e. an ordering  $(x_1, x_2, \dots, x_m)$  of the alternatives in  $X$ , where we assume that  $m \geq 2$ .

#### *Amendment Procedure*

Given an agenda  $(x_1, \dots, x_m)$ , the binary voting tree  $(X, N, \triangleright, \phi)$  for the amendment procedure is such that the first vote is over  $x_1$  and  $x_2$ , the second vote is over the winner of the first vote and  $x_3$ , the third vote is over the winner of the second vote and  $x_4$ , and so on until all alternatives are exhausted. Figure 1 shows the binary voting tree for the amendment procedure in the case

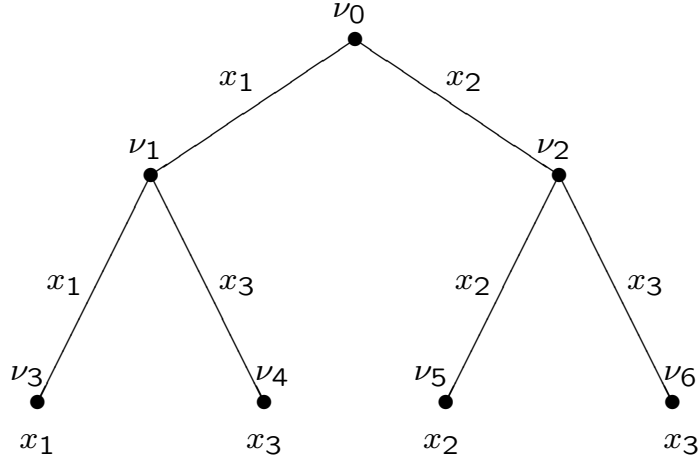


Figure 1: Binary voting tree for the amendment procedure with agenda  $(x_1, x_2, x_3)$ .

where  $m = 3$ . Observe that the agenda also yields a natural labeling of the two successor nodes of every non-terminal decision nodes: at every decision node  $\nu$  there is a vote over two alternatives,  $x_i$  and  $x_j$ , where  $i < j$ . The left successor,  $l(\nu)$ , then is the node reached if alternative  $x_i$  wins, and  $r(\nu)$  is the node reached if alternative  $x_j$  wins.

Consider now the sequential binary voting game  $(X, N, \triangleright, \phi, \mathcal{P}, q)$  for the amendment procedure. By Theorem 2.1 the game is dominance solvable and we have seen that there is a simple backwards induction procedure to derive the unique outcome of the game. To determine this outcome, we let  $P$  denote the social preference relation on  $X$  under sincere voting with quota  $q$ , i.e. for all  $x, y \in X$ ,

$$xPy \iff \#\{i \mid x\mathcal{P}_iy\} \geq q. \quad (1)$$

Observe that for given quota  $q$ ,  $P$  is either complete or asymmetric or both. In the latter case  $P$  defines a *tournament*.<sup>9</sup> By allowing  $P$  to be incomplete or to violate asymmetry we depart from the tournament literature and provide a more general analysis of sequential voting games. Let  $o^A(x_1, x_2, \dots, x_m)$  denote the

<sup>9</sup>This is the case if  $n$  is odd and  $q = (n + 1)/2$ .

outcome of the sequential binary voting game for the amendment procedure with a given agenda  $(x_1, \dots, x_m)$ . Then  $o^A(x_1, x_2, \dots, x_m)$  is inductively defined over the number of alternatives in the agenda as follows.

1. If  $m = 2$ , then

$$o^A(x_1, x_2) = \begin{cases} x_1, & \text{if } \neg x_2 P x_1, \\ x_2, & \text{if } x_2 P x_1. \end{cases} \quad (2)$$

2. Let  $m \geq 3$  and suppose the outcome has been defined for any agenda with up to  $m - 1$  alternatives. Consider the agenda  $(x_1, x_2, \dots, x_m)$ . Then

$$\begin{aligned} & o^A(x_1, x_2, \dots, x_m) \\ &= \begin{cases} o^A(x_1, x_3, \dots, x_m), & \text{if } \neg o^A(x_2, x_3, \dots, x_m) P o^A(x_1, x_3, \dots, x_m), \\ o^A(x_2, x_3, \dots, x_m), & \text{if } o^A(x_2, x_3, \dots, x_m) P o^A(x_1, x_3, \dots, x_m). \end{cases} \end{aligned} \quad (3)$$

Note that in (2) and (3) we use a forward looking tie-breaking rule according to which the alternative that is introduced later in the agenda proceeds to the next vote if and only if the final outcome that is reached in this case dominates the final outcome that is reached if the alternative introduced earlier proceeds to the next vote.

### *Successive Procedure*

Given an agenda  $(x_1, \dots, x_m)$ , the binary voting tree  $(X, N, \triangleright, \phi)$  for the successive procedure is such that the first vote is over the approval of  $x_1$ . If  $x_1$  is approved, the voting is over and the outcome is  $x_1$ . If  $x_1$  is rejected, the next vote is over the approval of  $x_2$ . If  $x_2$  is approved, the voting is over and the outcome is  $x_2$ . Otherwise, if  $x_2$  is rejected the next vote is over the approval of  $x_3$ , and so on. If  $x_{m-1}$  is rejected, the outcome is  $x_m$ . Figure 2 shows the binary voting tree for the successive procedure in the case where  $m = 3$ . Again, the agenda yields a natural labeling of the two successor nodes of every non-terminal decision nodes: at every decision node  $\nu$  there is a vote over approving or rejecting an alternative  $x_i$ . The left successor,  $l(\nu)$ , then is the node reached if  $x_i$  is approved, and  $r(\nu)$  is the node reached if  $x_i$  is rejected.

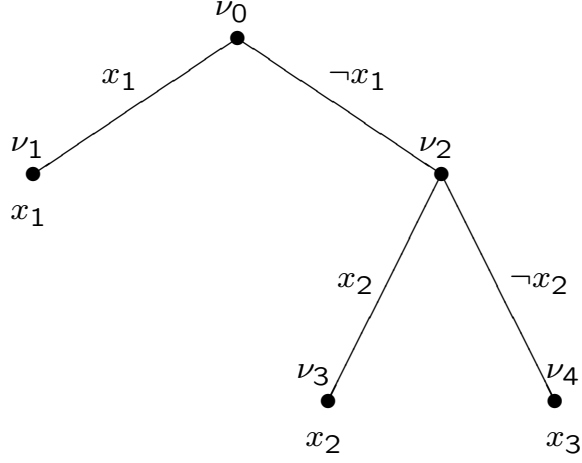


Figure 2: Binary voting tree for the successive procedure with agenda  $(x_1, x_2, x_3)$ .

As for the amendment procedure we consider the sequential binary voting game  $(X, N, \triangleright, \phi, \mathcal{P}, q)$  for the successive procedure. Again, let  $P$  be the social preference relation on  $X$  as defined in (1). Then the outcome  $o^S(x_1, x_2, \dots, x_m)$  for the successive procedure is inductively defined over the number of alternatives in the agenda as follows.

1. If  $m = 2$ , then

$$o^S(x_1, x_2) = \begin{cases} x_1, & \text{if } \neg x_2 P x_1, \\ x_2, & \text{if } x_2 P x_1. \end{cases} \quad (4)$$

2. Let  $m \geq 3$  and suppose the outcome has been defined for any agenda with up to  $m - 1$  alternatives. Consider the agenda  $(x_1, x_2, \dots, x_m)$ . Then

$$o^S(x_1, x_2, \dots, x_m) = \begin{cases} x_1 & , \text{ if } \neg o^S(x_2, x_3, \dots, x_m) P x_1, \\ o^S(x_2, x_3, \dots, x_m), & \text{ if } o^S(x_2, x_3, \dots, x_m) P x_1. \end{cases} \quad (5)$$

Again note the use of a forward looking tie-breaking rule in (4) and (5).

The inductive definitions of  $o^A(x_1, x_2, \dots, x_m)$  and  $o^S(x_1, x_2, \dots, x_m)$  for the amendment and successive procedure in equations (2)-(5) reveal that the outcome of an agenda only depends on the social preference relation  $P$  and is invariant with respect to changes in the individual preferences  $\mathcal{P}_i$  that leave  $P$  unchanged. Hence, in the following we will consider the general case, where society makes binary choices according to an arbitrary binary relation  $P$  on  $X$ , which is not necessarily derived from majority voting with quota  $q$ . We refer to  $P$  as a *dominance relation* and continue to use the term *social preference relation* if  $P$  is derived from  $q$ -majority voting. As long as we assume that society is forward looking in the sense that at every decision node in the binary voting tree it chooses the consequence that is preferred according to  $P$ , the outcome of an agenda for the amendment and successive procedures is still given by (2) and (3), respectively by (4) and (5).

The following two sections will provide characterizations of the outcomes that an agenda setter can achieve under the amendment and successive procedure for a given dominance relation  $P$  of society.

### 3 Choosing with the Amendment Procedure

In this section we consider the case where society uses the amendment procedure for a given agenda in order to choose an alternative from  $X$ . Hence, we assume that society has a dominance relation  $P$  on  $X$  and that the outcome of an agenda is determined according to (2) and (3). In order to characterize the set of alternatives that can be supported as the outcome for some agenda, we will assume that  $P$  is complete or asymmetric.<sup>10</sup> We first derive some auxiliary results. All proofs are in the appendix.

The first lemma provides a recursive procedure for deriving the outcome of an agenda if  $P$  is complete.<sup>11</sup>

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<sup>10</sup>These are the cases that arise if  $P$  is derived from majority voting with quota  $q$ . By  $\lfloor c \rfloor$  ( $\lceil c \rceil$ ) we denote the largest (smallest) integer less (larger) than or equal to  $c \in \mathbb{R}$ . Then  $P$  is complete if  $q \leq \lceil \frac{n}{2} \rceil$ , and  $P$  is asymmetric if  $q \geq \lfloor \frac{n}{2} \rfloor + 1$ .

<sup>11</sup>Shepsle and Weingast (1984) have proved a similar result for the special case of tournaments, i.e. where  $P$  is derived from simple majority voting.

**Lemma 3.1** *Let  $P$  be complete. Then,  $o^A(x_1, \dots, x_m) = \hat{x}_1$ , where the auxiliary alternatives  $\hat{x}_1, \dots, \hat{x}_m$ , are recursively defined as follows.*

$$\hat{x}_m = x_m,$$

$$\text{and for } k = m - 1, \dots, 1, \hat{x}_k = \begin{cases} x_k & , \text{ if } \neg \hat{x}_l P x_k \text{ for all } l = k + 1, \dots, m \\ \hat{x}_{k+1}, & \text{ otherwise.} \end{cases}$$

Observe that we cannot dispense with the completeness assumption in Lemma 3.1. To see this, consider the following example.

**Example 3.1** *Let  $X = \{x_1, x_2, x_3\}$  and consider the incomplete dominance relation  $P$  given by  $x_3 P x_1$ . Let the agenda be given by  $(x_1, x_2, x_3)$ . Then, applying the recursive procedure in Lemma 3.1 we get*

$$\hat{x}_3 = x_3, \hat{x}_2 = x_2, \hat{x}_1 = \hat{x}_2 = x_2.$$

However,  $\hat{x}_1 \neq o^A(x_1, x_2, x_3) = x_3$ .<sup>12,13</sup>

In the case where  $P$  is complete, from Lemma 3.1 we can derive the following necessary condition for an alternative to be the outcome of an agenda.

**Corollary 3.1** *Let  $P$  be complete and let  $x = o^A(x_1, \dots, x_m)$ . Then, for all  $y \in X$  with  $y \neq x$ , at least one of the following two conditions is satisfied.*

- (i)  $\neg y P x$ .
- (ii) There exists  $z \in X$  with  $z P y$  and  $\neg z P x$ .

Note that in the special case of a tournament, i.e. when  $P$  is complete and asymmetric, an alternative  $x$  belongs to the *uncovered set* (Miller, 1980) if and

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<sup>12</sup>This is a counterexample to Theorem 3.4 in Banks and Bordes (1988) who claim that the recursive procedure in Lemma 3.1 applies to an incomplete dominance relation  $P$ .

<sup>13</sup>The following preference profile  $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$  for three voters generates  $P$  for majority voting with quota  $q = 3$ :  $x_3 \mathcal{P}_i x_1 \mathcal{P}_i x_2$  for  $i = 1, 2$ , and  $x_2 \mathcal{P}_3 x_3 \mathcal{P}_3 x_1$ .

only if condition (i) and/or (ii) in Corollary 3.1 are satisfied for all alternatives  $y \neq x$ . Hence, Corollary 3.1 recovers Miller's (1980) result that the set of sophisticated voting outcomes for the amendment procedure is a subset of the uncovered set in the case of a tournament.

The following two lemmas hold without imposing any assumptions on  $P$ . For a given agenda  $(x_1, \dots, x_m)$  define the auxiliary alternatives  $\bar{x}_1, \dots, \bar{x}_m$ , by  $\bar{x}_m = x_m$  and

$$\bar{x}_k = o^A(x_k, x_{k+1}, \dots, x_m), \text{ for } k = 1, \dots, m-1. \quad (6)$$

**Lemma 3.2** *For all  $k = 1, \dots, m-1$ ,*

$$\bar{x}_k = x_k \iff \neg \bar{x}_l P x_k \text{ for all } l = k+1, \dots, m.$$

**Lemma 3.3** *If  $x_k = o^A(x_1, \dots, x_m)$  for some  $k \leq m-1$ , then  $x_k = \bar{x}_k$ .*

We are now ready to state our main characterization result that provides a necessary and sufficient condition for an alternative to be the outcome of some agenda under the amendment procedure.

**Theorem 3.1** *Let  $P$  be complete or asymmetric. Let  $x \in X$  and let*

$$Y(x) = \{y \in X \mid y P x \text{ and } \neg x P y\}.$$

*Then there exists an agenda  $(x_1, \dots, x_m)$  with  $x = o^A(x_1, \dots, x_m)$  if and only if for all  $y \in Y(x)$ , there is an alternative  $z(y) \in X$ , such that the following two conditions are satisfied.*

(i)  $z(y) P y$  and  $\neg z(y) P x$ .

(ii) *There exists an ordering  $(z_1, \dots, z_t)$  of the alternatives in*

$$Z(x) = \{z \mid z = z(y) \text{ for some } y \in Y(x)\},$$

*such that  $\neg z_l P z_k$  for all  $k = 1, \dots, t-1$ , and for all  $l > k$ .*

The proof of the theorem is in the Appendix. Let us briefly hint at the major ideas behind it. Regarding necessity, it is clear that the choice of  $x$  is threatened by the existence of the elements of  $Y(x)$ , that would eliminate  $x$  if ever really confronted against it. Hence, alternatives that do not beat  $x$  but beat those in  $Y(x)$  are needed, and these are the ones in the set  $Z(x)$ . The additional conditions on the dominance relation among the alternatives in  $Z(x)$  are also needed to ensure that they can be presented in an appropriate order, so as to fulfill their role as deterrents of alternatives in  $Y(x)$ . The sufficiency part consists in exhibiting a way to order the alternatives that would deliver  $x$  as an outcome, given that the conditions are satisfied. These orders depend on whether we consider the case of a complete or an asymmetric dominance relation. For the complete case, if  $Y(x)$  is empty, then use any order where  $x$  is the last alternative in the agenda. Otherwise, use the order

$$(x_1, \dots, x_{m-r-t-1}, x, y_1, \dots, y_r, z_1, \dots, z_t),$$

where here the order of the  $y_i$ 's is any, and the  $x_i$ 's stand for those alternatives other than  $x$  that do not belong to either  $Y(x)$  or to  $Z(x)$ . Similarly, for the asymmetric case, if  $Y(x)$  is empty use any order where  $x$  is the first alternative in the agenda, and if  $Y(x)$  is nonempty, use the order

$$(x, x_1, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t),$$

where again the order of the  $y_i$ 's is any, and the  $x_i$ 's stand for those alternatives other than  $x$  that do not belong to either  $Y(x)$  or to  $Z(x)$ .

For later use we provide the following alternative characterization of the set of attainable alternatives under the amendment procedure. It is immediate to see that the following characterization is equivalent to the one in Theorem 3.1.

**Theorem 3.2** *Let  $P$  be complete or asymmetric. Let  $x \in X$  and let*

$$Y(x) = \{y \in X \mid yPx \text{ and } \neg xPy\}.$$

*Then there exists an agenda  $(x_1, \dots, x_m)$  with  $x = o^A(x_1, \dots, x_m)$  if and only if there is a set of alternatives  $Z(x)$  with  $x \notin Z(x)$  and  $\neg zPx$  for all  $z \in Z(x)$ , such that the following two conditions are satisfied.*



- (i) For all  $y \in Y(x)$  there exists a  $z \in Z(x)$  such that  $zPy$ .
- (ii) There exists an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x)$  such that  $\neg z_l P z_k$  for all  $k = 1, \dots, t-1$ , and for all  $l > k$ .

Clearly, in many cases one can attain a given alternative through several orders. Therefore, no uniqueness claim is placed on the orders that we use in the sufficiency part of the proof. However, it is interesting to realize that, in the asymmetric case, placing in first place the alternative that one wants to obtain is always effective, in the following sense.

**Corollary 3.2** *Let  $P$  be asymmetric and let  $(x_1, \dots, x_m)$  be an agenda. If for some  $k > 1$ ,  $x_k = o^A(x_1, \dots, x_m)$ , then there exists an agenda  $(x'_1, \dots, x'_m)$  with  $x'_1 = x_k$  and*

$$x_k = o^A(x'_1, \dots, x'_m).$$

The following example shows that it is not sufficient to move the outcome of an agenda one or only a few steps forward. Unless it is moved to the first position in the agenda, it need not remain the outcome.

**Example 3.2** Let  $X = \{x_1, x_2, x_3\}$  and consider the asymmetric and incomplete dominance relation  $P$  given by

$$x_2 P x_1 \text{ and } x_3 P x_1.^{14}$$

Then  $x_3 = o^A(x_1, x_2, x_3)$  and  $x_3 = o^A(x_3, x_1, x_2)$ . However,  $x_2 = o^A(x_1, x_3, x_2) = o^A(x_2, x_3, x_1)$ .

If  $P$  is asymmetric, then Theorem 3.1 provides an alternative characterization of the set of possible outcomes to the one given in Banks and Bordes, (1988,

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<sup>14</sup>The following preference profile  $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$  for three voters generates  $P$  for majority voting with quota  $q = 3$ :  $x_2 \mathcal{P}_i x_3 \mathcal{P}_i x_1$  for  $i = 1, 2$ , and  $x_3 \mathcal{P}_3 x_2 \mathcal{P}_3 x_1$ .

Theorem 3.7).<sup>15</sup> To state their result, we need some additional definitions. The pair  $(X', d)$  with  $X' \subseteq X$  is a *trajectory* if  $d : X' \rightarrow \{1, \dots, m\}$  is one-to-one and  $d(x) > d(y)$  implies that  $\neg yPx$ . A trajectory  $(X', d)$  is *maximal* if for all  $y \in X \setminus X'$  the pair  $(X' \cup \{y\}, d')$  is not a trajectory, where  $d'(x) = d(x)$  for all  $x \in X'$  and  $d'(y) = \#X' + 1$ .

**Corollary 3.3 (Banks and Bordes, 1988)** *If  $P$  is asymmetric, then  $x = o^A(x_1, \dots, x_m)$  for some agenda  $(x_1, \dots, x_m)$  if and only if there exists a maximal trajectory  $(X', d)$  with  $d(x) = t$ , where  $t = \#X'$ .*

## 4 Choosing with the Successive Procedure

We now turn to the case where society uses the successive procedure for a given agenda in order to choose an alternative from  $X$ . Hence, we assume that society has a dominance relation  $P$  on  $X$  and that the outcome of an agenda is determined according to (4) and (5). Again we first derive some auxiliary results before presenting the characterization of the set of alternatives that can be achieved as the outcome for some agenda. To this end, we define the auxiliary alternatives  $\bar{x}_k$  by

$$\bar{x}_k = o^S(x_k, x_{k+1}, \dots, x_m) \text{ for } k = 1, \dots, m - 1.$$

The first lemma shows that an alternative which was eliminated at some stage will never return.

**Lemma 4.1** *Let  $(x_1, \dots, x_m)$  be an agenda. If  $\bar{x}_k \neq x_s$  for some  $s \geq k$ , then  $\bar{x}_l \neq x_s$  for all  $l < k$ .*

Lemma 4.1 immediately implies the following result.

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<sup>15</sup>Observe, however, that the proof of Theorem 3.7 in Banks and Bordes (1988) is incorrect whenever  $P$  is not complete, since it relies on the recursive procedure in Lemma 3.1. In Example 3.1 we showed that this procedure does not yield the outcome of an agenda if  $P$  is not complete.

**Lemma 4.2**  $x_k = o^S(x_1, \dots, x_m)$  for some  $1 \leq k \leq m$  if and only if  $\bar{x}_l = x_k$  for all  $l \leq k$ .

We are now ready to present our main characterization result for the successive procedure.

**Theorem 4.1** *Let  $P$  be complete or asymmetric. Let  $x \in X$  and let*

$$Y(x) = \{y \in X \mid yPx \text{ and } \neg xPy\}.$$

*Then there exists an agenda  $(x_1, \dots, x_m)$  with  $x = o^S(x_1, \dots, x_m)$  if and only if there is a set of alternatives  $Z(x)$  with  $x \notin Z(x)$  such that the following two conditions are satisfied.*

- (i) For all  $y \in Y(x)$  there exists a  $z \in Z(x)$  such that  $zPy$ , if  $P$  is complete, and such that  $\neg yPz$ , if  $P$  is asymmetric.*
- (ii) There exists an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x)$  such that  $\neg z_{l+1}Pz_l$  for all  $l = 1 \dots, t-1$ , and  $\neg z_1Px$ .*

Again let us briefly dwell on the major ideas of the proof which is in the Appendix. For necessity, any alternative in  $Y(x)$  which threatens  $x$  must be eliminated before it meets  $x$ . This is achieved by the alternatives in  $Z(x)$  which may in turn threaten  $x$ , but which can be placed in such an order that the alternative which actually meets  $x$  does not eliminate  $x$ . For sufficiency, we must find an order of the alternatives which delivers  $x$  as an outcome, given that the conditions are satisfied. This order again depends on whether we consider the case of a complete or an asymmetric dominance relation. For the complete case, if  $Y(x)$  is empty, then use any order where  $x$  is the last alternative in the agenda. Otherwise, use the order

$$(x_1, \dots, x_{m-r-1}, x, w_1, \dots, w_r).$$

Here, the  $x_i$ 's are all alternatives other than  $x$  that do not belong to either  $Y(x)$  or to  $Z(x)$ , and the  $w_i$ 's are alternatives that belong either to  $Y(x)$  or to  $Z(x)$ ,

and their order has to be selected in a delicate manner that is explained along the inductive proof. Similarly, for the asymmetric case, if  $Y(x)$  is empty use any order where  $x$  is the first alternative in the agenda, and if  $Y(x)$  is nonempty, use the order

$$(x, x_1, \dots, x_{m-r-1}, w_1, \dots, w_r),$$

where again the  $x_i$ 's are all alternatives other than  $x$  that do not belong to either  $Y(x)$  or to  $Z(x)$  and the  $w_i$ 's are alternatives that belong either to  $Y(x)$  or to  $Z(x)$  and that are ordered in a specific manner.

Like for the amendment procedure, if  $P$  is asymmetric and an alternative  $x$  can be obtained as the outcome for some agenda, then  $x$  is the outcome of an agenda where  $x$  is placed first.

**Corollary 4.1** *Let  $P$  be asymmetric and let  $(x_1, \dots, x_m)$  be an agenda. If for some  $k > 1$ ,  $x_k = o^S(x_1, \dots, x_m)$ , then there exists an agenda  $(x'_1, \dots, x'_m)$  with  $x'_1 = x_k$  and*

$$x_k = o^S(x'_1, \dots, x'_m).$$

We omit the proof of this corollary as it is similar to the proof of Corollary 3.2.

For the special case where  $P$  is a tournament, i.e. complete and asymmetric, Theorem 4.1 recovers the well-known result that the set of attainable outcomes under the successive procedure coincides with the top cycle (Miller, 1977). This result can be generalized to any asymmetric dominance relation  $P$ . In order to define the top cycle for an asymmetric dominance relation  $P$ , let  $R$  be the binary relation on  $X$  given by  $xRy$  if and only if  $\neg yPx$  for  $x, y \in X$ . Observe that  $R$  is complete since  $P$  is asymmetric. The *top cycle* of  $P$  then is the set of all alternatives  $x$  such that for all  $y \neq x$ , there exists a sequence of alternatives  $z_0, z_1, \dots, z_s$ , with  $z_0 = x, z_s = y$ , and  $z_l R z_{l+1}$  for all  $l = 1, \dots, s - 1$ . We then have the following corollary to Theorem 4.1.

**Corollary 4.2** *If  $P$  is asymmetric, then  $x = o^S(x_1, \dots, x_m)$  for some agenda  $(x_1, \dots, x_m)$  if and only if  $x$  is in the top cycle of  $P$ .*

## 5 On the Forms and Extent of Agenda Manipulation

In this section we focus on the possibilities that an agenda setter may find to use her power to determine the order of vote in her own favor, in order to get a most preferred alternative. In its most demanding version, non-manipulability would require that whoever is chosen as an agenda setter could not change the outcome at all, because it is the same regardless of the order of vote.

**Definition 5.1** *A sequential voting procedure is **non-manipulable** by any agenda setter at a given dominance relation  $P$  if it yields the same outcome regardless of the agenda.*

Note that the definition applies to any potential agenda setter.

It turns out that both the amendment and successive procedure are non-manipulable whenever there exists a (generalized) Condorcet winner, i.e. an alternative that dominates all others and in turn is not dominated. Hence, both procedures are non-manipulable on the same set of preference profiles. In order to state this result, for any dominance relation  $P$  we let  $O^A(P)$  ( $O^S(P)$ ) denote the set of alternatives that are outcomes for some agenda under the amendment (successive) procedure given  $P$ .

**Theorem 5.1** *Let  $P$  be complete or asymmetric and let  $x \in X$ . Then the following statements are equivalent.*

(i)  $O^S(P) = \{x\}$ .

(ii)  $O^A(P) = \{x\}$ .

(iii) For all  $y \neq x$  it is true that  $xPy$  and  $\neg yPx$ .

For those profiles where several outcomes could be reached, depending on the order of vote, it is possible to compare the choice flexibility that an agenda setter may obtain from alternative rules, as expressed by the following definition.

**Definition 5.2** *Given two sequential voting procedures, we say that one is **more agenda manipulable** than the other if, for any complete or asymmetric dominance relation  $P$ , the set of alternatives that are attainable by agenda manipulation under the latter is a subset of the former, and it is a strict subset for at least one dominance relation  $P$ .*

We can now state our first result on agenda manipulation.

**Proposition 5.1** *The successive procedure is more agenda manipulable than the amendment procedure.*

The claim that  $O^A(P) \subseteq O^S(P)$  for all preference relations  $P$  is an immediate implication of Theorem 3.2 and Theorem 4.1. To get an intuition for this result observe that the amendment procedure imposes stronger conditions on an alternative for it to survive the sequential voting procedure than the successive procedure. In order to obtain  $x$  as the outcome of an agenda for the successive procedure it is sufficient that  $x$  is not dominated by the outcome of some agenda for the remaining alternatives. Hence, it is sufficient to find some ordering  $(x_1, \dots, x_{m-1})$  of the alternatives different from  $x$ , such that  $\neg o^S(x_1, \dots, x_{m-1})Px$  (see (4) and (5)). By contrast, in order for  $x$  to be the outcome of the agenda  $(x, x_1, \dots, x_{m-1})$  under the amendment procedure,  $x$  must be the outcome of any agenda  $(x, x_k, \dots, x_{m-1})$  for  $k = 1, \dots, m - 1$  (see (2) and (3)).

Observe that Proposition 5.1 generalizes a known result for tournaments to arbitrary preference relations or arbitrary quotas, respectively.<sup>16</sup>

To verify that there exist relations  $P$  with  $O^A(P) \subsetneq O^S(P)$  consider the following example.

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<sup>16</sup>For tournaments, the result follows from the fact that the Banks set is a subset of the top cycle.

**Example 5.1** Let  $X = \{x, y, w, z\}$  and let  $P$  be given by

$$xPw, yPx, yPw, wPz, zPx \text{ and } zPy.^{17}$$

Then  $x = o^S(x, w, z, y)$ , but  $x \notin O^A(P)$ . In fact, only  $y, w$ , and  $z$  satisfy conditions (i) and (ii) in Theorem 3.1.

In what follows we analyse the role of the quota in determining the degree of manipulability of our rules for the special case where the social relation  $P$  is derived from a vote under a given quota. It turns out that the set of preference profiles at which the amendment and successive procedures are non-manipulable is maximized at simple majority voting. To state this result, we denote by  $\Phi(q)$  the set of profiles  $\mathcal{P}$  such that there exists a generalized Condorcet winner under majority voting with quota  $q$ , i.e.  $\Phi(q)$  is the set of profiles at which the amendment and the successive procedures are non-manipulable given  $q$  (cf. Theorem 5.1).

**Proposition 5.2** Let  $1 \leq q < q' \leq \lfloor \frac{n}{2} \rfloor + 1$  or  $\lfloor \frac{n}{2} \rfloor + 1 \leq q' < q \leq n$ . Then

$$\Phi(q) \subseteq \Phi(q').$$

In particular,  $\Phi(q)$  is maximal for  $q = \lfloor \frac{n}{2} \rfloor + 1$ , i.e. for simple majority voting.

We now fix a preference profile and compare the degree of manipulability across different quotas. Let  $O^A(\mathcal{P}, q)$  ( $O^S(\mathcal{P}, q)$ ) denote the set of alternatives that are outcomes under majority voting with quota  $q$  at profile  $\mathcal{P}$  for some agenda under the amendment (successive) procedure.

We first consider the amendment procedure. The following example shows that the sets  $O^A(\mathcal{P}, q)$  are not nested in general.

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<sup>17</sup>The following preference profile  $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$  for three voters generates  $P$  for simple majority voting:  $y\mathcal{P}_1x\mathcal{P}_1w\mathcal{P}_1z, w\mathcal{P}_2z\mathcal{P}_2y\mathcal{P}_2x, z\mathcal{P}_3y\mathcal{P}_3x\mathcal{P}_3w$ .

**Example 5.2** Let  $X = \{x, y, z\}$  and let there be five voters with the following preferences.

$$\begin{aligned} z\mathcal{P}_i y\mathcal{P}_i x & \text{ for } i = 1, 2, \\ y\mathcal{P}_i x\mathcal{P}_i z & \text{ for } i = 3, 4, \\ x\mathcal{P}_5 z\mathcal{P}_5 y & \end{aligned}$$

Using Theorem 3.1 it is straightforward to verify that

$$O^A(\mathcal{P}, 1) = O^A(\mathcal{P}, 3) = O^A(\mathcal{P}, 5) = \{x, y, z\} \text{ and } O^A(\mathcal{P}, 2) = O^A(\mathcal{P}, 4) = \{y, z\}.$$

While there is no quota which minimizes the degree of manipulability for the amendment procedure, unanimity turns out to be the one that maximizes it.

**Proposition 5.3** *For every preference profile  $\mathcal{P}$  and for all  $q = 1, \dots, n - 1$ , it is true that*

$$O^A(\mathcal{P}, q) \subseteq O^A(\mathcal{P}, n).$$

The situation is somewhat different for the successive procedure. There, the sets  $O^S(\mathcal{P}, q)$  are nested for supermajority and simple majority quotas. Hence, simple majority is a manipulation minimizer among all supermajority and simple majority quotas. However, nestedness does not hold for submajority quotas. We summarize these results in the following proposition.

**Proposition 5.4** *Let  $\mathcal{P}$  be an arbitrary preference profile. Then the following is true.*

1. *For all  $q, q'$ , with  $\lfloor \frac{n}{2} \rfloor + 1 \leq q < q' \leq n$  it is true that  $O^S(\mathcal{P}, q) \subseteq O^S(\mathcal{P}, q')$ .*
2. *For  $q, q'$ , with  $1 \leq q < q' \leq \lfloor \frac{n}{2} \rfloor + 1$  the sets  $O^S(\mathcal{P}, q)$  and  $O^S(\mathcal{P}, q')$  are not necessarily nested, i.e. there exist a set of alternatives  $X$  and a preference profile  $\mathcal{P}$  such that neither  $O^S(\mathcal{P}, q) \subseteq O^S(\mathcal{P}, q')$  holds for all  $q, q'$ , with  $1 \leq q < q' \leq \lfloor \frac{n}{2} \rfloor + 1$ , nor  $O^S(\mathcal{P}, q') \subseteq O^S(\mathcal{P}, q)$  holds for all  $q, q'$ , with  $1 \leq q < q' \leq \lfloor \frac{n}{2} \rfloor + 1$ .*



The first claim in Proposition 5.4 is proved in the appendix. To prove the second claim, observe that for the preferences in Example 5.2 we obtain

$$O^S(\mathcal{P}, 1) = O^S(\mathcal{P}, 3) = O^S(\mathcal{P}, 4) = O^S(\mathcal{P}, 5) = \{x, y, z\}$$

and

$$O^S(\mathcal{P}, 2) = \{y, z\}.$$

The preference profile in Example 5.2 also demonstrates that it is not true that simple majority voting always minimizes the degree of manipulability, neither for the amendment nor for the successive procedure. In particular and quite surprisingly, in this example the submajority quota  $q = 2$  minimizes the power of the agenda setter for both procedures.

One source of the difference in the set of attainable outcomes under the amendment and successive procedure is that the former always selects an outcome in the Pareto set while the latter may also have inefficient outcomes as we will show below. However, this is not the only reason why  $O^A(\mathcal{P}, q)$  and  $O^S(\mathcal{P}, q)$  differ, as shown by Example 5.2.<sup>18</sup> The following proposition summarizes the relation of the Pareto set with the attainable set for the amendment procedure.

**Proposition 5.5**

1. *No Pareto dominated alternative is attainable under the amendment procedure for any  $q = 1, \dots, n$ , i.e.*

$$O^A(\mathcal{P}, q) \subseteq \{x \mid \text{there exists no } y \text{ with } y \mathcal{P}_i x \text{ for all } i\}.$$

2. *For  $q \in \{1, n\}$  the set of outcomes  $O^A(\mathcal{P}, q)$  coincides with the set of alternatives which are not Pareto dominated by any other, i.e.*

$$O^A(\mathcal{P}, 1) = O^A(\mathcal{P}, n) = \{x \mid \text{there exists no } y \text{ with } y \mathcal{P}_i x \text{ for all } i\}.$$

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<sup>18</sup>In this example,  $x \in O^S(\mathcal{P}, 4)$  and  $x \notin O^A(\mathcal{P}, 4)$ , but  $x$  is not Pareto dominated.

Next we consider the successive procedure. Again,  $O^S(\mathcal{P}, 1)$  is the set of alternatives that are not Pareto dominated by any other alternative. However, different from the amendment procedure, for the successive procedure and  $q > 1$  an alternative can be the outcome for some agenda even if it is Pareto dominated. In particular, it is not true in general that  $O^S(\mathcal{P}, 1) = O^S(\mathcal{P}, n)$ .

**Proposition 5.6**

1.  $O^S(\mathcal{P}, 1) = \{x \mid \text{there exists no } y \text{ with } y\mathcal{P}_i x \text{ for all } i\}$ .
2. Let  $n \geq 3$  and let  $2 \leq q \leq n$ . Then there exist a set of alternatives  $X$  and voters' preferences  $\mathcal{P}$  such that  $x \in O^S(\mathcal{P}, q)$  for some Pareto dominated alternative  $x \in X$ .

The first claim in Proposition 5.6 is proved in the appendix. The second claim is proved by the following example.

**Example 5.3** Let  $n \geq 3$  and  $2 \leq q \leq n$ . Let  $X = \{x, y, w, z\}$  and let there be  $n$  voters with the following preferences:

$$\begin{aligned} z\mathcal{P}_1 w\mathcal{P}_1 y\mathcal{P}_1 x, \\ w\mathcal{P}_i y\mathcal{P}_i x\mathcal{P}_i z \quad \text{for all } i = 2, \dots, q, \end{aligned}$$

If  $q < n$ , let

$$y\mathcal{P}_i x\mathcal{P}_i z\mathcal{P}_i w, \quad \text{for all } i = q + 1, \dots, n.$$

Then  $x$  is Pareto dominated by  $y$  and  $x = o^S(x, z, y, w)$  for majority voting with quota  $q$ , i.e.  $x \in O^S(\mathcal{P}, q)$ .

## 6 Conclusion

It is well known that sequential voting procedures are prone to agenda manipulation except for very special cases, where there is a unique alternative which is the outcome under every agenda at a given profile of voters' preferences. Nevertheless, to the best of our knowledge our paper is the first to provide a comprehensive analysis of whether and how the voting procedures derived from the amendment and successive procedure with different majority quotas differ with respect to the scope of manipulation they permit.

Our analysis builds upon a characterization of the attainable sets for the amendment and successive procedure for arbitrary majority quotas. Using this characterization we can show that a well-known result for tournaments extends to arbitrary majority quotas, namely that the successive procedure is uniformly more vulnerable towards agenda manipulation than the amendment procedure. This gives support to using the amendment rather than the successive procedure if the possibility of agenda manipulation is a concern in a committee or, more general, in any democratic institution. We have also shown that the set of preference profiles for which neither procedure is manipulable is maximal under simple majority voting. However, when manipulation is possible, the connection between the degree of manipulability and the choice of a quota is a complex one. In particular, simple majority need no longer be the quota that minimizes the size of choices available to the agenda setter.

## Appendix

**Proof of Lemma 3.1:** The proof is by induction over  $m$ . For  $m = 2$  the claim is obvious. Hence, let  $m \geq 3$  and assume that the claim is true for any agenda with up to  $m - 1$  alternatives. Consider an agenda with  $m$  alternatives,  $(x_1, x_2, \dots, x_m)$ . Denote by  $\hat{y}_i, i = 1, 3, 4, \dots, m$ , the auxiliary alternatives when applying the recursive procedure to the agenda  $(x_1, x_3, x_4, \dots, x_m)$ , and denote by  $\hat{z}_i, i = 2, 3, \dots, m$ , the auxiliary alternatives when applying the recursive procedure to the agenda  $(x_2, x_3, \dots, x_m)$ . Then, by assumption  $o^A(x_1, x_3, x_4, \dots, x_m) = \hat{y}_1$  and  $o^A(x_2, x_3, \dots, x_m) = \hat{z}_2$ . Moreover, let  $\hat{x}_i$  be the auxiliary alternatives when applying the recursive procedure to the agenda  $(x_1, \dots, x_m)$ . We have to show that  $\hat{x}_1 = o^A(x_1, \dots, x_m)$ .

First observe that  $\hat{x}_k = \hat{y}_k = \hat{z}_k$  for  $k = 3, \dots, m$ , and that  $\hat{x}_2 = \hat{z}_2$ . Consider the following cases:

**Case 1:**  $\hat{y}_1 = \hat{y}_3$ . In this case, there exists  $k \geq 3$  with  $\hat{y}_k P x_1$ , i.e. with  $\hat{x}_k P x_1$  and hence  $\hat{x}_1 = \hat{x}_2$ . If  $\hat{z}_2 P \hat{y}_1$ , then by (3)  $o^A(x_1, \dots, x_m) = \hat{z}_2 = \hat{x}_2 = \hat{x}_1$ . If  $\neg \hat{z}_2 P \hat{y}_1$ , then by (3)  $o^A(x_1, \dots, x_m) = \hat{y}_1 = \hat{y}_3 = \hat{x}_3$ . Since  $\hat{z}_2 = \hat{x}_2$  and  $\hat{y}_1 = \hat{y}_3 = \hat{x}_3$ ,  $\neg \hat{z}_2 P \hat{y}_1$  means  $\neg \hat{x}_2 P \hat{x}_3$ , from which it follows that  $\hat{x}_3 P \hat{x}_2$ , if  $\hat{x}_2 \neq \hat{x}_3$  since  $P$  is complete. However,  $\hat{x}_2 \neq \hat{x}_3$  implies  $\hat{x}_2 = x_2$  which is impossible if  $\hat{x}_3 P x_2$ . Hence,  $\hat{x}_3 = \hat{x}_2 = \hat{x}_1$  which proves the claim for this case.

**Case 2:**  $\hat{y}_1 = x_1$ . In this case, for all  $k \geq 3$ ,  $\neg \hat{x}_k P x_1$ . If  $\hat{z}_2 P \hat{y}_1$ , then by (3)  $o^A(x_1, \dots, x_m) = \hat{z}_2 = \hat{x}_2$ . Since  $\hat{z}_2 = \hat{x}_2$  and  $\hat{y}_1 = x_1$ ,  $\hat{z}_2 P \hat{y}_1$  is equivalent to  $\hat{x}_2 P x_1$  from which it follows that  $\hat{x}_1 = \hat{x}_2 = o^A(x_1, \dots, x_m)$ . If  $\neg \hat{z}_2 P \hat{y}_1$ , then by (3)  $o^A(x_1, \dots, x_m) = \hat{y}_1 = x_1$ . Since  $\hat{z}_2 = \hat{x}_2$  and  $\hat{y}_1 = x_1$ ,  $\neg \hat{z}_2 P \hat{y}_1$  is equivalent to  $\neg \hat{x}_2 P x_1$  from which it follows that  $\hat{x}_1 = x_1 = o^A(x_1, \dots, x_m)$  which proves the claim for this case.

□

**Proof of Corollary 3.1:** Let  $o^A(x_1, \dots, x_m) = x_k$ . Consider  $x_l$  with  $l > k$ . If  $\hat{x}_l = x_l$ , then by Lemma 3.1  $\neg x_l P x_k$  and (i) holds. If  $\hat{x}_l = \hat{x}_{l+1}$ , then again by Lemma 3.1 there exists  $l' > l$  with  $x_{l'} P x_l$  and  $\neg x_{l'} P x_k$ , and hence (ii) holds. Consider  $x_l$  with  $l < k$ . Since  $x_k = o^A(x_1, \dots, x_m)$ , by Lemma 3.1 there exists  $l' > l$  with  $x_{l'} P x_l$  and  $\neg x_{l'} P x_k$ . Therefore, also in this case (ii) holds.

□

**Proof of Lemma 3.2:** The proof is by induction over  $m$ . For  $m = 2$  the claim immediately follows from (2). So assume that the claim has been proved for all agendas with at most  $m - 1$  alternatives and consider an agenda with  $m$  alternatives,  $(x_1, \dots, x_m)$ . By assumption, the claim holds for all  $k = 2, \dots, m$ , and it remains to consider  $k = 1$ .

To prove necessity assume that  $\bar{x}_1 = x_1$ . By (3),  $\bar{x}_1 = x_1$  implies that  $x_1 = o^A(x_1, x_3, \dots, x_m)$ . Since there are  $m - 1$  alternatives in the agenda  $(x_1, x_3, \dots, x_m)$ , it follows that  $\neg \bar{x}_k P x_1$  for all  $k = 3, \dots, m$  (observe that the auxiliary variables for agenda  $(x_1, x_3, \dots, x_m)$  defined in (6) are identical to those for agenda  $(x_1, \dots, x_m)$  whenever  $k \geq 3$ ). Moreover, by (3),  $o^A(x_1, \dots, x_m) = o^A(x_1, x_3, \dots, x_m) = x_1$  if and only if  $\neg o^A(x_2, \dots, x_m) P x_1$ , where the latter is equivalent to  $\neg \bar{x}_2 P x_1$ .

For sufficiency assume that  $\neg \bar{x}_l P x_1$  for all  $l = 2, \dots, m$ . Then, by (3),  $o^A(x_1, \dots, x_m) \neq x_1$  implies that either  $x_1 \neq o^A(x_1, x_3, \dots, x_m)$  or  $x_1 = o^A(x_1, x_3, \dots, x_m)$  and  $o^A(x_2, \dots, x_m) P o^A(x_1, x_3, \dots, x_m)$  which holds if and only if  $\bar{x}_2 P x_1$ . The latter case immediately leads to a contradiction since we have assumed that  $\neg \bar{x}_2 P x_1$ . It remains to consider the case where  $x_1 \neq o^A(x_1, x_3, \dots, x_m)$ . Because the agenda  $(x_1, x_3, \dots, x_m)$  has  $m - 1$  alternatives we conclude that there must exist a  $k \geq 3$  with  $\bar{x}_k P x_1$  which contradicts our assumption that  $\neg \bar{x}_l P x_1$  for all  $l = 2, \dots, m$ . This proves the claim. □

**Proof of Lemma 3.3:** The proof is by induction over  $m$ . For  $m = 2$  we only have to consider the case  $k = 1$ , where nothing has to be proved. Hence, assume that the claim is true for any agenda with up to  $m \geq 2$  alternatives and consider an agenda with  $m + 1$  alternatives. Let  $x_k = o^A(x_1, \dots, x_{m+1})$  for some  $k \leq m$ . If  $k = 1$ , nothing has to be proved. If  $k \geq 2$ , then by definition of the outcome of an agenda

$$x_k \in \{o^A(x_1, x_3, \dots, x_{m+1}), o^A(x_2, \dots, x_{m+1})\}.$$

Since both agendas,  $(x_1, x_3, \dots, x_{m+1})$  and  $(x_2, \dots, x_{m+1})$  have  $m$  alternatives and  $k \geq 2$ , it follows in either case that  $x_k = o^A(x_k, \dots, x_{m+1})$ . □

**Proof of Theorem 3.1:** We have to consider the cases, where  $P$  is complete and where  $P$  is asymmetric.

**Case 1:**  $P$  is complete.

*Necessity:* Let  $(x_1, \dots, x_m)$  be an agenda with  $o^A(x_1, \dots, x_m) = x$ . Nothing has to be proved if  $Y(x) = \emptyset$ . Hence, let  $Y(x) \neq \emptyset$  and let  $y \in Y(x)$ . For any alternative  $w$  we denote by  $\hat{w}$  the corresponding auxiliary alternative defined in the recursive procedure in Lemma 3.1. If  $\hat{y} = y$ , then  $x$  cannot be the outcome of any agenda: If  $y$  is a successor of  $x$ , then  $yPx$  implies that  $\hat{x} \neq x$  and hence  $x$  is not the outcome of the agenda. If  $y$  is a predecessor of  $x$ , then  $\hat{y} = y$  immediately implies that the outcome is different from  $x$ . Hence,  $\hat{y} \neq y$  which implies that there exists an alternative  $z(y)$  with  $\widehat{z(y)} = z(y)$  and  $z(y)Py$ . If  $x$  is the outcome of the agenda, then from  $\widehat{z(y)} = z(y)$  it follows that  $\neg z(y)Px$ . This proves (i). Let  $Z(x) = \{z \mid z = z(y) \text{ for some } y \in Y(x)\}$  and let  $(z_1, \dots, z_t)$  be the ordering of the alternatives in  $Z(x)$  in the agenda of which  $x$  is the outcome. Since we have shown that  $\hat{z}_k = z_k$  for all  $k = 1, \dots, t$ , we conclude that (ii) must hold.

*Sufficiency:* The proof is by construction. Let  $x$  be an alternative such that for all  $y \in Y(x)$  there exists an alternative  $z(y) \in X$  such that conditions (i) and (ii) are satisfied. If  $Y(x) = \emptyset$ , then by completeness of  $P$ ,  $xPy$  for all alternatives  $y \neq x$  and hence  $x$  is the outcome of any agenda  $(x_1, \dots, x_m)$  with  $x_m = x$ . If  $Y(x) \neq \emptyset$ , let  $(z_1, \dots, z_t)$  be the ordering of the alternatives in  $Z(x)$  with the property as given in (ii). Observe that  $z_k \neq x$  for all  $k = 1, \dots, t$ , since  $yPx$  and  $\neg xPy$  for all  $y \in Y(x)$ . Take an arbitrary order  $(y_1, \dots, y_r)$  of the alternatives in  $Y(x)$ . If  $r + t + 1 < m$ , let  $(x_1, \dots, x_{m-r-t-1})$  be an arbitrary order of the set of alternatives in  $X \setminus (Y(x) \cup Z(x) \cup \{x\}) \neq \emptyset$ . Consider the agenda  $(x_1, \dots, x_{m-r-t-1}, x, y_1, \dots, y_r, z_1, \dots, z_t)$  (if  $r + t + 1 = m$ , the agenda is  $(x, y_1, \dots, y_r, z_1, \dots, z_t)$ ). We will now verify that

$$x = o^A(x_1, \dots, x_{m-r-t-1}, x, y_1, \dots, y_r, z_1, \dots, z_t).$$

We use the recursive procedure in Lemma 3.1. By construction,  $\hat{z}_l = z_l$  for all  $l = 1, \dots, t$ , and  $\hat{y}_l = z_1$  for all  $l = 1, \dots, r$ , since for all  $l = 1, \dots, r$ , there exists  $k \in \{1, \dots, t\}$  such that  $\hat{z}_k Py_l$ . Since  $\neg \hat{z}_l Px$  for all  $l = 1, \dots, t$ , it follows that

$\hat{x} = x$ . None of the  $x_k, k = 1, \dots, m-r-t-1$ , is in  $Y(x)$ . Hence, by completeness of  $P$ ,  $xPx_k$  for all  $k = 1, \dots, m-r-t-1$ . This implies that  $\hat{x}_1 = \hat{x} = x$  and hence  $x = o^A(x_1, \dots, x_{m-r-t-1}, x, y_1, \dots, y_r, z_1, \dots, z_t)$ .

**Case 2:**  $P$  is asymmetric.

*Necessity:* Let  $x_k = o^A(x_1, \dots, x_m)$  for some  $k \in \{1, \dots, m\}$ . If  $k < m$ , then, from Lemma 3.3 it follows that  $\bar{x}_k = x_k$  and then Lemma 3.2 implies that  $\neg\bar{x}_lPx_k$  for all  $l > k$ . We will now show that for all  $l < k$ ,  $x_k = \bar{x}_l$  or  $\neg\bar{x}_lPx_k$ . Suppose by way of contradiction that there exists  $l < k$  with  $x_k \neq \bar{x}_l$  and  $\bar{x}_lPx_k$ . We will prove that this implies, that  $\bar{x}_s \neq x_k$  for all  $s = 1, \dots, l-1$ , where the case  $s = 1$  yields a contradiction to the assumption that  $x_k = \bar{x}_1 = o^A(x_1, \dots, x_m)$ :

The proof is by backwards induction over  $s$ : Let  $s = l-1$  and suppose by way of contradiction that  $\bar{x}_{l-1} = x_k$ . Since

$$\bar{x}_{l-1} \in \{o^A(x_{l-1}, x_{l+1}, \dots, x_m), \bar{x}_l\}$$

this implies  $x_k = \bar{x}_{l-1} = o^A(x_{l-1}, x_{l+1}, \dots, x_m)$  and  $\neg\bar{x}_lPx_k$  which is a contradiction. Hence,  $\bar{x}_{l-1} \neq x_k$ . Assume we have shown that  $\bar{x}_s \neq x_k$  for all  $s$  with  $t \leq s \leq l-1$ , where  $2 \leq t \leq l-1$ . Suppose by way of contradiction that  $\bar{x}_{t-1} = x_k$ . Since

$$\bar{x}_{t-1} \in \{o^A(x_{t-1}, x_{t+1}, \dots, x_m), \bar{x}_t\}$$

this implies  $x_k = \bar{x}_{t-1} = o^A(x_{t-1}, x_{t+1}, \dots, x_m)$ . Since

$$o^A(x_{t-1}, x_{t+1}, \dots, x_m) \in \{o^A(x_{t-1}, x_{t+2}, \dots, x_m), \bar{x}_{t+1}\}$$

and  $\bar{x}_{t+1} \neq x_k$  it follows that  $x_k = o^A(x_{t-1}, x_{t+2}, \dots, x_m)$ . Continuing in this manner we conclude that

$$x_k \in \{o^A(x_{t-1}, x_{l+1}, \dots, x_m), \bar{x}_l\}$$

and hence  $x_k = o^A(x_{t-1}, x_{l+1}, \dots, x_m)$  which is impossible given that  $\bar{x}_lPx_k$ .

Summarizing, we have shown that for all  $l \neq k$ ,  $x_k = \bar{x}_l$  or  $\neg\bar{x}_lPx_k$ . Returning to the proof of necessity we first note that nothing has to be proved if  $Y(x_k) = \emptyset$ . Hence, let  $Y(x_k) \neq \emptyset$  and let  $x_l \in Y(x_k)$ , i.e.  $x_lPx_k$  and  $\neg x_kPx_l$ . Then by our previous argument  $\bar{x}_l \neq x_l$ . Hence, from Lemma 3.2 it follows that there exists

$l' > l$  with  $\bar{x}_{l'}Px_l$ . By what we have shown above  $\neg\bar{x}_{l'}Px_k$ . Moreover, either  $x_{l'} = \bar{x}_{l'}$  and  $\neg x_{l'}Px_k$  or there exists  $l'' > l'$  with  $\bar{x}_{l''} = x_{l''} = \bar{x}_{l'}$ . Also in this case  $\neg x_{l''}Px_k$ . This proves that for all  $y \in Y(x_k)$  there exists  $z(y) \in X$  with  $\overline{z(y)} = z(y)$ ,  $z(y)Py$  and  $\neg z(y)Px_k$ , i.e. in particular (i) holds. Let  $Z(x_k) = \{z \mid z = z(y) \text{ for some } y \in Y(x_k)\}$  and let  $(z_1, \dots, z_t)$  be the ordering of the alternatives in  $Z(x_k)$  in the agenda of which  $x_k$  is the outcome. Since we have shown that  $\bar{z}_s = z_s$  for all  $s = 1, \dots, t$ , we conclude that (ii) must hold.

*Sufficiency:* The proof is again by construction. Let  $x$  be an alternative such that for all  $y \in Y(x)$  there exists an alternative  $z(y) \in X$  such that conditions (i) and (ii) are satisfied. If  $Y(x) = \emptyset$ , then from Lemma 3.2 it follows that  $x$  is the outcome of any agenda  $(x_1, \dots, x_m)$  with  $x_1 = x$ . If  $Y(x) \neq \emptyset$ , let  $(y_1, \dots, y_r)$  be an arbitrary ordering of the alternatives in  $Y(x)$ . Moreover, let  $(z_1, \dots, z_t)$  be the ordering of the alternatives in  $Z(x)$  with the property as given in (ii). As in case 1 observe that  $z_k \neq x$  for all  $k = 1, \dots, t$ , since  $yPx$  and  $\neg xPy$  for all  $y \in Y(x)$ . If  $r + t + 1 < m$ , let  $(x_1, \dots, x_{m-r-t-1})$  be an arbitrary ordering of the set of alternatives in  $X \setminus (Y(x) \cup Z(x) \cup \{x\})$ . Consider the agenda  $(x, x_1, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t)$  (if  $r + t + 1 = m$ , the agenda is  $(x, y_1, \dots, y_r, z_1, \dots, z_t)$ ). We will now verify that

$$x = o^A(x, x_1, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t).$$

By Lemma 3.2 it is sufficient to show that

1.  $\neg o^A(z_k, \dots, z_t)Px$  for all  $k = 1, \dots, t$ .
  2.  $\neg o^A(y_k, \dots, y_r, z_1, \dots, z_t)Px$  for all  $k = 1, \dots, r$ ,
  3.  $\neg o^A(x_k, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t)Px$  for all  $k = 1, \dots, m-r-t-1$ ,
1. follows from the fact that  $o^A(z_k, \dots, z_t) = z_k$  and  $\neg z_kPx$  for all  $k = 1, \dots, t$ .  
2. will follow from the fact that  $o^A(y_k, \dots, y_r, z_1, \dots, z_t) \in \{z_1, \dots, z_t\}$  for all  $k = 1, \dots, r$ , and  $\neg z_sPx$  for all  $s = 1, \dots, t$ . We prove the former by showing that

$$o^A(y'_1, \dots, y'_l, z_1, \dots, z_t) \in \{z_1, \dots, z_t\} \quad (7)$$

for any agenda with  $y'_1, \dots, y'_l \in Y(x)$  and  $l = 1, \dots, r$ . The proof is by induction over  $l$ . Let  $l = 1$ . Then  $o^A(y'_1, z_1, \dots, z_t) \in \{z_1, \dots, z_t\}$  because otherwise, by



Lemma 3.2,  $\neg z_k P y'_1$  for all  $k = 1, \dots, t$ , contradicting the definition of the set  $Z(x)$ . Suppose (7) has been shown for all subsets of  $Y(x)$  with at most  $l \geq 1$  alternatives, where  $l \leq r - 1$ . Consider now the agenda  $(y'_1, \dots, y'_{l+1}, z_1, \dots, z_t)$  with  $l + 1$  alternatives from  $Y(x)$ . By definition of the outcome of an agenda,

$$\begin{aligned} & o^A(y'_1, \dots, y'_{l+1}, z_1, \dots, z_t) \\ & \in \{o^A(y'_1, y'_3, \dots, y'_{l+1}, z_1, \dots, z_t), o^A(y'_2, \dots, y'_{l+1}, z_1, \dots, z_t)\}. \end{aligned}$$

Since there are  $l$  alternatives from  $Y(x)$  in the agendas  $(y'_1, y'_3, \dots, y'_{l+1}, z_1, \dots, z_t)$  and  $(y'_2, \dots, y'_{l+1}, z_1, \dots, z_t)$  it follows that the outcome of these agendas is an alternative in  $Z(x)$  and hence also  $o^A(y'_1, \dots, y'_{l+1}, z_1, \dots, z_t) \in \{z_1, \dots, z_t\}$ . This proves 2.

To prove 3. suppose by way of contradiction that

$$o^A(x_k, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t) P x$$

for some  $k \in \{1, \dots, m - r - t - 1\}$ . This implies

$$o^A(x_k, \dots, x_{m-r-t-1}, y_1, \dots, y_r, z_1, \dots, z_t) \in \{y_1, \dots, y_r\}.$$

Using Lemma 3.3 we conclude that there exists  $k \in \{1, \dots, r\}$  such that  $y_k = o^A(y_k, \dots, y_r, z_1, \dots, z_t)$ . However, this contradicts (7) and hence 3. holds as claimed. □

**Proof of Corollary 3.2:** Let  $x_k = o^A(x_1, \dots, x_m)$  and let  $Y(x_k) = \{y \in X \mid y P x_k\}$ .<sup>19</sup> Then by Theorem 3.1 for all  $y \in Y(x_k)$ , there exists an alternative  $z(y) \in X$ , such that  $z(y) P y$  and  $\neg z(y) P x_k$ , and there exists an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x_k) = \{z \mid z = z(y) \text{ for some } y \in Y(x_k)\}$ , such that  $\neg z_l P z_s$  for all  $s = 1, \dots, t - 1$ , and for all  $l > s$ . Given the latter condition is satisfied, the proof of Theorem 3.1 has shown that there exists an agenda  $(x'_1, \dots, x'_m)$  with  $x'_1 = x_k$  and  $x_k = o^A(x'_1, \dots, x'_m)$ . □

**Proof of Corollary 3.3:** Let  $P$  be asymmetric and let  $x$  be an alternative such that there exists a maximal trajectory  $(X', d)$  with  $d(x) = t$ , where  $t = \#X'$ . Let

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<sup>19</sup>Observe that  $y P x_k$  implies  $\neg x_k P y$  since  $P$  is asymmetric.

$z_l = d^{-1}(l)$  for  $l = 1, \dots, t-1$ . Since  $(X', d)$  is a trajectory, it follows that

$$\neg z_l P x \text{ for all } l = 1, \dots, t-1, \quad (8)$$

and

$$\neg z_k P z_l \text{ for all } k = 1, \dots, t-2, \text{ and for all } t-1 \geq l > k. \quad (9)$$

Consider the set

$$Y(x) = \{y \mid y P x\}.$$
<sup>20</sup>

If  $Y(x) = \emptyset$ , then by Theorem 3.1  $x = o^A(x_1, \dots, x_m)$  for some agenda. If  $Y(x) \neq \emptyset$ , then for all  $y \in Y(x)$ ,  $z_l P y$  for some  $l \in \{1, \dots, t-1\}$  since  $(X', d)$  is a maximal trajectory. Since  $\neg z_l P x$  by (8),  $z(y) := z_l$  fulfills condition (i) in Theorem 3.1. Moreover, if we let  $Z(x) = \{z_1, \dots, z_{t-1}\}$ , then by (9) condition (ii) in Theorem 3.1 is satisfied for the ordering  $(z_1, \dots, z_{t-1})$ . Hence,  $x = o^A(x_1, \dots, x_m)$ . This proves the first part of the claim.

Let  $x = o^A(x_1, \dots, x_m)$  for some agenda  $(x_1, \dots, x_m)$ . If  $Y(x) = \{y \mid y P x\} = \emptyset$ , then let  $(X', d)$  be a maximal trajectory on the set  $X \setminus \{x\}$ . Then  $(X' \cup \{x\}, d')$  with  $d'(x') = d(x')$  for all  $x' \in X'$  and  $d'(x) = \#X' + 1$  is a maximal trajectory. If  $Y(x) \neq \emptyset$ , let  $(z_1, \dots, z_t)$  satisfy condition (ii) in Theorem 3.1. Then  $(\{z_1, \dots, z_t, x\}, d)$  with  $d(z_l) = t - l + 1$  for  $l = 1, \dots, t$ , and  $d(x) = t + 1$  is a trajectory. If for all  $y \notin \{z_1, \dots, z_t, x\}$  with  $\neg y P x$  it holds that  $x P y$  or  $z_l P y$  for some  $l \in \{1, \dots, t\}$ , then  $(\{z_1, \dots, z_t, x\}, d)$  is a maximal trajectory and we are done. If, instead  $\neg x P y$  and  $\neg z_l P y$  for all  $l \in \{1, \dots, t\}$ , then consider the trajectory  $(\{z_1, \dots, z_t, y, x\}, d')$  with  $d'(z_l) = t - l + 1$  for  $l = 1, \dots, t$ ,  $d'(y) = t + 1$  and  $d'(x) = t + 2$ . If for all  $y' \notin \{z_1, \dots, z_t, y, x\}$  with  $\neg y' P x$  it holds that  $x P y'$  or  $y P y'$  or  $z_l P y'$  for some  $l \in \{1, \dots, t\}$ , then  $(\{z_1, \dots, z_t, y, x\}, d')$  is a maximal trajectory and we are done. If, instead  $\neg x P y'$ ,  $\neg y P y'$  and  $\neg z_l P y'$  for all  $l \in \{1, \dots, t\}$ , then consider the trajectory  $(\{z_1, \dots, z_t, y, y', x\}, d'')$  with  $d''(z_l) = t - l + 1$  for  $l = 1, \dots, t$ ,  $d''(y) = t + 1$ ,  $d''(y') = t + 2$  and  $d''(x) = t + 3$ . As before, either  $(\{z_1, \dots, z_t, y, y', x\}, d'')$  is a maximal trajectory and we are done or we can add another alternative in the same manner as above. In any case, after a finite number of steps we end up with a maximal trajectory that has  $x$  as the last alternative. This proves the claim. □

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<sup>20</sup>Observe that  $y P x$  implies  $\neg x P y$  since  $P$  is asymmetric.

**Proof of Lemma 4.1:** Let  $(x_1, \dots, x_m)$  be an agenda with  $\bar{x}_k \neq x_s$  for some  $s \geq k$ . The proof is by backwards induction over  $l < k$ . Let  $l = k - 1$ . Then by definition  $\bar{x}_{k-1} \in \{x_{k-1}, \bar{x}_k\}$  and since  $\bar{x}_k \neq x_s$  by assumption, it follows that  $\bar{x}_{k-1} \neq x_s$ . Suppose the claim has been proven for all  $l$  with  $t \leq l < k$ , where  $2 \leq t \leq k - 1$ . Since by definition  $\bar{x}_{l-1} \in \{x_{l-1}, \bar{x}_l\}$  and since  $\bar{x}_l \neq x_s$  by assumption, it follows that  $\bar{x}_{l-1} \neq x_s$ . □

**Proof of Lemma 4.2:** Let  $x_k = o^S(x_1, \dots, x_m)$  for some  $1 \leq k \leq m$ , and suppose by way of contradiction that  $\bar{x}_l \neq x_k$  for some  $l \leq k$ . By Lemma 4.1 this implies that  $\bar{x}_s \neq x_k$  for all  $s < l$  contradicting the fact that  $x_k = \bar{x}_1$ .

Let  $1 \leq k \leq m$  and let  $\bar{x}_l = x_k$  for all  $l \leq k$ . In particular, we have  $\bar{x}_1 = o^S(x_1, \dots, x_m) = x_k$  which proves the claim. □

**Proof of Theorem 4.1:**

1. Let  $P$  be complete.

*Necessity:* Let  $x_k = o^S(x_1, \dots, x_m)$ . Nothing has to be proved if  $Y(x_k) = \emptyset$ . Hence, let  $Y(x_k) \neq \emptyset$  and let  $Y(x_k) = \{x_{l(1)}, \dots, x_{l(r)}\}$ , where  $l(1) < l(2) < \dots < l(r)$ . From Lemma 4.2 it follows that  $k < l(1)$ . We now construct a sequence  $(z_1, \dots, z_t)$  with the following properties:

- $\neg z_l P z_{l+1}$  for all  $l = 1, \dots, t - 1$ .
- $\neg z_t P x_k$ .
- $x \neq z_s$  for all  $s = 1, \dots, t$ .
- For all  $j = 1, \dots, r$ , there exists an  $s, 1 \leq s \leq t$ , with  $z_s P x_{l(j)}$ .

Renumbering the alternatives such that  $z'_s = z_{t-s+1}$  for  $s = 1, \dots, t$ , and defining  $Z(x_k) = \{z'_1, \dots, z'_t\}$ , this will prove necessity.

Define  $z_1 = o^S(x_{l(r)}, x_{l(r)+1}, \dots, x_m)$  and

$$s(1) = \min\{h \mid o^S(x_h, \dots, x_m) = z_1\}.$$

Since  $x_k = o^S(x_1, \dots, x_m)$  it follows that  $s(1) > k$ .

Suppose  $s(1) \leq l(1)$ . Then  $z_1 P x_{l(j)}$  for all  $j = 1, \dots, r-1$ . If  $\neg z_1 P x_k$  it follows that  $z_1 \neq x_{l(r)}$  and hence  $z_1 P x_{l(r)}$ . In this case we are done because the sequence  $(z_1)$  has all the properties specified above. If  $z_1 P x_k$ , define  $z_2 = x_{s(1)-1}$ . Then, by definition of  $s(1)$ ,  $\neg z_1 P z_2$  which implies  $z_2 P z_1$  since  $P$  is complete. Moreover, either  $z_1 \neq x_{l(r)}$  and hence  $z_1 P x_{l(r)}$ , or  $z_1 = x_{l(r)}$  and  $z_2 P x_{l(r)}$ . If  $\neg z_2 P x_k$  we are done because the sequence  $(z_1, z_2)$  has all the properties specified above. If  $z_2 P x_k$  there exists an  $s(2)$  with  $k < s(2) < s(1)$  such that  $\neg z_2 P x_{s(2)}$ . Let  $z_3 = x_{s(2)}$ . If  $\neg z_3 P x_k$  we are done because the sequence  $(z_1, z_2, z_3)$  has all the properties specified above. Otherwise, we continue in the same manner. Since  $x_k = o^S(x_k, \dots, x_m)$ , after finitely many steps we arrive at an alternative  $z_t$  with  $\neg z_t P x_k$ . The sequence  $(z_1, \dots, z_t)$  has all the properties specified above.

Suppose now that  $l(j) < s(1) \leq l(j+1)$  for some  $j$  with  $1 \leq j \leq r-1$ . Then  $z_1 P x_{l(i)}$  for all  $i = j+1, \dots, r-1$ . Define  $z_2 = x_{s(1)-1}$ . Then  $\neg z_1 P z_2$  and again either  $z_1 P x_{l(r)}$ , or  $z_2 P x_{l(r)}$ . Define

$$s(2) = \min\{h \mid o^S(x_h, \dots, x_m) = z_2\}.$$

Observe that  $s(2) < s(1)$ . If  $s(2) \leq l(1)$  we can use the same argument as in the case where  $s(1) \leq l(1)$  to construct a sequence  $(z_1, \dots, z_t)$  with the desired properties. If  $l(i) < s(2) \leq l(i+1)$  for some  $i$  with  $1 \leq i \leq r-1$ , define  $z_3 = x_{s(2)-1}$ . Then  $\neg z_2 P z_3$ , and if  $i < j$ , then  $z_2 P x_{l(h)}$  for all  $h = i+1, \dots, j$ . Define

$$s(3) = \min\{h \mid o^S(x_h, \dots, x_m) = z_3\}.$$

Again, either  $s(3) \leq l(1)$  and we can follow the proof for the case where  $s(1) \leq l(1)$ , or  $l(h) < s(3) \leq l(h+1)$  for some  $h$  with  $1 \leq h \leq r-1$ . Continuing in this manner we see that after finitely many steps we arrive at an index  $s(K)$  with  $s(K) \leq l(1)$  and we can follow the argument in the proof for the case where  $s(1) \leq l(1)$ . This proves the existence of a sequence  $(z_1, \dots, z_t)$  with the desired properties.

*Sufficiency:* Let  $x \in X$ . If  $Y(x) = \emptyset$ , then  $xPy$  for all  $y \neq x$  since  $P$  is complete. Hence,  $x = o^S(x_1, \dots, x_m)$  for any agenda  $(x_1, \dots, x_m)$  with  $x_m = x$  and we are done.

Let  $Y(x) \neq \emptyset$ . Then there exists a set of alternatives  $Z(x)$  with  $x \notin Z(x)$  and an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x)$  such that

- for all  $y \in Y(x)$ , there exists an  $s, 1 \leq s \leq t$ , with  $z_sPy$ ,
- $\neg z_{l+1}Pz_l$  for all  $l = 1, \dots, t-1$ ,
- $\neg z_1Px$ .

We now define an agenda  $(w_1, \dots, w_r)$  with  $Y(x) \cup Z(x) = \{w_1, \dots, w_r\}$  and  $o^S(w_1, \dots, w_r) = z_1$ . Let  $Y' = Y(x) \setminus Z(x)$ . If  $Y' = \emptyset$ , then let  $r = t$  and  $w_s = z_s$  for all  $s = 1, \dots, t$ . In this case it immediately follows that  $o^S(w_1, \dots, w_r) = z_1$ . If  $Y' \neq \emptyset$  let  $(w_1, \dots, w_r)$  be the agenda that is obtained if all  $y \in Y'$  with  $z_tPy$  (if any) are placed between  $z_{t-1}$  and  $z_t$ , all  $y \in Y'$  with  $\neg z_tPy$  and  $z_{t-1}Py$  (if any) are placed between  $z_{t-2}$  and  $z_{t-1}$ , and so on, and finally all  $y \in Y'$  with  $\neg z_sPy$  for all  $s = 2, \dots, t$ , and  $z_1Py$  (if any) are placed before  $z_1$ . Then, by definition  $Y(x) \cup Z(x) = \{w_1, \dots, w_r\}$ , and  $o^S(w_1, \dots, w_r) = z_1$ .

If  $Y(x) \cup Z(x) \cup \{x\} = X$ , then it follows that

$$x = o^S(x, w_1, \dots, w_r).$$

If  $X \setminus (Y(x) \cup Z(x) \cup \{x\}) = \{x_1, \dots, x_{m-r-1}\}$ , where  $r \leq m-2$ , then

$$x = o^S(x_1, \dots, x_{m-r-1}, x, w_1, \dots, w_r).$$

This proves sufficiency.

2. Let  $P$  be asymmetric. The proof is very similar to the proof for a complete relation  $P$ .

*Necessity:* Let  $x_k = o^S(x_1, \dots, x_m)$ . Nothing has to be proved if  $Y(x_k) = \emptyset$ . Hence, let  $Y(x_k) \neq \emptyset$  and let  $Y(x_k) = \{x_{l(1)}, \dots, x_{l(r)}\}$ , where  $l(1) < l(2) < \dots < l(r)$ . From Lemma 4.2 it follows that  $k < l(1)$ . We now construct a sequence  $(z_1, \dots, z_t)$  with the following properties:

- $\neg z_l P z_{l+1}$  for all  $l = 1, \dots, t - 1$ .
- $\neg z_t P x_k$ .
- $x \neq z_s$  for all  $s = 1, \dots, t$ .
- For all  $j = 1, \dots, r$ , there exists an  $s, 1 \leq s \leq t$ , with  $\neg x_{l(j)} P z_s$ .

Again, renumbering the alternatives such that  $z'_s = z_{t-s+1}$  for  $s = 1, \dots, t$ , and defining  $Z(x_k) = \{z'_1, \dots, z'_t\}$ , this will prove necessity.

Define  $z_1 = o^S(x_{l(r)}, x_{l(r)+1}, \dots, x_m)$  and

$$s(1) = \min\{h \mid o^S(x_h, \dots, x_m) = z_1\}.$$

Since  $x_k = o^S(x_1, \dots, x_m)$  it follows that  $s(1) > k$ .

Suppose  $s(1) \leq l(1)$ . Then  $z_1 P x_{l(j)}$ , which implies  $\neg x_{l(j)} P z_1$  for all  $j = 1, \dots, r - 1$ , by asymmetry of  $P$ . If  $\neg z_1 P x_k$  it follows that  $z_1 \neq x_{l(r)}$  and hence  $z_1 P x_{l(r)}$ . Since  $P$  is asymmetric, the latter implies that  $\neg x_{l(r)} P z_1$ . In this case we are done because the sequence  $(z_1)$  has all the properties specified above. If  $z_1 P x_k$ , define  $z_2 = x_{s(1)-1}$ . Then  $\neg z_1 P z_2$ , and either  $z_1 \neq x_{l(r)}$  and hence  $z_1 P x_{l(r)}$ , which implies that  $\neg x_{l(r)} P z_1$  since  $P$  is asymmetric. Or  $z_1 = x_{l(r)}$  and hence  $\neg x_{l(r)} P z_2$ . If  $\neg z_2 P x_k$  we are done because the sequence  $(z_1, z_2)$  has all the properties specified above. If  $z_2 P x_k$  there exists an  $s(2)$  with  $k < s(2) < s(1)$  such that  $\neg z_2 P x_{s(2)}$ . Let  $z_3 = x_{s(2)}$ . If  $\neg z_3 P x_k$  we are done because the sequence  $(z_1, z_2, z_3)$  has all the properties specified above. Otherwise, we continue in the same manner. Since  $x_k = o^S(x_k, \dots, x_m)$ , after finitely many steps we arrive at an alternative  $z_t$  with  $\neg z_t P x_k$ . The sequence  $(z_1, \dots, z_t)$  has all the properties specified above.

Suppose now that  $l(j) < s(1) \leq l(j + 1)$  for some  $j$  with  $1 \leq j \leq r - 1$ . Then  $z_1 P x_{l(i)}$  for all  $i = j + 1, \dots, r - 1$ , which implies that  $\neg x_{l(i)} P z_1$  for all  $i = j + 1, \dots, r - 1$ , since  $P$  is asymmetric. Define  $z_2 = x_{s(1)-1}$ . Then  $\neg z_1 P z_2$  and again either  $\neg x_{l(r)} P z_1$ , or  $\neg x_{l(r)} P z_2$ . Define

$$s(2) = \min\{h \mid o^S(x_h, \dots, x_m) = z_2\}.$$

Observe that  $s(2) < s(1)$ . If  $s(2) \leq l(1)$  we can use the same argument as in the case where  $s(1) \leq l(1)$  to construct a sequence  $(z_1, \dots, z_t)$  with the

desired properties. If  $l(i) < s(2) \leq l(i+1)$  for some  $i$  with  $1 \leq i \leq r-1$ , define  $z_3 = x_{s(2)-1}$ . Then  $\neg z_2 P z_3$ , and if  $i < j$ , then  $z_2 P x_{l(h)}$  for all  $h = i+1, \dots, j$ , which implies that  $\neg x_{l(h)} P z_2$  for all  $h = i+1, \dots, j$ . Define

$$s(3) = \min\{h \mid o^S(x_h, \dots, x_m) = z_3\}.$$

Again, either  $s(3) \leq l(1)$  and we can follow the proof for the case where  $s(1) \leq l(1)$ , or  $l(h) < s(3) \leq l(h+1)$  for some  $h$  with  $1 \leq h \leq r-1$ . Continuing in this manner we see that after finitely many steps we arrive at an index  $s(K)$  with  $s(K) \leq l(1)$  and we can follow the argument in the proof for the case where  $s(1) \leq l(1)$ . This proves the existence of a sequence  $(z_1, \dots, z_t)$  with the desired properties.

*Sufficiency:* Let  $x \in X$ . If  $Y(x) = \emptyset$ , then  $\neg y P x$  for all  $y \neq x$ . Hence,  $x = o^S(x_1, \dots, x_m)$  for any agenda  $(x_1, \dots, x_m)$  with  $x_1 = x$  and we are done.

Let  $Y(x) \neq \emptyset$ . Then there exists a set of alternatives  $Z(x)$  with  $x \notin Z(x)$  and an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x)$  such that

- for all  $y \in Y(x)$ , there exists an  $s$ ,  $1 \leq s \leq t$ , with  $\neg y P z_s$ ,
- $\neg z_{l+1} P z_l$  for all  $l = 1, \dots, t-1$ ,
- $\neg z_1 P x$ .

We now define an agenda  $(w_1, \dots, w_r)$  with  $Y(x) \cup Z(x) = \{w_1, \dots, w_r\}$  and  $o^S(w_1, \dots, w_r) = z_1$ . Let  $Y' = Y(x) \setminus Z(x)$ . If  $Y' = \emptyset$ , then let  $r = t$  and  $w_s = z_s$  for all  $s = 1, \dots, t$ . In this case it immediately follows that  $o^S(w_1, \dots, w_r) = z_1$ . If  $Y' \neq \emptyset$  let  $(w_1, \dots, w_r)$  be the agenda that is obtained if all  $y \in Y'$  with  $\neg y P z_t$  (if any) are placed after  $z_t$ , all  $y \in Y'$  with  $y P z_t$  and  $\neg y P z_{t-1}$  (if any) are placed between  $z_{t-1}$  and  $z_t$ , and so on, and finally all  $y \in Y'$  with  $y P z_s$  for all  $s = 2, \dots, t$ , and  $\neg y P z_1$  (if any) are placed between  $z_1$  and  $z_2$ . Then, by definition  $Y(x) \cup Z(x) = \{w_1, \dots, w_r\}$ , and  $o^S(w_1, \dots, w_r) = z_1$ .

If  $Y(x) \cup Z(x) \cup \{x\} = X$ , then

$$x = o^S(x, w_1, \dots, w_r).$$

If  $X \setminus (Y(x) \cup Z(x) \cup \{x\}) = \{x_1, \dots, x_{m-r-1}\}$ , where  $r \leq m - 2$ , then  $\neg x_s P x$  for all  $s = 1, \dots, m - r - 1$ , which implies that

$$x = o^S(x, x_1, \dots, x_{m-t(r)-1}, w_1, \dots, w_r).$$

This proves sufficiency. □

**Proof of Corollary 4.2:** Let  $P$  be asymmetric. Assume first that  $x = o^S(x_1, \dots, x_m)$  for some agenda  $(x_1, \dots, x_m)$  and let  $y \in X, y \neq x$ . If  $\neg y P x$ , then define  $z_0 = x$  and  $z_1 = y$ . If  $y P x$ , then by Theorem 4.1 there exists a sequence of alternatives  $(z_1, \dots, z_t)$  with the following properties:

- There exists an  $s, 1 \leq s \leq t$ , such that  $\neg y P z_s$ ,
- $\neg z_{l+1} P z_l$  for all  $l = 1, \dots, t - 1$ ,
- $\neg z_1 P x$ .

This proves that  $x$  is in the top cycle of  $P$ .

For the reverse, let  $x$  be in the top cycle of  $P$  and let  $Y(x) = \{y \mid y P x \text{ and } \neg x P y\}$ . If  $Y(x) = \emptyset$ , then  $x = o^S(x_1, \dots, x_m)$  for some agenda  $(x_1, \dots, x_m)$  by Theorem 4.1 and we are done. Suppose  $Y(x) = \{y_1, \dots, y_r\}$  for some  $r \geq 1$ . Since  $x$  is in the top cycle of  $P$ , for all  $l = 1, \dots, r$ , there exists a sequence of distinct alternatives  $(w_1^l, \dots, w_{s(l)}^l)$  with  $w_1^l = y_l, \neg w_k^l P w_{k+1}^l$  for all  $k = 1, \dots, s(l)$ , and  $\neg w_{s(l)}^l P x$ .

For  $l = 1, \dots, r$ , we now inductively define sequences of distinct alternatives  $z^l$  as follows: For  $l = 1$ , define  $\bar{s}(1) = s(1)$  and

$$z^1 = (w_{\bar{s}(1)}^1, \dots, w_1^1).$$

For  $l = 2$ , let  $z^2 = z^1$ , if  $y_2 \in \{w_1^1, \dots, w_{\bar{s}(1)}^1\}$ . Otherwise, if  $y_2 \notin \{w_1^1, \dots, w_{\bar{s}(1)}^1\}$  let  $\bar{s}(2) \in \{1, \dots, s(2) - 1\}$  be the minimal  $s$  with the property that

$$w_{s+1}^2 \in \{w_1^1, \dots, w_{\bar{s}(1)}^1\}.$$

If there is no such  $s$  define  $\bar{s}(2) = s(2)$ . Then define

$$z^2 = (w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2).$$



For ease of presentation we assume that  $y_2 \notin \{w_1^1, \dots, w_{\bar{s}(1)}^1\}$  and then continue to define  $z^3$ . If  $y_3 \in \{w_1^1, \dots, w_{\bar{s}(1)}^1, w_1^2, \dots, w_{\bar{s}(2)}^2\}$ , define  $z^3 = z^2$ . Otherwise, if  $y_3 \notin \{w_1^1, \dots, w_{\bar{s}(1)}^1, w_1^2, \dots, w_{\bar{s}(2)}^2\}$  let  $\bar{s}(3) \in \{1, \dots, s(3) - 1\}$  be the minimal  $s$  with the property that

$$w_{s+1}^3 \in \{w_1^1, \dots, w_{\bar{s}(1)}^1, w_1^2, \dots, w_{\bar{s}(2)}^2\}.$$

If there is no such  $s$  define  $\bar{s}(3) = s(3)$ . Then define

$$z^3 = (w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2, w_{\bar{s}(3)}^3, \dots, w_1^3).$$

Continuing in this manner and assuming that  $y_k \notin \bigcup_{l=1}^{k-1} \{w_1^l, \dots, w_{\bar{s}(l)}^l\}$  for all  $k = 2, \dots, r$ ,<sup>21</sup> we arrive at the sequence

$$z^r = (w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2, \dots, w_{\bar{s}(r)}^r, \dots, w_1^r).$$

Observe that by construction  $z^r$  has the property that  $y_k \in \bigcup_{l=1}^r \{w_1^l, \dots, w_{\bar{s}(l)}^l\}$  for all  $k = 1, \dots, r$ , and that

$$o^S(w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2, \dots, w_{\bar{s}(r)}^r, \dots, w_1^r) \in \{w_{\bar{s}(1)}^1, w_{\bar{s}(2)}^2, \dots, w_{\bar{s}(r)}^r\}.$$

Since  $\neg w_{\bar{s}(l)}^l P x$  for all  $l = 1, \dots, r$ , it follows that

$$x = o^S(x, w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2, \dots, w_{\bar{s}(r)}^r, \dots, w_1^r).$$

If  $X \setminus \{x\} = \bigcup_{l=1}^r \{w_1^l, \dots, w_{\bar{s}(l)}^l\}$ , we are done. Otherwise, let

$$X \setminus \left( \{x\} \cup \bigcup_{l=1}^r \{w_1^l, \dots, w_{\bar{s}(l)}^l\} \right) = \{x_1, \dots, x_t\}.$$

Since  $P$  is asymmetric and  $Y(x) \subset \bigcup_{l=1}^r \{w_1^l, \dots, w_{\bar{s}(l)}^l\}$ , it follows that  $\neg x_s P x$  for all  $s = 1, \dots, t$ . Hence,

$$x = o^S(x, x_1, \dots, x_t, w_{\bar{s}(1)}^1, \dots, w_1^1, w_{\bar{s}(2)}^2, \dots, w_1^2, \dots, w_{\bar{s}(r)}^r, \dots, w_1^r).$$

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<sup>21</sup>The proof for the case where  $y_k \in \bigcup_{l=1}^{k-1} \{w_1^l, \dots, w_{\bar{s}(l)}^l\}$  for some  $k$  and hence,  $z^k = z^{k-1}$  is similar and hence omitted.

This proves the claim that any alternative  $x$  in the top cycle is an outcome for some agenda under the successive procedure.

□

**Proof of Theorem 5.1:** The theorem is proved by showing that (i) and (ii) are equivalent to (iii).

**(i)  $\iff$  (iii):** Assume (iii), i.e.  $xPy$  and  $\neg yPx$  for all  $y \neq x$ . Then, by Theorem 4.1,  $O^S(P) = \{x\}$ , i.e. (i) holds. Assume (i), i.e.  $O^S(P) = \{x\}$ . Then,  $x = o^S(x_1, \dots, x_m)$  for any agenda with  $x_m = x$ . By definition of the successive procedure this implies that  $x = o^S(x_k, \dots, x_m)$  for all  $k = 1, \dots, m - 1$ , and hence,  $xPx_k$  for all  $k = 1, \dots, m - 1$ . This already establishes the proof when  $P$  is asymmetric. As for the complete case, suppose by way of contradiction that  $yPx$  for some  $y \neq x$ . Then  $o^S(x, y) = y$  which implies that  $x \neq o^S(x_1, \dots, x_m)$  for any agenda with  $x_{m-1} = x$  and  $x_m = y$  by Lemma 4.1. This contradicts our assumption that  $O^S(P) = \{x\}$ . Hence, also for  $P$  complete,  $O^S(P) = \{x\}$  implies that (iii) holds.

**(ii)  $\iff$  (iii):** Assume (iii), i.e.  $xPy$  and  $\neg yPx$  for all  $y \neq x$ . Then, by Theorem 3.1,  $O^A(P) = \{x\}$ , i.e. (ii) holds. Assume (ii), i.e.  $O^A(P) = \{x\}$ . First consider the case where  $P$  is complete. Then  $x = o^A(x_1, \dots, x_m)$  for all agendas  $(x_1, \dots, x_m)$ . Suppose by way of contradiction that there exists  $y$  with  $yPx$ . Then, by Lemma 3.1,  $x \neq o^A(x_1, \dots, x_{m-2}, x, y)$ , where  $(x_1, \dots, x_{m-2})$  is an arbitrary ordering of the alternatives different from  $x$  and  $y$ . This contradicts our assumption that  $O^A(P) = \{x\}$ . Hence,  $O^A(P) = \{x\}$  implies that  $\neg yPx$  for all  $y \neq x$ . Since  $P$  is complete, this implies (iii).

Next consider the case where  $P$  is asymmetric. Suppose by way of contradiction that there exists  $y$  with  $\neg xPy$ . We then claim that  $x \neq o^A(x_1, \dots, x_m)$  for any agenda with  $x_1 = y$  and  $x_m = x$ . The claim is proved by induction over  $m$ . If  $m = 2$  the claim is immediate. Suppose now that the claim is true for  $m \geq 2$  and consider the agenda  $(x_1, \dots, x_{m+1})$  with  $x_1 = y$  and  $x_{m+1} = x$ . Suppose by way of contradiction that  $x = o^A(x_1, \dots, x_{m+1})$ . By definition of the amendment

procedure,

$$o^A(x_1, \dots, x_{m+1}) \in \{o^A(x_1, x_3, \dots, x_{m+1}), o^A(x_2, \dots, x_{m+1})\}.$$

Since the agenda  $(x_1, x_3, \dots, x_{m+1})$  has  $m$  alternatives, it follows that  $x \neq o^A(x_1, x_3, \dots, x_{m+1})$ . Then,  $x = o^A(x_1, \dots, x_{m+1})$  implies that  $x = o^A(x_2, \dots, x_{m+1})$  and

$$xPo^A(x_1, x_3, \dots, x_{m+1}). \quad (10)$$

Since  $x_1 = y$  and  $\neg xPy$  it follows that  $o^A(x_1, x_3, \dots, x_{m+1}) = x_k$  for some  $k$  with  $3 \leq k \leq m$ . Lemma 3.3 then implies that  $x_k = o^A(x_k, \dots, x_{m+1}) = \bar{x}_k$ . But then,  $\neg \bar{x}_{m+1}Px_k$  by Lemma 3.2. Since  $\bar{x}_{m+1} = x_{m+1} = x$  this is a contradiction to (10). This proves our claim.

Hence,  $O^A(P) = \{x\}$  implies that  $xPy$  for all  $y \neq x$ . Finally, by asymmetry of  $P$  we conclude that  $\neg yPx$  for all  $y \neq x$ , i.e. (iii) holds. □

**Proof of Proposition 5.2:** Consider first the case where  $1 \leq q < q' \leq \lfloor \frac{n}{2} \rfloor + 1$  and let  $\mathcal{P} \in \Phi(q)$ . Then, by definition of  $\Phi(q)$  there exists an alternative  $x$  such that for all  $y \neq x$ ,

$$\#\{i \mid x\mathcal{P}_iy\} \geq q \quad \text{and} \quad \#\{i \mid y\mathcal{P}_ix\} < q. \quad (11)$$

Observe that  $\#\{i \mid y\mathcal{P}_ix\} = n - \#\{i \mid x\mathcal{P}_iy\} < q$  implies that  $\#\{i \mid x\mathcal{P}_iy\} \geq q$  since  $q < \lfloor \frac{n}{2} \rfloor + 1$ . Hence, (11) is satisfied if and only if

$$\#\{i \mid x\mathcal{P}_iy\} > n - q.$$

This immediately implies that  $\Phi(q) \subseteq \Phi(q')$  if  $q < q' < \lfloor \frac{n}{2} \rfloor + 1$ . It remains to consider the case where  $q = \lfloor \frac{n}{2} \rfloor$  and  $q' = \lfloor \frac{n}{2} \rfloor + 1$ . If  $n$  is odd, then by what we have shown above,  $\mathcal{P} \in \Phi(\lfloor \frac{n}{2} \rfloor)$  if and only if there exists an alternative  $x$  such that for all  $y \neq x$ ,

$$\#\{i \mid x\mathcal{P}_iy\} > n - \lfloor \frac{n}{2} \rfloor = \frac{n+1}{2} = \lfloor \frac{n}{2} \rfloor + 1.$$

This implies

$$\#\{i \mid y\mathcal{P}_ix\} < \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor < \lfloor \frac{n}{2} \rfloor + 1.$$

Hence,  $\mathcal{P} \in \Phi(\lfloor \frac{n}{2} \rfloor + 1)$ , i.e.  $\Phi(\lfloor \frac{n}{2} \rfloor) \subseteq \Phi(\lfloor \frac{n}{2} \rfloor + 1)$ . If  $n$  is even, then by the above  $\mathcal{P} \in \Phi(\lfloor \frac{n}{2} \rfloor)$  if and only if there exists an alternative  $x$  such that for all  $y \neq x$ ,

$$\#\{i \mid x\mathcal{P}_iy\} > n - \lfloor \frac{n}{2} \rfloor = \frac{n}{2}.$$

This implies

$$\#\{i \mid x\mathcal{P}_iy\} \geq \frac{n}{2} + 1$$

and

$$\#\{i \mid y\mathcal{P}_ix\} < \frac{n}{2} < \frac{n}{2} + 1.$$

Hence, also for  $n$  even we conclude that  $\mathcal{P} \in \Phi(\lfloor \frac{n}{2} \rfloor + 1)$ , which implies that  $\Phi(\lfloor \frac{n}{2} \rfloor) \subseteq \Phi(\lfloor \frac{n}{2} \rfloor + 1)$ .

Next, consider the case where  $\lfloor \frac{n}{2} \rfloor + 1 \leq q' < q \leq n$  and let  $\mathcal{P} \in \Phi(q)$ . Since the majority relation is asymmetric for quotas greater than or equal to  $\lfloor \frac{n}{2} \rfloor + 1$  by definition of  $\Phi(q)$  there exists a unique alternative  $x$  such that for all  $y \neq x$ ,

$$\#\{i \mid x\mathcal{P}_iy\} \geq q.$$

Since  $q > q'$  this immediately implies that  $\Phi(q) \subseteq \Phi(q')$ .

□

**Proof of Proposition 5.3:** Suppose by way of contradiction that there exists a  $q \in \{1, \dots, n-1\}$  and an alternative  $x \in X$  such that  $x \in O^A(\mathcal{P}, q)$  and  $x \notin O^A(\mathcal{P}, n)$ . By Theorem 3.1 the latter implies that there exists an alternative  $y$  with  $y\mathcal{P}_ix$  for all  $i = 1, \dots, n$ . Since  $x \in O^A(\mathcal{P}, q)$ , by Theorem 3.1 there exists  $z(y)$  with

$$\#\{i \mid z(y)\mathcal{P}_iy\} \geq q \tag{12}$$

$$\text{and } \#\{i \mid z(y)\mathcal{P}_ix\} < q \tag{13}$$

However,  $y\mathcal{P}_ix$  for all  $i = 1, \dots, n$ , and (12) imply that  $\#\{i \mid z(y)\mathcal{P}_ix\} \geq q$  contradicting (13). This proves the claim that  $O^A(\mathcal{P}, q) \subseteq O^A(\mathcal{P}, n)$ .

□

**Proof of Proposition 5.4, 1.:** Let  $q, q'$  be given with  $\lfloor \frac{n}{2} \rfloor + 1 \leq q < q' \leq n$ . Then, the dominance relation  $P$  derived from majority voting with quota  $q$  is asymmetric. Let  $x \in O^S(\mathcal{P}, q)$ . If, for all  $y \in X$ ,  $\#\{i \mid y\mathcal{P}_i x\} < q'$ , then  $x \in O^S(\mathcal{P}, q')$  and we are done. Otherwise, let  $Y(x) = \{y \mid \#\{i \mid y\mathcal{P}_i x\} \geq q'\} \neq \emptyset$ . Then, for all  $y \in Y(x)$ ,  $\#\{i \mid y\mathcal{P}_i x\} \geq q$ , and by Theorem 4.1 there exists a sequence of distinct alternatives  $(z_1, \dots, z_t)$  with the following properties:

- for all  $y$  with  $\#\{i \mid y\mathcal{P}_i x\} \geq q$  there exists an  $s, 1 \leq s \leq t$ , with  $\#\{i \mid y\mathcal{P}_i z_s\} < q$ ,
- $\#\{i \mid z_{l+1}\mathcal{P}_i z_l\} < q$  for all  $l = 1, \dots, t-1$ ,
- $\#\{i \mid z_1\mathcal{P}_i x\} < q$ .

Since  $q' > q$  this implies that

- for all  $y$  with  $\#\{i \mid y\mathcal{P}_i x\} \geq q'$  there exists an  $s, 1 \leq s \leq t$ , with  $\#\{i \mid y\mathcal{P}_i z_s\} < q'$ ,
- $\#\{i \mid z_{l+1}\mathcal{P}_i z_l\} < q'$  for all  $l = 1, \dots, t-1$ ,
- $\#\{i \mid z_1\mathcal{P}_i x\} < q'$ .

Hence, by Theorem 4.1  $x \in O^S(\mathcal{P}, q')$  which proves the claim. □

**Proof of Proposition 5.5:**

1. Let  $q \in \{1, \dots, n\}$  and let  $x \in O^A(\mathcal{P}, q)$ . Suppose by way of contradiction that  $x$  is Pareto dominated by some alternative  $y$ , i.e.  $y\mathcal{P}_i x$  for all voters  $i$ . Then  $y \in Y(x)$  and by Theorem 3.1 there exists an alternative  $z(y)$  such that

$$\#\{i \mid z(y)\mathcal{P}_i y\} \geq q \tag{14}$$

$$\text{and } \#\{i \mid z(y)\mathcal{P}_i x\} < q. \tag{15}$$

However, since  $y\mathcal{P}_i x$  for all voters  $i$ , (14) implies that  $\#\{i \mid z(y)\mathcal{P}_i x\} \geq q$  contradicting (15). Hence,  $x$  is not Pareto dominated by any alternative  $y$ .

2. Let  $x \in X$  and let  $q \in \{1, n\}$ . Then  $Y(x) = \{y \mid y \mathcal{P}_i x \text{ for all } i\}$ , i.e.  $Y(x)$  is the set of all alternatives that Pareto dominate  $x$ . From 1. we know that  $O^A(\mathcal{P}, q) \subseteq \{x \mid \text{there exists no } y \text{ with } y \mathcal{P}_i x \text{ for all } i\}$ . Hence, it remains to show that any alternative  $x$ , which is not Pareto dominated, is an element of  $O^A(\mathcal{P}, q)$ . If  $x$  is not Pareto dominated by any other alternative, then  $Y(x) = \emptyset$  and Theorem 3.1 implies that  $x \in O^A(\mathcal{P}, q)$ .

□

**Proof of Proposition 5.6, 1.:** If  $q = 1$ , then  $Y(x) = \{y \mid y \mathcal{P}_i x \text{ for all } i\}$ . Hence, if  $x$  is an alternative that is not Pareto dominated by any other alternative, then  $Y(x) = \emptyset$ , and Theorem 4.1 implies that  $x \in O^S(\mathcal{P}, 1)$ . Now let  $x \in O^S(\mathcal{P}, 1)$  and suppose by way of contradiction that  $x$  is Pareto dominated by  $y$ . Then  $y \in Y(x)$  and by Theorem 4.1 there exists a set of alternatives  $Z(x)$  and an ordering  $(z_1, \dots, z_t)$  of the alternatives in  $Z(x)$  such that the following conditions are satisfied:

- there exists an  $s \in \{1, \dots, t\}$  and a voter  $j$  with  $z_s \mathcal{P}_j y$ ,
- $z_l \mathcal{P}_i z_{l+1}$  for all voters  $i$  and for all  $l = 1, \dots, t - 1$ ,
- $x \mathcal{P}_i z_1$  for all voters  $i$ .

However, this implies that

$$x \mathcal{P}_j z_1 \mathcal{P}_j z_2 \dots \mathcal{P}_j z_s \mathcal{P}_j y$$

contradicting the assumption that  $y$  Pareto dominates  $x$ . Hence, no alternative in  $O^S(\mathcal{P}, 1)$  is Pareto dominated, which proves the claim.

□

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