The Outcome of Competitive Equilibrium Rules In Buyer-seller Markets When the Agents Play Strategically

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THE OUTCOME OF COMPETITIVE EQUILIBRIUM RULES IN BUYER-SELLER MARKETS WHEN THE AGENTS PLAY STRATEGICALLY[1]

by

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ABSTRACT

We analyze the two-stage games induced by competitive equilibrium rules for the buyer-seller market of Shapley and Shubik (1972). In these procedures, first sellers and then buyers report their valuation and the outcome is determined by a competitive equilibrium outcome for the market reported by the agents. We provide results concerning buyers and sellers' equilibrium strategies. In particular, our results point out that, by playing first, sellers are able to instigate an outcome that corresponds to the sellers' optimal competitive equilibrium allocation for the true market.

Key words: assignment game, competitive price, optimal matching, competitive rule

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1. INTRODUCTION

We analyze a one-to-one buyer-seller market where a set of possibly heterogeneous sellers and a set of possibly heterogeneous buyers meet. Each seller owns one indivisible object, for which he has a certain valuation. Each buyer places a monetary value on each of the objects and she is interested in acquiring at most one of them. This market is a version of the assignment game, introduced in Shapley and Shubik (1972). While we refer to the model as the buyer-seller market, many other markets enter into our framework, including the labor markets in which workers sell their services for salaries.

For the buyer-seller market an outcome consists of two elements: a matching function that states who buys from whom (and which agent does not sell or does not buy), and a vector of prices for the objects. Gale (1960) proposed the competitive equilibrium as a reasonable solution concept for this market. In a competitive equilibrium, the demand of every buyer is satisfied (that is, each buyer receives an object that maximizes her surplus, given the prices, whenever this surplus is non-negative), the price of each unsold object is its seller's reservation price, and no two buyers are assigned the same object. Gale (1960) also proved the existence of competitive equilibrium outcomes. Shapley and Shubik (1972) showed that a competitive equilibrium matching is an optimal matching, in the sense that it maximizes the sum of the gains of the whole set of agents. They also proved that the set of competitive equilibrium prices forms a complete lattice whose extreme points are the minimum and the maximum equilibrium prices, which are called buyer-optimal and seller-optimal competitive prices, respectively.[4] Finally, it is possible to define a cooperative model for the buyer-seller market and the previous results also apply to the cooperative model because the core coincides with the set of competitive equilibrium payoffs (Shapley and Shubik, 1972).

Competitive equilibria provide efficient, stable, and envy-free allocations. However, they may be very difficult to obtain through decentralized processes with contracts, bids, offers, and counter offers. In this paper we explore the idea of using competitive equilibria as the basis for centralized mechanisms that set the prices for the

[4] Kelso and Crawford (1982) extend the analysis to many-to-one matching models. Sotomayor (2007) introduces the concept of a competitive equilibrium payoff for the multiple-partners assignment game and extends the previous results for this environment.

objects and allocate them to the buyers. In any such mechanism, the designer announces a competitive equilibrium rule, that is, a function that selects a particular competitive equilibrium for every possible market.

The adoption of a centralized mechanism requires the designer to request the valuations of sellers and buyers who may have an incentive to manipulate their report. This issue was partially addressed in the literature by studying the incentives for truthful reporting by the agents of one side of the market, considering that the agents of the other side do not have any room for strategic behavior. Demange (1982) and Leonard (1983) provide a "non-manipulability theorem" for the assignment game: if the designer uses the buyer-optimal (respectively, the seller-optimal) competitive equilibrium rule then no buyer (respectively, seller) can profit by misstating her (or his) true valuations.[5] However, Demange and Gale (1985), Roth and Sotomayor (1990), and Pérez-Castrillo and Sotomayor (2013) show that agents have an incentive to manipulate their report if they do not obtain their most preferred competitive equilibrium allocation.[6]

In this paper, we propose and analyze a mechanism (a "game") where both the sellers and the buyers report their valuation and the outcome is given by a competitive equilibrium outcome for the market reported by the agents. In addition, given that there may be several competitive matchings in the market, the buyers are also requested to send a "signal" that will only be used to select among the optimal matchings, whenever several optimal matchings exist. The game has two stages because sellers, simultaneously and non-cooperatively, report their valuations first and, once these are known, buyers, simultaneously and non-cooperatively, are asked to make their reports. Once the agents have played, the competitive price rule maps the matrix of valuations announced by the sellers and the buyers to a competitive price vector for the corresponding market and the matching rule determines a competitive allocation of the objects to the buyers. Although we know that in this game some agents will typically have an incentive to misreport their valuation, we show that the equilibria of the game are, in general, competitive equilibrium outcomes for the true market.

[5] Demange and Gale (1985) extend the theorem to a model where the utilities are continuous in money, but are not necessarily linear. Pérez-Castrillo and Sotomayor (2013) prove that buyers (respectively, sellers) do not have an incentive to misreport their valuation if the buyer-optimal (respectively, seller-optimal) competitive equilibrium is used by the designer in a one-to-many (respectively, many-to-one) buyer-seller market.

[6] Papers analyzing the consequences of manipulation in marriage and the college admission models, that is, in models where there are no prices, include Gale and Sotomayor (1985a, 1985b), Roth (1985), Roth and Sotomayor (1990), Sotomayor (2008), Kojima and Pathak (2009), Ma (2010), Sotomayor (2012), and Jaramillo, Kayi, and Klijn (2013).

After the sellers report their valuations, the second stage begins. We develop the analysis of this stage in two parts. In the first part, we construct specific strategies for the buyers that satisfy that a buyer's report is always lower than her true valuation (which guaranties that she will never pay more for an object than her valuation) and show that they constitute a Nash equilibrium (NE) that leads to the minimum competitive price for the market where the buyers' valuations are the true valuations and the sellers' valuations correspond to those reported. That is, by choosing their reports, buyers can non-cooperatively "select" their best competitive outcome, given the sellers' reports.

Since truth-telling is a dominant strategy for the buyers when the minimum competitive price vector is selected, the second part of the analysis of the buyers' behavior concentrates on competitive price rules that do not select the minimum competitive price, whenever several competitive prices exist. For these rules, we fully characterize the set of the buyers' NE. We provide a condition under which the set of Nash equilibrium allocations coincides with the set of competitive equilibrium allocations. This holds, for example, for every market where the number of sellers is larger than the number of buyers. For the other markets, the set of competitive equilibria that are sustained as NE outcomes is smaller than the set of all competitive equilibria.

Finally, we look for the subgame perfect equilibria (SPE) of the two-stage game. Thus, we analyze sellers' strategies and the outcome of the whole game. The main results reveal that, by playing first, sellers are able to achieve an outcome that corresponds to the sellers' optimal competitive equilibrium allocation for the true market. We construct a vector of strategies that constitute an SPE for any competitive equilibrium rule and that lead to the maximum competitive prices. Furthermore, we provide reasonable conditions under which every SPE outcome selects the maximum competitive prices.

In addition to the papers studying the manipulability of competitive equilibrium rules which we reviewed above, our paper is related to the literature that looks for mechanisms (unrelated to competitive equilibrium rules) that implement stable or competitive allocations. For the assignment game, Kameke (1989) and Pérez-Castrillo and Sotomayor (2002) propose variants of sequential mechanisms where sellers choose prices first and then buyers choose objects that implement the maximum competitive equilibrium outcome. In the (many-to-one) job market matching, Alcalde, Pérez-Castrillo, and Romero-Medina (1998) offer simple mechanisms that implement the set of stable allocations in SPE when there are at least two firms, and Hayashi and Sakai (2009) study the Nash implementation of the competitive equilibrium correspondence, in addition to proposing mechanisms that lead to this correspondence.

This paper is organized as follows. In sections 2 and 3 we present the model and the structure of the competitive equilibria. In section 4 we introduce the game that we analyze. Section 5 is devoted to a study of the buyers' strategies, and section 6 analyzes the sellers' strategies and the equilibrium of the game. Finally, section 7 concludes.

2. THE BUYER-SELLER MARKET

In the buyer-seller market, there is a set B with m buyers, $B = \{b_1, b_2,\ldots, b_m\}$, and a set S with n sellers, $S = \{s_1, s_2,\ldots,s_n\}$. Each seller s_k owns one indivisible object and each buyer b_j wants to buy, at most, one object. We use the same notation for the seller and his object. Letters j and k are assigned to index buyers and objects (or sellers), respectively.

Each seller s_k values his object in $r_k \geq 0$. Concerning the valuation of the objects for buyers, for each pair (b_j, s_k), there is a number a_{jk} (possibly negative) representing the value of object s_k for b_j. We denote by a_j the vector of values a_{jk}'s. The valuation matrix of the buyers and the valuation vector of the sellers are denoted by a and r, respectively. We use the notation $\boldsymbol{M(a, r)}$ for the market $\boldsymbol{(B, S; a, r)}$ where a and r may vary but B and S are fixed.

The potential gains from trade between buyer b_j and seller s_k are given by $a_{jk} - r_k$. We say that object s_k is *acceptable* to b_j if $a_{jk} - r_k \geq 0$ and it is *unacceptable* otherwise. If an object is unacceptable to a buyer then there is no price higher than the seller's valuation of that object at which the buyer wishes to buy the object. If buyer b_j purchases object s_k at price $p_k \geq r_k$, her payoff is $a_{jk} - p_k$ and the payoff of seller s_k is $p_k - r_k$. We denote by $\underline{a}(r)_{jk}$ the potential gains from trade for the pair (b_j, s_k), that is, $\underline{a}(r)_{jk} \equiv a_{jk} - r_k$ if $a_{jk} - r_k \geq 0$ and $\underline{a}(r)_{jk} \equiv 0$ otherwise. When each seller's reservation price is 0 and all objects are acceptable to every buyer, the corresponding model is the Shapley and Shubik's (1972) well-known assignment game.[7]

A matching is an assignment of the objects to the buyers. Formally,[8]

[7] See Roth and Sotomayor (1990) for an overview of this model.
[8] We use the notation $\sum_j$ for the sum over all b_j in B, $\sum_k$ for the sum over all s_k in S and $\sum_{j,k}$ for the sum over all b_j in B and s_k in S.

Definition 1. *A **matching** for $M(a, r)$ is a matrix $x = (x_{jk})$ of zeros and ones. A matching x for $M(a, r)$ is **feasible** if it satisfies (i) $\sum_j x_{jk} \leq 1$ for all $s_k \in S$, (ii) $\sum_k x_{jk} \leq 1$ for all $b_j \in B$, and (iii) $\underline{a}(r)_{jk} \geq 0$ if $x_{jk} = 1$.*

Conditions (*i*) and (*ii*) in Definition 1 state, respectively, that a feasible matching assigns an object to at most one buyer and a buyer to at most one object. Condition (*iii*) requires that the object assigned to a buyer is acceptable to her. If $x_{jk} = 1$, we say that b_j is *matched* to s_k or s_k is matched to b_j, in which case both agents are *active* at x. If $x_{jk} = 0$ for all $s_k \in S$ (respectively, $b_j \in B$), we say that b_j (respectively, s_k) is *unmatched.*

Definition 2. *A matching x for $M(a, r)$ is **optimal** if it is feasible and (i) $\sum_{j,k} \underline{a}(r)_{jk}\, x_{jk} \geq \sum_{j,k} \underline{a}(r)_{jk}\, x'_{jk}$ for all feasible matchings x' and (ii) if b_j and s_k are both unmatched, then s_k is not acceptable to b_j.*

Remark 1. It is easy to check that if x is optimal for $M(a, r)$ and y satisfies (*i*) and (*ii*) of Definition 1, but it does not satisfy (*iii*), $\sum_{j,k} \underline{a}(r)_{jk}\, x_{jk} \geq \sum_{j,k} \underline{a}(r)_{jk}\, y_{jk}$ still holds.

We now define feasible prices and feasible allocations.

Definition 3. *A **feasible price vector** p (feasible price, for short) for market $M(a, r)$ is a function from S to R such that $p_k \equiv p(s_k) \geq r_k$ for all $s_k \in S$.*

Definition 4. *A **feasible allocation** for $M(a, r)$ is a pair (p, x), where p is a feasible price and x is a feasible matching. The payoff vector of the buyers corresponding to a feasible allocation (p, x) is defined as $u_j = a_{jk} - p_k$ if $x_{jk} = 1$ and $u_j = 0$ if b_j is unmatched. If (p, x) is a feasible allocation then $(u, p - r, x)$ is called a **feasible payoff,** and we say that x is **compatible** with $(u, p - r)$ and vice versa.*

The *demand set* of buyer b_j at the feasible price p is the set $D(b_j, p)$ defined as

$$D(b_j, p) = \{s_k \in S;\ a_{jk} - p_k \geq 0 \text{ and } a_{jk} - p_k \geq a_{jt} - p_t \text{ for all } s_t \in S\}.$$

Thus, among all the acceptable objects that buyer b_j can acquire given the price vector p, she demands those that maximize her payoff. Once we have introduced the buyers' demand sets, we can define a competitive equilibrium for the buyer-seller market.

Definition 5. *A feasible allocation (p, x) for $M(a, r)$ is a **competitive equilibrium** if (i) every active buyer b_j is assigned to some $s_k \in D(b_j, p)$; (ii) for all unmatched buyers b_j we have that $a_{jk} - p_k \leq 0$ for all $s_k \in S$, and (iii) $p_k = r_k$ if s_k is unmatched.*

If (p, x) is a competitive equilibrium allocation for $M(a, r)$, p is called a *competitive equilibrium price vector* or simply a *competitive price* and x is called a *competitive matching*. We denote by **E(*a*, *r*)** the set of *competitive equilibrium allocations* for $M(a, r)$.

Definition 6. *(i) The allocation (p^*, x) is a maximum **competitive equilibrium price** allocation for $M(a, r)$ if it is competitive and $p^*_k \geq p_k$ for all $s_k \in S$ and for any competitive price p; (ii) the allocation (p_*, x) is a **minimum competitive equilibrium price** allocation for $M(a, r)$ if it is competitive and $p_{*k} \leq p_k$ for all $s_k \in S$ and for any competitive price p.*

We denote $p^*(a, r)$ and $p_*(a, r)$ the maximum and the minimum competitive prices for $M(a, r)$.

By using linear programming, Shapley and Shubik (1972) prove that $E(a, r)$ is always *non-empty* for the case where $r = (0,\ldots, 0)$ and all values a_{jk}'s are non-negative.[9] They also show the existence of a maximum and of a minimum competitive equilibrium allocation. The same results apply to $M(a, r)$ for any reservation price vector r and any valuation matrix a.

3. THE STRUCTURE OF THE SET OF COMPETITIVE EQUILIBRIA

In this section we present adaptations of some well-known propositions, which we will use in the following sections. We also provide some intermediate results.

Proposition 1. (Shapley and Shubik, 1972). *(i) If x is an optimal matching for $M(a, r)$ then it is compatible with any competitive equilibrium allocation for $M(a, r)$; (ii) if (p, x) is a competitive equilibrium allocation for $M(a, r)$ then x is an optimal*

[9] This result was also proved in Sotomayor (2000) by using combinatorial arguments.

matching for $M(a, r)$.

Consequently, if (p, x) is a competitive equilibrium allocation for $M(a, r)$ then (p', x) is also a competitive equilibrium allocation for $M(a, r)$, for any competitive price p'.

Proposition 2 follows straightforwardly from Definition 5 and Proposition 1.

Proposition 2. (Demange and Gale, 1985). *Let (p, x) be a competitive equilibrium allocation for $M(a, r)$. Then, if $p_k - r_k > 0$ (respectively, $a_{jt} - p_t > 0$ for some s_t), it is the case that s_k (respectively, b_j) is matched at any optimal matching for $M(a, r)$.*

The following proposition states that the set of competitive equilibria always contains more than one element.

Proposition 3. (Sotomayor, 2002). *Consider the market $M(a, r)$. Suppose $a_{jk} > r_k$ for at least one pair $(b_j, s_k) \in BxS$. If there is only one optimal matching for $M(a, r)$ then the set of competitive equilibrium prices has more than one element.*

Define $\boldsymbol{V_{a,r}(B, S)} \equiv \max \sum_{BxS} \underline{a}(r)_{jk}.x_{jk}$, with the maximum to be taken over all feasible matchings x for $M(a, r)$. The value $V_{a,r}(B, S)$ expresses the maximum total worth of the market $(B, S; a, r)$. It is very helpful in the computation and interpretation of the maximum and minimum competitive prices.

Proposition 4. (Demange, 1982; Leonard, 1983). *For all s_k in S,*
(i) $p^*_k(a, r) = V_{a,r}(B, S) - V_{a,r}(B, S-\{s_k\}) + r_k$
(ii) $p_{*k}(a, r) = V_{a,r}(B-\{b_j\}, S) - V_{a,r}(B-\{b_j\}, S-\{s_k\}) + r_k$ if $x_{jk} = 1$ and $p_{*k}(a, r) = r_k$ if s_k is unmatched at x.

Propositions 5 and 6 show some comparative statics properties of the maximum and minimum competitive prices.

Proposition 5. (Demange and Gale, 1985) *Let $r \leq r'$. Then, $p^*(a, r) \leq p^*(a, r')$.*

Proposition 6. (Demange and Gale, 1985). *Consider the market* $M(a, r)$. *If a set of buyers leaves the market then* $p^*(a, r)$ *and* $p_*(a, r)$ *do not increase.*

The next result states that either some minimum competitive prices are equal to the sellers' valuations or the number of buyers is necessarily larger than the number of objects sold at equilibrium.

Proposition 7. (Demange and Gale, 1985; Pérez-Castrillo and Sotomayor, 2013). *Let* (p_*, x) *be the minimum competitive equilibrium for* $M(a, r)$. *Set* $S' \equiv \{s_k \in S;\ p_{k*} > r_k\}$. *Then* $|B| > |S'|$.

4. THE TWO-STAGE GAME

Competitive equilibria satisfy, among others, the desirable properties of efficiency and envy-freeness – no buyer envies the situation of another one. Thus, it is reasonable to use them as mechanisms for allocating objects to buyers. In any such mechanisms, the designer needs to know the valuations of the sellers and buyers. However, sellers and buyers may have an incentive to report valuations that are not the true ones. We consider this situation as a game, where each seller and buyer is requested to report her or his valuations and the outcome is given by a competitive equilibrium allocation for the reported market. The game also includes a mechanism that allows a selection among the optimal matchings, when their number for the reported market is larger than one.

We consider a two-stage game where first sellers and then buyers are asked to report their valuations. Given the reports, a *competitive equilibrium rule* (Π, f), composed by a *competitive price rule* Π and a *matching rule* f, will select a competitive allocation for the reported market. Formally, we consider the following two-stage game $\Gamma(\Pi, f)$, where the set of players is $B \cup S$.

First stage: Sellers play simultaneously. A strategy for seller s_k consists of choosing a reservation price $r'_k \geq 0$ for his object.

Second stage: Knowing the choices of the sellers, buyers play simultaneously. A strategy for buyer b_j is a pair of functions (a'_j, σ_j) defined as follows. For each vector of reservation prices $r' = (r'_1, \ldots, r'_m)$, the function a'_j selects a valuation vector $a'_j(r') \equiv (a'_{j1}(r'), \ldots, a'_{jn}(r'))$ and the function σ_j gives a *signal vector* $\sigma_j(r') = (\sigma_{j1}(r'), \ldots, \sigma_{jn}(r'))$ of zeros and ones.

To explain the outcome of the game, denote $a'(r') \equiv (a'_1(r'),\ldots, a'_m(r'))$ and $\underline{a}'_{jk}(r') \equiv a'_{jk}(r') - r'_k$ if $a'_{jk}(r') - r'_k \geq 0$ and $\underline{a}'_{jk}(r') \equiv 0$ otherwise. Similary, denote the matrix of signals by $\sigma(r')$. We say that an optimal matching for $M(a'(r'), r')$ is *signalized by* $\sigma(r')$ if it maximizes $\Sigma_{j,k}\ \sigma_{jk}(r')\ y_{jk}$ over all optimal matchings y in $M(a'(r'), r')$.[10] Then, given the profile of decisions $(r'; a', \sigma)$ (where $a' = a'(r')$ and $\sigma = \sigma(r')$), the function Π associates the competitive price $\Pi(a', r') = (\Pi_1(a', r'),\ldots, \Pi_n(a', r'))$ for the market $M(a', r')$. Moreover, the function f associates the optimal matching $f(a', r', \sigma)$ for $M(a', r')$, which is a matching signalized by σ. When there are several signalized optimal matchings, the function f uses some deterministic criterion specified *a priori* (e.g., all matchings are indexed and the matching rule chooses x_i if x_i is present and $x_1, x_2,\ldots,x_{i-1}$ are not present).[11]

If $x = f(a', r', \sigma)$ and $x_{jk} = 1$, buyer b_j pays $\Pi_k(a', r')$ to seller s_k and gets the true payoff of $a_{jk} - \Pi_k(a', r')$. The seller's true payoff is $\Pi_k(a', r') - r_k$. That is, the *true payoffs of buyer* b_j *and seller* s_k under the allocation $(\Pi(a', r'); x)$ are, respectively,

$U_j(\Pi(a', r'); x) = a_{jk} - \Pi_k(a', r')$ if $x_{jk} = 1$ and

$U_j(\Pi(a', r'); x) = 0$ if b_j is unmatched at x.

$V_k(\Pi(a', r'); x) = \Pi_k(a', r') - r_k$ if s_k is matched at x and

$V_k(\Pi(a', r'); x) = 0$ otherwise.

We notice that the sincere strategy profile of the sellers corresponds to $r' = r$. The sincere strategy profile of the buyers is given by $a'_j(r') = a$ for every r' together with an arbitrary σ.

Remark 2. If s_k is acceptable to b_j in the market $M(a, r)$ then $a_{jk} - r_k \geq 0$. Thus, if x is chosen when (a', σ) is selected and s_k is acceptable to b_j in $M(a, r)$ for all pairs (b_j, s_k) such that $x_{jk} = 1$, the true payoff vector $(U_j(\Pi(a', r'); x), V(\Pi(a', r'); x))$ is a feasible payoff for $M(a, r)$.

We are interested in analyzing the class of all two-stage games $\Gamma(\Pi, f)$. Given that

[10] The definition of $\sigma(r')$ implies that the buyers can signalize any optimal matching x for $M(a'(r'), r')$ by choosing $\sigma(r') = x$. More generally, they can signalize any subset S of optimal matchings for $M(a'(r'), r')$ by selecting $\sigma_{jk}(r') = 1$ if there is some matching x in S such that $x_{jk} = 1$ and $\sigma_{jk}(r') = 0$ otherwise.

[11] We can also consider that each matching in this set has the same probability of being selected.

the games have two stages, we use Subgame Perfect Equilibrium (SPE) as the solution concept. Thus, we first analyze the buyers' behavior once the sellers have taken their decision and then we study the sellers' equilibrium behavior.

5. BUYERS' STRATEGIES

In this section we study the subgame $G(r', \Pi, f)$, the strategic game that starts once the sellers' profile of strategies r' has been selected and the outcome function is given by a competitive equilibrium rule (Π, f). This corresponds to the analysis of the second stage of the game Γ. Moreover, this study is interesting in itself because it allows us to understand the buyers' behavior when the sellers' valuations are given, for example, because they are public knowledge or because sellers cannot manipulate them (in which cases, $r' = r$).

Since r' is fixed throughout this section, we drop it from our notations when no confusion is caused and we denote $a' \equiv a'(r')$, $M(a') \equiv M(a'(r'), r')$, and so on. Also, $p^*(a')$ and $p_*(a')$ are the maximum and the minimum competitive prices for $M(a')$ and $u_*(a')$ and $u^*(a')$ are the corresponding buyers' payoffs.

We write a'_{-T} to indicate the restriction of a' to the set $B-T$, where $T \subseteq B$. If $T = \{b_j\}$ we simply write a'_{-j} to denote the decision profile for the buyers other than b_j. We look for the NE of $G(\Pi, f)$. Before stating our main results concerning the NE in the game $G(r', \Pi, f)$, we state and discuss some preliminary results.

5.1. PRELIMINARY RESULTS ON THE BUYERS' STRATEGIES

Previous results in the literature ensure that if the competitive price rule Π is $\Pi(a) = p_*(a)$ then (a_j, σ_j) is a dominant strategy for every buyer b_j, for any signal σ. On the other hand, (a, σ) is not an NE if $\Pi(a) \neq p_*(a)$. In this case, there always exist some buyer b_j who can profitably deviate from a_j if the other players report truthfully. We state here these existing results, which help us understand the strategic incentives of the buyers to manipulate the game or not.

Proposition 8. (Demange and Gale, 1985; Sotomayor, 1986, 1990). *Let $(p_*(a, r), x)$ be a minimum competitive equilibrium allocation for $M(a, r)$ and let u^* be the corresponding buyer payoff vector. Let a' be such that $a'_j = a_j$ for all $b_j \notin B'$, where $B' \subseteq B$. Let $(p(a', r), x')$ be any competitive equilibrium price for $M(a', r)$ and (U, V)*

be the true payoffs under $(p(a', r), x')$. *Then,* $u^*_j \geq U_j$ *for all* $b_j \in B'$.

Proposition 9. (Pérez-Castrillo and Sotomayor, 2013) *Let* (Π, f) *be any competitive equilibrium rule. Let* $M(a, r)$ *be a market with more than one competitive price vector. Then, any* $b_j \in B$ *whose vector of individual payoffs at* $(\Pi(a, r); f(a, r))$ *is different from her vector of individual payoffs under the minimum competitive equilibrium allocation* $(p_*(a, r), x)$ *can manipulate* (Π, f).

Propositions 8 and 9 show that the incentives for the buyers to misrepresent their true valuations depend on whether the rule leads them to their minimum competitive equilibrium allocation. Therefore, while we develop part of the analysis for any competitive equilibrium rule, some of the results focus on the games $G(\Pi, f)$ such that $\Pi(a) \neq p_*(a)$ when $M(a)$ has more than one competitive price, a set that we denote by $\mathbf{C}^+$. For example, the competitive price rules given by a convex combination of the maximum and the minimum competitive price rules are in C^+. That is, $\{\Pi; \Pi = \lambda\, \Pi^* + (1 - \lambda)\, \Pi_*$, with $\lambda \in (0, 1]\} \subseteq C^+$.

If the competitive price rule is in C^+, the NE certainly involve a misrepresentation of the valuations by the buyers. Thus, it is natural to expect that the competitiveness of the NE allocations (when they exist) under the true valuations are affected when buyers behave strategically (see Example 1 below).

We present two examples that motivate our results. Example 1 illustrates a situation in which (*i*) not every competitive equilibrium for $M(a)$ (in particular, the maximum competitive price) is reached through NE strategies satisfying $a' \leq a$; (*ii*) in cases in which $M(a')$ has several competitive prices then a' is not an NE if $\Pi(a') \neq p_*(a')$; (*iii*) some NE allocations are not competitive equilibrium allocations for $M(a)$; (*iv*) when the NE strategy profile satisfies $a' \leq a$ and $M(a')$ has only one competitive price then the NE is a competitive equilibrium allocation for $M(a)$ and $\Pi(a) = p_*(a)$; and (*v*) the fact that a profile of strategies is or is not an NE under some competitive price rule is independent of which prices are associated with other strategy profiles under this rule.

Example 1. The market is given by $B = \{b_1, b_2\}$, $S = \{s_1\}$, $r_1 = 0$ and $a = (8, 7)$. The set of competitive prices is $[7, 8]$. Assume that $\Pi(a) \neq p_*(a) = 7$. It is a matter of verification that if a' is an NE and $a' \leq a$ then either $a'_1 = a'_2 = 7$ or $a'_1 = 7$ and $a'_2 < 7$. In the first case, there are two optimal matchings: x with $x_{11} = 1$ and x' with $x'_{21} = 1$. The profile (a', σ) is an NE if and only if σ signalizes the matching x. In this case, $\Pi(a') = 7 = \pi_*(a') = \pi_*(a)$ and (a', σ) is an NE for the game $G(\Pi, f)$ for every competitive price rule Π. In the second case, x is the only optimal matching for $M(a')$. Then any σ signalizes x. We can check that for any σ, (a', σ) is an NE for the game $G(\Pi, f)$ if and only if $\Pi(a') = a'_2 = p_*(a')$.

In both Nash equilibria, $\Pi(a') = p_*(a')$ and $p_*(a)$ is competitive for $M(a')$. However, in the second case $a'_2 < 7$, so $\Pi(a')$ is not a competitive price for $M(a)$. Also, there are competitive prices for $M(a)$ that are not the outcome of any NE a' with $a' \leq a$ as, for example, $p^*(a) = 8$. If we relax the assumption that $a' \leq a$ then every competitive price of $M(a)$ can be reached via NE strategies. Indeed, if $(p, x) \in \mathrm{E}(a)$, then (a', σ), with $a'_{jk} = p_k$ for all (b_j, s_k) and σ signalizes the matching x, is an NE and (p, x) is the resulting NE allocation. Finally, this example also illustrates that for any two competitive price rules Π_1 and Π_2 such that $\Pi_1(a') = \Pi_2(a')$, (a', σ) is an NE under Π_1 if and only if (a', σ) is an NE under Π_2. ■

When buyers select $a'_j \leq a_j$ and there is only one seller, it is easy to verify that the only NE payoff is (u^*, v_*). Example 1 has shown that this is not always true when $|S| > 1$ because, in that example, some NE allocations are not competitive equilibrium allocations for $M(a)$. Example 2 illustrates a market where there are NE outcomes $(\Pi(a'), f(a', \sigma))$ that satisfy that $\Pi(a')$ is a competitive price for $M(a)$ but it is different from $p_*(a)$, even when $a'_j \leq a_j$ for all b_j.

Example 2. **((a', σ) is an NE, $a'_j \leq a_j$ for all b_j, $\Pi(a') \neq p_*(a)$ is competitive for $M(a)$)** Consider $B = \{b_1, b_2, b_3\}$, $S = \{s_1, s_2\}$, $a_1 = (8, 7)$, $a_2 = (5, 6)$, $a_3 = (4, 5)$ and $r = (0, 0)$. The minimum competitive equilibrium for this market is $(p_* = (4, 5), x)$, where x allocates s_1 to b_1 and s_2 to b_2. Let (Π, f) be any competitive equilibrium rule. Let buyers choose (a', σ) where $\sigma = x$ and $a'_1 = (5, 5)$, $a'_2 = (5, 5)$ and $a'_3 = (4, 5)$. Then, $p_*(a') = p^*(a') = (5, 5)$, so there is only one competitive price in $M(a')$, and so $\Pi(a') = (5, 5)$. Matching x is optimal for $\mathrm{M}(a')$, so it is the only matching signalized by σ. The corresponding true payoff vector for the buyers is

$U(\Pi(a'); x) = (3, 1, 0)$. Clearly $(\Pi(a'); x)$ is a competitive equilibrium under the true valuations. Moreover, we can check that (a', σ) is an NE of the game $G(\Pi, f)$ and $\Pi(a') = (5, 5)$, which differs from $p_*(a)$. ■

Theorems 1 and 2 show that the phenomena described in (*ii*) and (*v*) of Example 1, respectively, are not accidents.

Theorem 1. *Let* (a', σ) *be an NE for* $G(\Pi, f)$. *Then,* $\Pi(a') = p_*(a')$.

Proof. Suppose by way of contradiction that the set of competitive equilibria for $M(a')$ is not a singleton and $\Pi(a') \neq p_*(a')$. Then, $\Pi_k(a') > p_{k*}(a') \geq 0$ for an object $s_k \in S$, and s_k must be matched to some b_j under $x \equiv f(a', \sigma)$. Let $u(a')$ and $u^*(a')$ be the payoff vectors for the buyers corresponding to $\Pi(a')$ and $p_*(a')$, respectively. We have that $u_j^*(a') > u_j(a')$. Let $\lambda \in R^n_+$ be such that

$$u^*_j(a') > \lambda_j > u_j(a') \qquad (1)$$

Define a'' as follows: $a''_{jk} = a'_{jk} - \lambda_j$, $a''_{jt} < 0$ if $s_t \neq s_k$, and $a''_t = a'_t$ if $b_t \neq b_j$. By (1) and because x is compatible with $p_*(a')$ we can write that $a''_{jk} = a'_{jk} - \lambda_j > a'_{jk} - u^*_j(a') = p_{k*}(a') \geq r'_k$. Then,

$$a''_{jk} > p_{k*}(a') \geq r'_k \qquad (2)$$

and s_k is acceptable to b_j in $M(a'')$. We claim that b_j is matched under any optimal matching for $M(a'')$. In fact, arguing by contradiction, suppose that b_j is unmatched under some optimal matching x' for $M(a'')$. By definition of the function $\boldsymbol{V}_{a'}$ we can write: $\boldsymbol{V}_{a'}(B-\{b_j\}, S) = \boldsymbol{V}_{a''}(B, S) \geq (a''_{jk} - r'_k) + \boldsymbol{V}_{a''}(B-\{b_j\}, S-\{s_k\}) > p_{k*}(a') - r'_k + \boldsymbol{V}_{a'}(B-\{b_j\}, S-\{s_k\})$, where (2) was used in the last inequality. However, using Proposition 4(*ii*) we have that $p_{k*}(a') - r'_k = \boldsymbol{V}_{a'}(B-\{b_j\}, S) - \boldsymbol{V}_{a'}(B-\{b_j\}, S-\{s_k\}) > p_{k*}(a') - r'_k$, which is a contradiction.

Since any object other than s_k is not acceptable to b_j in $M(a'')$, b_j must be matched to s_k at any optimal matching for $M(a'')$. Then, $\Pi_k(a'') \leq a''_{jk} = a'_{jk} - \lambda_j$. However, $a'_{jk} - \lambda_j < a'_{jk} - u_j(a')$ by (1). Therefore, $\Pi_k(a'') < a'_{jk} - u_j(a') = \Pi_k(a')$. Thus, for every optimal matching y for $M(a'')$, we have that $U_j(\Pi(a''), y) = a_{jk} - \Pi_k(a'') > a''_{jk} - \Pi_k(a') = U_j(\Pi(a'), x)$, which contradicts the fact that (a', σ) is an NE for $G(\Pi, f)$. Hence, $\Pi(a') = p_*(a')$ if $M(a')$ has more than one competitive equilibrium. ■

As stated before, in addition to its intrinsic interest, Theorem 1 helps us to better understand some of the facts discussed in Example 1. In that example, the profile of strategies $a'_1 = 7$ and $a'_2 < 7$ constitute an NE if and only if the competitive price rule Π satisfies $\Pi(a') = a'_2 = p_*(a')$. Moreover, in the example, where $\Pi \notin C^+$, the matching x is the only optimal matching for $M(a')$ and any σ signalizes it. However, when $\Pi \in C^+$, Proposition 10 below implies that, aside very special cases in which no agent is able to obtain a positive payoff, for a vector of reports a' to be an NE, the number of optimal matchings must be greater than one.

Proposition 10. *Let (a', σ) be an NE of $G(\Pi, f)$, where $\Pi \in C^+$. Suppose that $a'_{jk} - r_{jk} > 0$ for at least one pair $(b_j, s_k) \in BxS$. Then, there is more than one optimal matching in $M(a')$.*

Proof. By Theorem 1, $\Pi(a') = p_*(a')$. Given that $\Pi \in C^+$, $\Pi(a') = p_*(a')$ is possible only if the set of competitive prices for $M(a')$ is a singleton. The result then follows from Proposition 3 applied to market $M(a')$. ■

Theorem 2 states a relationship between the NE of two related games.

Theorem 2. *Let Π_1 and Π_2 be competitive price rules. Let (a', σ) be a profile of strategies such that $\Pi_1(a') = \Pi_2(a')$. Then, (a', σ) is an NE of $G(\Pi_1, f)$ if and only if (a', σ) is an NE of $G(\Pi_2, f)$.*

Proof. Suppose by way of contradiction that (a', σ) is an NE of $G(\Pi_1, f)$ but it is not an NE of $G(\Pi_2, f)$. Then there is a buyer b_j and (a'', σ') with $a''_{-j} = a'_{-j}$ and $\sigma'_{-j} = \sigma_{-j}$ such that $U_j(\Pi_2(a''), f(a'', \sigma')) > U_j(\Pi_2(a'), f(a', \sigma))$. Then, b_j must be matched at $f(a'', \sigma')$ to some s_k and for some $\lambda > 0$ we have that $U_j(\Pi_2(a''), f(a'', \sigma')) - \lambda > U_j(\Pi_2(a'), f(a', \sigma))$. Moreover, by Theorem 1, $p_*(a') = \Pi_1(a') = \Pi_2(a')$, so

$$U_j(\Pi_2(a''), f(a'', \sigma')) - \lambda > U_j(\Pi_1(a'), f(a', \sigma)) \qquad (3)$$

Define a''' such that $a'''_{jk} = \Pi_{2k}(\beta) + \lambda$, $a'''_{jt} = 0$ for all $s_t \neq s_k$ and $a'''_{-j} = a'_{-j}$. It is clear that $(\Pi_2(a''), f(a'', \sigma'))$ is a competitive equilibrium allocation for $M(a''')$, so $f(a'', \sigma')$ is an optimal matching for $M(a''')$. In addition, the corresponding payoff for b_j is $a'''_{jk} - \Pi_{2k}(a'') = \lambda > 0$, so b_j is matched under every optimal matching for

$M(a''')$ by Proposition 2. Also, $a'''_{jk} - \Pi_{2k}(a'') > a'''_{jt} - \Pi_{2t}(a'')$ for every $s_t \neq s_k$, so b_j only demands s_k at prices $\Pi_2(a'')$. Hence, any competitive matching for $\Pi_2(a'')$ must allocate s_k to b_j. Given that, by Proposition 1, any optimal matching for $M(a''')$ is competitive for $\Pi_2(a'')$, we must have that b_j is matched to s_k at $f(a''', \sigma')$. This implies that $\Pi_{2k}(a''') \leq a'''_{jk} = \Pi_{2k}(a'') + \lambda$. Then, $U_j(\Pi_2(a'''), f(a''', \sigma)) = a_{jk} - \Pi_{2k}(a''') \geq a_{jk} - (\Pi_{2k}(a'') + \lambda) = U_j(\Pi_2(a''), f(a'', \sigma')) - \lambda > U_j(\Pi_1(a'), f(a', \sigma))$, where the last inequality follows from (3). But then $U_j(\Pi_2(a'''), f(a''', \sigma)) > U_j(\Pi_1(a'), f(a', \sigma))$, which contradicts the hypothesis that (a', σ) is an NE of $G(\Pi_1, f)$. ■

Corollaries 1 and 2 below follow straightforwardly from theorems 1 and 2, respectively.

Corollary 1. *Let (a', σ) be an NE for $G(\Pi, f)$, where $\Pi \in C^+$. Then, there is one and only one competitive equilibrium price for $M(a')$.*

Corollary 2. *Let Π_1 and Π_2 be competitive price rules in C^+. Then, the profile of strategies (a', σ) is an NE of $G(\Pi_1, f)$ if and only if (a', σ) is an NE of $G(\Pi_2, f)$.*

5.2. MAIN RESULTS ON THE BUYERS' STRATEGIES

In this subsection, we address the existence and characteristics of the NE of the subgame $G(\Pi, f)$. For that purpose, we develop two complementary analyses. In the first analysis, we concentrate on strategies that satisfy $a' \leq a$, which may be a reasonable restriction in several environments. We conclude with the existence result in Theorem 5, which constructs an NE that leads to the minimum competitive price for the market $M(a)$. Then, in the second analysis, whose final results are theorems 6 and 7, we fully characterize the set of NE of $G(\Pi, f)$ without restrictions on the type of strategies for any market. In the latest theorems, we focus on competitive price rules in C^+.

Before presenting these two analyses, we provide a theorem that is interesting by itself and that will be used in the rest of the section. As Example 1 illustrates, the NE allocation does not always yield a competitive equilibrium for $M(a)$. Theorem 3 shows that the NE is a competitive equilibrium if there is only one competitive price vector in the market defined by the vector of reports.

Theorem 3. *Let* (a', σ) *be an NE of* $G(\Pi, f)$. *If* $M(a')$ *has only one competitive equilibrium price then* $\Pi(a')$ *is a competitive equilibrium price for* $M(a)$.

Proof. Suppose that $(\Pi(a'), x)$ is not a competitive equilibrium allocation for $M(a)$. For any pair (b_j, s_k) with $x_{jk} = 1$, $a_{jk} - \Pi_k(a') \geq 0$ because b_j could obtain a zero true payoff by selecting $a''_{jt} < 0$ for every s_t, which would contradict that a' is an NE. Then, there must exist some b_j, s_t and s_k, such that $x_{jt} = 1$ and $U_j(\Pi(a'), x) = a_{jt} - \Pi_t(a') < a_{jk} - \Pi_k(a')$. Let $\lambda > 0$ such that

$$\Pi_k(a') < a_{jk} - (U_j(\Pi(a'), x) + \lambda). \tag{4}$$

Define a'' as follows: $a''_{jk} = a_{jk} - (U_j(\Pi(a'), x) + \lambda)$, $a''_{jt} < 0$ if $s_t \neq s_k$ and $a''_t = a'_t$ if $b_t \neq b_j$. Choose any signal σ' and denote $y \equiv f(a'', \sigma')$. We have that $a''_{jk} - r'_k > 0$ by (4) and $a''_{jt} - r'_t < 0$ if $s_t \neq s_k$. Then, if b_j is active, he must be matched with s_k. We will show that $y_{jk} = 1$. Once this is established, it follows that $\Pi_k(a'') \leq a''_{jk}$ and then $U_j(\Pi(a''), y) = a_{jk} - \Pi_k(a'') \geq a_{jk} - a''_{jk} = U_j(\Pi(a'), x) + \lambda > U_j(\Pi(a'), x)$, which contradicts the assumption that a' is an NE for (Π, f).

To prove that $y_{jk} = 1$, consider the market $M' \equiv (B - \{b_j\}, S, a'_{-j}, r')$. Let $p^*(M')$ be the maximum competitive price for M'. If $y_{jk} = 0$, then b_j is unmatched under y and $V_{a''}(B, S) = V_{a''}(B - \{b_j\}, S)$. By definition of a'', we have that $V_{a''}(B, S - \{s_k\}) = V_{a''}(B - \{b_j\}, S - \{s_k\})$. Using Proposition 4(*i*), $p^*_k(a'') - r'_k = V_{a''}(B, S) - V_{a''}(B, S - \{s_k\}) = V_{a''}(B - \{b_j\}, S) - V_{a''}(B - \{b_j\}, S - \{s_k\}) = p^*_k(M') - r'_k$, so $p^*_k(a'') = p^*_k(M')$. On the other hand, Proposition 6 applied to markets $M(a')$ and M' implies that $p^*_k(a') \geq p^*_k(M')$. Therefore,

$$p^*_k(a') \geq p^*_k(a''). \tag{5}$$

$M(a')$ has only one competitive equilibrium price, hence

$$\Pi_k(a') = p^*_k(a') \tag{6}$$

Thus, $p^*_k(a'') \geq a''_{jk} > \Pi_k(a') = p^*_k(a') \geq p^*_k(a'')$, which is absurd, where the first inequality follows from the competitiveness of $(p^*(a''), y)$ and from the assumption that b_j is unmatched at y; in the second inequality we used (4) and the definition of a''_{jk}; the equality is given by (6); and the last inequality follows from (5). Then, b_j is necessarily matched under y, so $y_{jk} = 1$, which concludes the proof. ■

Corollary 3. *Let* (a', σ) *be an NE of* $G(\Pi, f)$. *If* $\Pi \in C^+$ *then* $\Pi(a')$ *is a competitive equilibrium price for* $M(a)$.

Proof. Corollary 1 implies that $M(a')$ has only one competitive equilibrium price. Accordingly, the result follows from Theorem 3. ■

We now proceed to construct strategies satisfying $a'_{jk} \leq a$ that constitute NE of $G(\Pi, f)$ and whose outcome is the best competitive equilibrium for the buyers. We start with two lemmas. They use the idea of a "super-optimal" matching with respect to two markets. We say that a matching x is *super-optimal* for $M(a')$ and $M(a)$ if it is optimal for both markets.

Lemma 1. *Let p be a competitive equilibrium price for $M(a)$ and u the corresponding payoff for the buyers. Let a' be defined by $a'_{jk} = a_{jk} - u_j$ for all $(b_j, s_k) \in BxS$. Then p is a competitive equilibrium price for $M(a')$ and all the optimal matchings for $M(a)$ are super-optimal matchings for $M(a)$ and $M(a')$.*

Proof. Let x be an optimal matching for $M(a)$. By Proposition 1(*i*), x is compatible with p in $M(a)$. For the first assertion, use the competitiveness of (p, x) in $M(a)$ to get that if $x_{jk} = 1$, $a'_{jk} - p_k = (a_{jk} - p_k) - u_j \geq (a_{jt} - p_t) - u_j = a'_{jt} - p_t$, for all $s_t \in S$. Also, $a'_{jk} - p_k = a_{jk} - u_j - p_k = 0$, so s_k is acceptable to b_j in $M(a')$. Therefore, (p, x) is a competitive equilibrium allocation for $M(a')$. For the second assertion, use Proposition 1(*ii*) to obtain that x is also optimal for $M(a')$. ■

Lemma 2. *Let p_* be the minimum competitive equilibrium price for $M(a)$ and u^* the corresponding payoff for the buyers. Let a' be defined by $a'_{jk} = a_{jk} - u^*_j$ for all $(b_j, s_k) \in BxS$. Then, the set of competitive equilibrium prices for $M(a')$ is a singleton and p_* is its only element.*

Proof. Let x be some optimal matching for $M(a)$. By Lemma 1, (p_*, x) is a competitive equilibrium allocation for $M(a')$. Let p be a competitive price for $M(a')$. Given that p is competitive for $M(a')$, if $x_{jk} = 1$ then $a_{jk} - p_k = a_{jk} - u^*_j + u^*_j - p_k = (a'_{jk} - p_k) + u^*_j \geq (a'_{jt} - p_t) + u^*_j = a_{jt} - u^*_j + u^*_j - p_t = a_{jt} - p_t$ for any $s_t \in S$. Also, $a_{jk} - p_k = (a'_{jk} - p_k) + u^*_j \geq 0$. Hence (p, x) is a competitive equilibrium allocation for $M(a)$, from which it follows that $p \geq p_*$. On the other hand, $p_k \leq a'_{jk} = a_{jk} - u^*_j = p_{k*}$ if $x_{jk} = 1$ and $p_k = p_{k*} = r'_k$ if s_k is unmatched at x. Then $p \leq p_*$. It follows that $p = p_*$. ■

Theorem 4 highlights the strong link between the equilibrium of the strategies constructed in Lemma 2 and the super-optimality of the matching generated by the strategies.

Theorem 4. *Let* (a', σ) *be a strategy profile for the game* $G(\Pi, f)$ *such that* $a'_{jk} = a_{jk} - u^*_j$ *for each pair* $(b_j, s_k) \in BxS$. *Then* (a', σ) *is an NE of* $G(\Pi, f)$ *if and only if* $f(a', \sigma)$ *is super-optimal for* $M(a')$ *and* $M(a)$.

Proof. Suppose that (a', σ) is an NE of $G(\Pi, f)$. Let $x \equiv f(a', \sigma)$. By Lemma 2, the set of competitive prices for $M(a')$ is a singleton and p_* is its only element. Therefore, Theorem 3 implies that $(\Pi(a'), x)$ is a competitive equilibrium allocation for $M(a)$. Then, x is optimal for both $M(a)$ and $M(a')$.

In the other direction, suppose by way of contradiction that x is a super-optimal matching for $M(a')$ and $M(a)$ but (a', σ) is not an NE of $G(\Pi, f)$. Then, there exists some buyer b_j, some strategy profile a'' with $a''_t = a'_t$ if $b_t \neq b_j$, and some signal σ', such that

$$U_j(\Pi(a''), x') > U_j(\Pi(a'), x), \tag{7}$$

where $x' = f(a'', \sigma')$.

If b_j is active, say $x_{jl} = 1$, then $u_j(a') = a''_{jl} - \Pi_l(a') = a_{jl} - \Pi_l(a') - u^*_j$, so $U_j(\Pi(a'), x) = a_{jl} - \Pi_l(a') = u_j(a') + u^*_j$. If b_j is unmatched at x then $u_j(a') = 0$ and $U_j(\Pi(a'), x) = 0$. By Lemma 1, x is optimal for $M(a)$, so $u^*_j = 0$. In both cases,

$$U_j(\Pi(a'), x) = u_j(a') + u^*_j \geq 0 \tag{8}$$

From (7) and (8) it follows that $U_j(\Pi(a''), x') > 0$, so b_j must be matched to some s_k under x'. Denote $U'_j(\Pi(a''), x')$ the transfer associated with x' in $M(a')$, that is, $U'_j(\Pi(a''), x') = a'_{jk} - \Pi_k(a'')$. Then, $U'_j(\Pi(a''), x') = (a_{jk} - u^*_j) - \Pi_k(a'') = U_j(\Pi(a''), x') - u^*_j > U_j(\Pi(a'), x) - u^*_j = u_j(a') + u^*_j - u^*_j = u_j(a')$, where the inequality and the second-to-last equality follow from (7) and (8), respectively. By Lemma 2, there is only one competitive price in $M(a')$. Hence, $u_j(a') = u^*_j(a')$, which implies $U'_j(\Pi(a''), x') > u^*_j(a')$, which contradicts Proposition 12 applied to $M(a')$. Hence, (a', σ) is an NE of $G(\Pi, f)$. ■

Theorem 5 shows that the strategies constructed above, which are based on and lead to the minimum competitive price, constitute an NE of the game $G(\Pi, f)$, for any competitive equilibrium rule. In this way, the theorem provides a constructive proof of the existence of equilibrium.

Theorem 5. *For each pair* $(b_j, s_k) \in BxS$, *let* $a'_{jk} = a_{jk} - u^*_j$ *and* $\sigma_{jk} = 1$ *if* $x_{jk} = 1$ *for some optimal matching* x *for* $M(a)$ *and* $\sigma_{jk} = 0$ *otherwise. Then,* (a', σ) *is an NE of any game* $G(\Pi, f)$. *Furthermore,* $U(\Pi(a'), f(a', \sigma)) = u^*$.

Proof. Consider any arbitrary game $G(\Pi, f)$. By Lemma 1, the optimal matchings for $M(a)$ are also optimal for $M(a')$ and vice-versa. By definition of σ, every optimal matching for $M(a)$ is signalized, which implies that the set of matchings signalized by σ coincides with the set of super-optimal matchings for $M(a')$. Then, if $f(a', \sigma) = x$ we have that x is super-optimal for $M(a)$ and $M(a')$; so (a', σ) is an NE for $G(\Pi, f)$. Also, $\Pi(a') = p_*$ by Lemma 2. Therefore, $U_j(\Pi(a'), x) = a_{jk} - p_{k*} = u^*_j$ if $x_{jk} = 1$ and $U_j(\Pi(a'), x) = 0 = u^*_j$ if b_j is unmatched at x, which completes the proof. ■

The signal vector is used in Theorem 5 to select a set of super-optimal matchings when $\Pi \in C^+$. Its role is crucial for guaranteeing the existence of NE of the game $G(\Pi, f)$.

Theorem 5 proposes strategies that yield the optimal competitive equilibrium for the buyers as an NE. Example 3 illustrates that there may be NE strategies (a', σ), where a' is not defined as in Theorem 5, that yields the optimal competitive equilibrium for the buyers. Thus, coordination problems may still exist.

Example 3. (***(a', σ) is a Nash equilibrium that is not defined as in Theorem 5, but $\Pi(a') = p_*(a)$***). Consider $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$, $a_j = (4, 3)$ for $j = 1, 2$, and $r_k = 0$ for $k = 1, 2$. The B-optimal competitive equilibrium payoff for this market is given by $u^*_1 = u^*_2 = 3$, $v_{*1} = 1$, $v_{*2} = 0$. For each pair $(b_j, s_k) \in B$xS, let $a'_{jk} = a_{jk} - \lambda_j$ where $\lambda_1 = 2$ and $\lambda_2 = 3$. Hence, (a', σ) is an NE for every competitive equilibrium rule and for every σ. Furthermore, (u^*, v_*) is the corresponding NE payoff. However, $\lambda \neq u^*$. ■

When we look for the NE of the game $G(\Pi, f)$ without restrictions on the type of strategies that buyers use, we have more precise information on the set of NE outcomes and the relationship between this set and the set of competitive equilibrium allocations of the market $M(a)$. In particular, we can provide a full characterization of the set of NE outcomes if the competitive price rule is in C^+. The main results, which are theorems 6 and 7 below, are immediate consequences of the following lemmas 3 and 4 and previous results.

Lemma 3. *Let (p, x) be a competitive equilibrium of $M(a)$. Denote $S'(p, x) \equiv \{s_k \in S;\, p_k > r'_k\}$. If $|B| > |S'(p, x)|$, then there is an NE of $G(\Pi, f)$ whose NE allocation is (p, x).*

Proof. Consider the following strategies: $a'_j = p$ and $\sigma_j = x_j$ for all $b_j \in B$. Given that p is a competitive price of $M(a)$, $p \geq r'$. Moreover, if $x_{jk} = 1$, then $V_{a'}(B, S) = V_{a'}(B-\{b_j\}, S) = \sum_t (p_t - r'_t)$ because $|B| > |S'(p, x)|$ allows all the objects in $S'(p, x)$ to be matched to a buyer (and the surplus of the other objects, according to a', is zero). Similarly, $V_{a'}(B, S-\{s_k\}) = V_{a'}(B-\{b_j\}, S-\{s_k\}) = \sum_{t\neq k} (p_t - r'_t)$ for all $s_k \in S$. Therefore, according to Proposition 4, $p^*_k(a') - r'_k = V_{a'}(B, S) - V_{a'}(B, S-\{s_k\}) = p_k - r'_k$ and $p_{*k}(a') - r'_k = V_{a'}(B-\{b_j\}, S) - V_{a'}(B-\{b_j\}, S-\{s_k\}) = p_k - r'_k$ for all $s_k \in S$, which implies that the outcome of the strategy profile (a', σ) is necessarily $(\Pi(a'), f(a', \sigma)) = (p, x)$.

We now prove that (a', σ) is an NE of $G(\Pi, f)$. Suppose that player b_j deviates to (a''_j, σ'_j) and denote $a'' = (a''_j, a'_{-j})$ and $\sigma' = (\sigma'_j, \sigma_{-j})$. Given that there are at least $|S'(p, x)|$ buyers reporting valuation p, any optimal matching for $M(a'')$ produces a value for the market at least equal to $\sum_{sk\in S\setminus S'(p,\, x)} (p_k - r'_k)$ and $\Pi_k(a'') \geq p_k$ for all $s_k \in S'(p, x)$. Moreover, $\Pi_k(a'') \geq r'_k = p_k$ because $\Pi(a'')$ is a competitive price vector. Therefore, $a_{jk} - \Pi_k(a'') \leq a_{jk} - p_k \leq U_j(p, x) = U_j(\Pi(a'), f(a', \sigma))$ for any $s_k \in S$. Thus, b_j cannot improve by deviating to a''. ■

Lemma 4. *Let (p, x) be a competitive equilibrium of $M(a)$. Denote $S'(p, x) \equiv \{s_k \in S;\, p_k > r'_k\}$. If $|B| \leq |S'(p, x)|$, then there is no NE of $G(\Pi, f)$ whose outcome is (p, x).*

Proof. Suppose (a', σ) is an NE of $G(\Pi, f)$ and $\Pi(a') = p$. By Theorem 1, we have that $p = p_*(a')$. Then, Proposition 8 implies that $|B| > |S'(p, x)|$, which is not possible. ■

For some markets, Lemma 3 allows us to state that the set of competitive equilibria coincides with the set of NE allocations. However, there are markets where the set of competitive equilibria that are sustained as NE outcomes of $G(\Pi, f)$ is smaller than the set of all competitive equilibria, although larger than the set of competitive equilibrium with minimum prices. We first illustrate that this can happen in Example 4 and then we state the general results.

Example 4. Let $B = \{b_1, b_2\}$, $S = \{s_1, s_2\}$, $a_{11} = 3$, $a_{12} = 3$, $a_{21} = 4$, $a_{22} = 8$, $r'_1 = r'_2 = 0$. It is easy to check that a vector of prices p is competitive if and only if $p_1 \leq 3$, $p_2 \geq p_1$ and $p_2 \leq 4 + p_1$. The minimum competitive price is $p_* = (0, 0)$ and the set of competitive prices that are outcome of some NE of $G(\Pi, f)$ is $\{(0, p_2); p_2 \in [0, 4]\}$, which is strictly smaller than the set of competitive equilibria. ■

Denote $S^o = \{s_k \in S; p_k > r'_k$ for some $(p, x) \in E(a)\}$. Sellers in S^o are active in every competitive equilibrium, so $|B| \geq |S^o|$ always.

Theorem 6. *If* $|B| > |S^o|$ *and* $\Pi \in C^+$*, then the set of NE allocations of* $G(\Pi, f)$ *coincides with the set of competitive equilibrium allocations of* $M(a)$.

Proof. For every $(p, x) \in E(a)$, it happens that $S'(p, x) \subseteq S^o$. Hence, $|B| > |S^o|$ implies $|B| > |S'(p, x)|$ for every $(p, x) \in E(a)$. According to Lemma 3, this implies that all competitive equilibrium allocations of $M(a)$ are NE allocations of $G(\Pi, f)$. On the other hand, Theorem 3 shows that all NE allocations of $G(\Pi, f)$ are competitive equilibrium allocations of $M(a)$ if $\Pi \in C^+$. ■

Corollary 4. *If* $|B| > |S|$ *and* $\Pi \in C^+$*, then the set of NE allocations of* $G(\Pi, f)$ *coincides with the set of competitive equilibrium allocations of* $M(a)$.

Theorem 7. *If* $|B| = |S^o|$ *and* $\Pi \in C^+$*, then the set of NE outcomes of* $G(\Pi, f)$ *corresponds to the set* $\{(p, x); (p, x) \in E(a)$ *and* $p_k = r'_k$ *for some* $s_k \in S^o\}$*, which contains the set of minimum competitive price allocations of* $M(a)$.

Proof. First, Theorem 3 shows that all NE allocations of $G(\Pi, f)$ are competitive equilibrium allocations of $M(a)$ if $\Pi \in C^+$. Second, consider $(p, x) \in E(a)$, for which $S'(p, x) \subseteq S^o$. If $p_k > r'_k$ for all $s_k \in S^o$, $S'(p, x) = S^o$ and, applying Lemma 4, there is no NE of $G(\Pi, f)$ whose outcome is (p, x). On the other hand, if $p_k = r'_k$ for some

$s_k \in S^o$, then $S'(p, x) \subset S^o$ and, applying Lemma 3, there is an NE of $G(\Pi, f)$ whose outcome is (p, x). Finally, Theorem 5 implies that the set $\{(p, x); (p, x) \in E(a)$ and $p_k = r'_k$ for some $s_k \in S^o\}$ includes the set of minimum competitive price allocations of $M(a)$.

6. SELLERS' STRATEGIES AND EQUILIBRIUM OF THE GAME

The analysis of section 5 shows that, for any sellers' choice r', buyers can get $u^*(r')$ by playing the equilibrium strategies a' identified in Theorem 5. We now show that, in equilibrium, sellers can reverse this situation.

We use some of the results in section 5 first to construct an SPE for the two-stage game $\Gamma(\Pi, f)$ for any competitive equilibrium rule (Π, f) and then to characterize the outcome of all the SPE of the game under certain conditions. The construction of an SPE uses two instrumental lemmas, where we denote p^* and p_*, respectively, the maximum and the minimum competitive prices for the true market $M(a, r)$.

Lemma 5. *p^* is the only competitive equilibrium price for market $M(a, r' = p^*)$.*

Proof. Let x be an optimal matching for $M(a, r)$. Then, x is compatible with p^* by Proposition 1(*i*). Clearly, (p^*, x) is a competitive equilibrium for $M(a, p^*)$, so x is an optimal matching for $M(a, p^*)$. Let p be some competitive price for $M(a, p^*)$. Then, x is compatible with p by Proposition 1(*ii*). We claim that $p = p^*$. In fact, the feasibility of p in $M(a, p^*)$ implies that

$$p_k \geq p^*_k \geq r_k \text{ for all } s_k \in S. \qquad (9)$$

Then p is feasible for $M(a, r)$. Moreover, if s_k is unmatched at x then $p_k = p^*_k = r_k$, so $p_k = r_k$. The competitiveness of (p, x) in $M(a, r)$ therefore follows from the competitiveness of (p, x) in $M(a, p^*)$. Then, the maximality of p^* in $M(a, r)$ implies

$$p^*_k \geq p_k \text{ for all } s_k \in S. \qquad (10)$$

By (9) and (10) we obtain that $p^* = p$. Hence, p^* is the only competitive price for $M(a, p^*)$. ■

Lemma 6. *Let r' be a vector of reservation prices for the sellers such that $r'_k > p^*_k$ for some $s_k \in S$ and $r'_t = p^*_t$ for all $s_t \in S-\{s_k\}$. Then, s_k is unmatched at any competitive equilibrium for $M(a, r')$.*

Proof. If (p, x) is a competitive equilibrium for $M(a, r')$ and s_k is matched at x, then (p, x) is also a competitive equilibrium for $M(a, p^*)$, and so $p = p^*$ by Lemma 5. However, $p_k \geq r'_k > p^*_k$, which is a contradiction. ■

Theorem 8 proposes strategies for the sellers and the buyers and shows that they constitute an SPE of the game $\Gamma(\Pi, f)$, for any competitive equilibrium rule (Π, f). In this SPE, the sellers report valuations that correspond to the maximum competitive prices for the true market and the buyers follow the strategy that we propose in Theorem 5, which leads to the minimum competitive prices for the "reported" market. The fact that sellers choose first gives them a crucial advantage and the SPE allocation corresponds to the sellers' optimal competitive equilibrium allocation for the true market.

Theorem 8. *Let* $(a'(.), \sigma(.))$ *be defined as follows:* $a'_{jk}(r') = a_{jk} - u^*_j(a, r')$ *for each* $(b_j, s_k) \in BxS$ *and for all* r', $\sigma_{jk}(r') = 1$ *if* $x_{jk} = 1$ *for some optimal matching* x *for* $M(a, r')$ *and* $\sigma_{jk}(r') = 0$ *otherwise. Then,* $(p^*, a'(.), \sigma(.))$ *constitutes an SPE of* $\Gamma(\Pi, f)$ *for any competitive equilibrium rule* (Π, f). *At this SPE, the true payoffs of buyers and sellers are, respectively,* u_* *and* $p^* - r$.

Proof. By Theorem 5, $(a'(r'), \sigma(r'))$ is an NE of the subgame $G(r', (\Pi, f))$, for every selection r' by the sellers. Moreover, $\Pi(a'(r'), r') = p_*(a, r')$ and $U(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = u^*(a, r')$. In particular, when $r' = p^*$, $(a'(p^*), \sigma(p^*))$ is an NE of the subgame $G(p^*, (\Pi, f))$, and $\Pi(a'(p^*)) = p_*(p^*)$, which is the minimum competitive price of the market $M(a, p^*)$. By Lemma 5, p^* is the only competitive price for $M(a, p^*)$, so $\Pi(a'(p^*)) = p^*$ and $u^*(p^*) = u_*(p^*) = u_*$. Also, $U(\Pi(a'(r')), f(a'(r'), r', \sigma(r'))) = u_*$ and $V(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = p^* - r$.

Thus, proving that $(p^*, a'(.), \sigma(.))$ is an SPE only requires showing that p^*_k is a best response for every seller $s_k \in S$. By playing p^*_k seller s_k gets p^*_k. If s_k selects any other r'_k, the competitive price is given by $\Pi(a'(r'), r') = p_*(r')$, where $r' \equiv (r'_k, p^*_{-k})$.

There are two cases. First, if $r'_k < p^*_k$ then Proposition 5 implies that $p_*(r') \leq p_*(p^*) = p^*$. Hence, $\Pi(a'(r'), r')_k = p_*(r')_k \leq p^*_k$, so s_k cannot profit by deviating from

p^*_k. Second, if $r'_k > p^*_k$, it follows from Lemma 6 that s_k is unmatched at any competitive equilibrium for $M(a, r')$, so $\Pi(a'(r'), r')_k = 0 \leq p^*_k$. Therefore, s_k cannot profit by deviating from p^*_k and $(a'(r'), p^*, \sigma(r'))$ is an SPE. ■

Theorem 9 shows a stronger result than Theorem 8 when we restrict attention to competitive price rules in the set C^+. For any such rule, the strategy vector where sellers select the maximum competitive prices for the true market and the buyers follow any NE of the continuation game is an SPE of the game.

Theorem 9. *Consider any competitive price rule $\Pi \in C^+$. Let $(a'(r'), \sigma(r'))$ be an NE of the subgame $G(r', (\Pi, f))$, for every selection r' by the sellers. Then, $(p^*, a'(.), \sigma(.))$ is an SPE of $\Gamma(\Pi, f)$. At this SPE, the true payoffs of buyers and sellers are, respectively, u_* and $p^* - r$.*

Proof. By Theorem 3, $(\Pi(a'(p^*), p^*), f(a'(p^*), p^*, \sigma(p^*)))$ is a competitive equilibrium for $M(a, p^*)$, so $\Pi(a'(p^*), p^*) = p^*$ and $x \equiv f(a'(p^*), p^*, \sigma(p^*))$ is optimal for $M(a, p^*)$. Then, $U(\Pi(a'(p^*), p^*), x) = u_*$ and $V(\Pi(a'(p^*), p^*), x) = p^* - r$.

The rest of the proof follows the same arguments as the proof of Theorem 8. ■

Theorems 8 and 9 construct SPE whose outcome corresponds to the sellers' optimal competitive equilibrium allocation for the true market. Is this allocation the unique SPE of the game $\Gamma(\Pi, f)$? Theorem 10, which constitutes our final result, states that this outcome is in fact the unique SPE outcome of any game $\Gamma(\Pi, f)$ for price rules Π in C^+ provided that the sellers' strategies satisfy two intuitive conditions: a seller's report cannot be lower than her true valuation, and it is equal to the true valuation if she does not sell her object at the SPE even reporting that valuation.

Theorem 10 makes use of Lemma 7, which states the outcome of any SPE when the price rule is in C^+ as a function of the equilibrium strategy by the sellers.

Lemma 7. *Consider any competitive price rule $\Pi \in C^+$. If $(r', a'(.), \sigma(.))$ is an SPE of $\Gamma(\Pi, f)$, then $V(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = p^*(a, r') - r$ and $f(a'(r'), r', \sigma(r'))$ is optimal for $M(a, r')$.*

Proof. Since $(a'(r'), \sigma(r'))$ is an NE for the buyers given r' and $\Pi \in C^+$, Theorem 3 implies that $(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r')))$ is a competitive equilibrium for $M(a, r')$, so $x' \equiv f(a'(r'), r', \sigma(r'))$ is optimal for $M(a, r')$, and it is compatible with $p^*(a, r')$. Now, suppose by contradiction that $\Pi(a'(r'), r') \neq p^*(r')$. Then, there is some seller s_k such that $p^*_k(a, r') > \Pi(a'(r'), r')_k \geq r'_k$ so $p^*_k(a, r') > r'_k$ and s_k is matched under x'. Choose r''_k so that

$$p_k^*(a, r') > r''_k > \Pi(a'(r'), r') \geq r'_k. \tag{11}$$

Suppose that seller s_k deviates from r' by choosing r''_k, and set $r'' \equiv (r''_k, r'_{-k})$. Because $(r', a'(.), \sigma(.))$ is an SPE of $\Gamma(\Pi, f)$, it is the case that $(a'(r''), \sigma(r''))$ is an NE for the buyers and $(\Pi(a'(r''), r''), f(a'(r''), r'',\sigma(r'')))$ is a competitive equilibrium for $M(a, r'')$. On the other hand, given that $r'' > r'$, Proposition 5 implies that $p^*_k(a, r'') \geq p^*_k(a, r')$, so $p^*_k(a, r'') > r''_k$ by (11), which implies that s_k is matched at $x'' \equiv f(a'(r''), r'', \sigma(r''))$. Therefore, s_k's true payoff is $V_k(\Pi(a'(r''), r''), x'') = \Pi(a'(r''), r'')_k - r_k \geq r''_k - r_k > \Pi(a'(r')\ , r')_k - r_k = V_k(\Pi(a'(r'), x)$, from which it follows that s_k is not playing his best response, contradicting the assumption that $(r', a'(.), \sigma(.))$ is an SPE. Hence, $\Pi(a'(r'), r') = p^*(a, r')$, so $V(\Pi(a'(r'), r'), x) = p^*(a, r') - r$ and the proof is complete. ■

Theorem 10. *Consider any competitive price rule $\Pi \in C^+$ and let $(r', a'(.), \sigma(.))$ be an SPE of $\Gamma(\Pi, f)$. If $r' \geq r$ and $r'_k = r_k$ for every s_k unmatched at $f(a'(r'), r', \sigma(r'))$, then $V(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = p^*(a, r) - r$ and $f(a'(r'), r', \sigma(r'))$ is optimal for $M(a, r)$.*

Proof. Denote $x' \equiv f(a'(r'), r', \sigma(r'))$. By Theorem 3 we know that x' is optimal for $M(a, r')$. We claim that $(p^*(a, r'), x')$ is a competitive equilibrium for $M(a, r)$. First, $p^*(a, r') \geq r' \geq r$, so $p^*(a, r')$ is feasible for $M(a, r)$. Second, the competitiveness of $p^*(a, r')$ in $M(a, r')$ implies its competitiveness in $M(a, r)$. Finally, every unsold object at x' gets its reservation price: $r'_k = r_k$. Then, x' is also optimal for $M(a, r)$.

We notice that on one hand $p^*(a, r') \leq p^*(a, r)$ by the maximality of p^* in $M(a, r)$. On the other hand, Proposition 5 implies that $p^*(a, r') \geq p^*(a, r)$ because $r' \geq r$. Hence, $p^*(a, r') = p^*(a, r)$. Now use Theorem 10 to obtain that $\Pi(a'(r'), r') = p^*(a, r') = p^*(a, r)$, and then $V(\Pi(a'(r'), r'), x') = p^*(a, r) - r$. ■

Our final example shows that if seller s_k is unmatched at some SPE outcome and $r'_k > r_k$, the conclusion of Theorem 10 is not always true. As we see in the example below, by playing $r'_k > r_k$ he might make some matched agent better off without making himself worse off.

Example 5 (***(r', a'(.), σ(.)) is an SPE of Γ(Π, f) with Π ∈ C⁺ but V(a'(r'), f(r', a'(.), σ(.)) > p*(a, r) − r; Π(a'(r'), r') is a competitive price for M(a, r') but it is not competitive for M(a, r)***). Consider $B = (b_1, b_2, b_3)$, $S = (s_1, s_2, s_3, s_4)$, $a_1 = (5, 4, 0, 1/2)$, $a_2 = (0, 5, 0, 0)$ and $a_3 = (4, 10, 5, 4.5)$, $r = (0, 0, 0, 0)$. Then, $p^* = (4.5, 5, 0, 0)$ and $p_* = (0, 5, 0, 0)$. Consider also the competitive price rule $\Pi \in C^+$ and let $r' = (0, 0, 0, 1/2)$. The optimal matchings for $M(a, r')$ are x and x', where $x_{11} = x_{22} = x_{33} = 1$ and $x'_{11} = x'_{23} = x_{32} = 1$.

Define (a', σ) as follows: $a'(r')_1 = a'(r')_2 = (5, 5, 0, 1/2)$, $a'(r')_3 = (4, 10, 5, 5)$, $\sigma(r')_{jk} = 1$ if $x_{jk} = 1$ or $x'_{jk} = 1$ and $\sigma(r')_{jk} = 0$ otherwise. For $r'' \neq r'$, let $(a'(r''), \sigma(r''))$ be the NE given by Theorem 5 by making $u^* = u^*(r'')$. We will show that $(r', a'(.), \sigma(.))$ is an SPE but $V(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = (5, 5, 0, 0) > p^* - r$.

First notice that the optimal matchings for $M(a'(r'), r')$ signalized by $\sigma(r')$ are x and x'. Both matchings are super-optimal for $M(a'(r'), r')$ and $M(a, r')$.

It is a matter of verification that in the market $M(a'(r'), r')$ we have that $p^*(a'(r'), r') = p_*(a'(r'), r') = (5, 5, 0, 1/2)$, so seller s_4 is unmatched at any optimal matching chosen by $f(a'(r'), r', \sigma(r'))$, $V(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = (5, 5, 0, 0)$ and $U(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r'))) = (0, 0, 5)$. To prove that $(a'(r'), \sigma(r'))$ is an NE for the buyers suppose, by way of contradiction, that some buyer b_j profits by choosing $(a''_j \neq a'_j(r'), \sigma'_j)$ when the other buyers keep their strategies. Denote $a'' \equiv (a''_j, a'(r')_{-j})$, $\sigma' \equiv (\sigma'_j, \sigma(r')_{-j})$ and $y \equiv f(a'', r', \sigma')$. If $b_j = b_1$, then b_1 must obtain a true payoff greater than zero by selecting (a''_1, σ'_1). This implies that b_1 is not matched to s_3 at y because $a_{13} = 0$ and b_1 is not matched to s_4 at y either because $a_{14} = 1/2 = r'_4 \leq \Pi(a'', r')_4$, so b_1 would not have a positive true payoff. If $y_{11} = 1$, the matching x defined above is optimal for $M(a'', r')$. Then, $10 - p(a'')_2 = a''_{32} - p(a'')_2 \leq a''_{33} - p(a'')_3 \leq a''_{33} = 5$ by the competitiveness *of* $p(a'')$, and so $p(a'')_2 \geq 5$. On the other hand, $x_{22} = 1$, so $p(a'')_2 \leq 5$. Then, $p(a'')_2 = 5$, so $a''_{21} - p(a'')_1 \leq a''_{22} - p(a'')_2 = 0$, so $p(a'')_1 \geq a''_{21} = 5$ by the competitiveness of $p(a'')$. Therefore, $U(\Pi(a'', r'), y, \sigma'(r')))_1 \leq 0$. If $y_{12} = 1$, then $y_{21} = y_{33} = 1$ and, by arguing as before, we get that $p(a'')_2 \geq 5$. Consequently, $U(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r')))_1 \leq 0$. Hence, b_1

cannot profit by deviating from a''_1, so $b_j \neq b_1$. With analogous arguments we can see that b_2 cannot profit by deviating from a''_2. Therefore, $b_j = b_3$, and so $U(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r')))_3 > 5$. This implies that b_3 must be matched to s_2 because otherwise her true payoff would be less than or equal to 5. In this case, the matching x' is optimal for $M(a'', r')$, so $u(a'')_2 = 0$ and $p(a'')_3 = 0$, and so $a'(r')_{22} - p(a'')_2 \leq u(a'')_2$ by the competitiveness of $p(a'')$. Then, $5 - p(a'')_2 \leq 0$, so $p(a'')_2 \geq 5$, and $U(\Pi(a'(r'), r'), f(a'(r'), r', \sigma(r')))_3 \leq 5$, which is a contradiction. Hence, no buyer can profit by deviating from her strategy and then $(a'(r'), \sigma(r'))$ is an NE for the buyers.

Now, we have to show that no seller s_k has an incentive to deviate from r'_k. Suppose, by way of contradiction, that there is some seller s_k who is better off by choosing r''_k. Denote $r'' \equiv (r''_k, r'_{-k})$, with $r''_k > 0$ if $k \neq 4$ and $r''_4 \neq 1/2$. Since $(a'(r''), \sigma(r''))$ is an NE and $\Pi \in \mathrm{C}^+$, it follows from Theorem 3 that $(\Pi(a'(r''), r''), f(a'(r''), r'', \sigma(r'')))$ is a competitive equilibrium allocation for $M(a, r'')$. Clearly $s_k \neq s_1$ (seller s_1 does not have any incentive to deviate from r'_1 because he is already receiving the highest possible payoff). Suppose that $s_k = s_2$. To have $V(\Pi(a'(r''), r''), f(a'(r''), r'', \sigma(r''))) > 5$, s_2 should be matched to b_3 under $z \equiv f(a'(r''), r'', \sigma(r''))$, which is an optimal matching for $M(a, r'')$. In this case s_3 cannot be matched to b_3, so $\Pi(a'(r''), r'')_3 = 0$. Therefore, by the competitiveness of $(\Pi(a'(r''), z)$ in $M(a, r'')$, $a_{32} - \Pi(a'(r''), r'')_2 \geq a_{33} - \Pi(a'(r''), r'')_3 = 5$, from which follows that $\Pi(a'(r''), r'')_2 \leq 10 - 5 = 5$. Then, $V(\Pi(a'(r''), r''), f(a'(r''), r'', \sigma(r'')))_2 = \Pi(a'(r''), r'')_2 - r_2 \leq 5$, which is a contradiction. If $s_k = s_3$, s_3 is not acceptable for b_1 and b_2 in $M(a'(r''), r'')$, since $a_{13} = a_{23} = 0 < r''_3$. Thus, in order to have $V(\Pi(a'(r''), r''), f(a'(r''), r'', \sigma(r'')))_3 > 0$, s_3 should be matched to b_3 at $f(a'(r''), r'', \sigma(r''))$, which is an optimal matching for $M(a, r'')$ by Theorem 3. The value of $f(a'(r''), r''' \sigma(r''))$ in $M(a, r'')$ is $15 - r''_3$. However, the value of the matching z' in $M(a, r'')$, where $z'_{11} = z'_{32} = 1$, is $\underline{a}(r'')_{11} + \underline{a}(r'')_{32} = 15 > 15 - r''_3$, which contradicts the optimality of $f(a'(r''), r'', \sigma(r''))$. Hence, s_3 cannot profit by deviating from r''_3. It remains to verify the case $s_k = s_4$. In order to have $V(\Pi(a'(r''), r''), f(a'(r''), r'', \sigma(r''))_4 > 0$, s_4 should be matched to some buyer at $f(a'(r''), r'', \sigma(r''))$, which is not possible because s_4 is unmatched at any optimal matching for $M(a, r'')$. Hence, we have proved that $(r', a'(.), \sigma(.))$ is an SPE of $\Gamma(\Pi, f)$.

It is a matter of verification that $p_*(r) = (0, 5, 0, 1/2)$. Therefore, this example also illustrates that $M(a, r)$ may have more than one competitive equilibrium. ■

7. CONCLUSION

We have analyzed the subgame perfect equilibria (SPE) of centralized mechanisms for the buyer-seller game. In any such mechanism, each seller is asked to report the valuation of his object and then buyers are requested to report their valuations for all the objects (together with a signal to break ties). Given the reports, a competitive price rule selects a competitive equilibrium vector of prices (for the reported market) and a competitive matching rule provides a competitive equilibrium matching, that is, an optimal assignment of objects to buyers.

At equilibrium, buyers and/or sellers often have an incentive not to report their valuations truthfully, as we know from previous literature. Still, we have shown that the SPE strategies typically lead to outcomes that are not only competitive equilibria for the reported economy but also for the true economy. While buyers' SPE strategies may lead to the minimum competitive equilibrium prices, sellers profit from the first-mover advantage and, under reasonable conditions, all the SPE outcomes select the maximum competitive equilibrium prices for the true economy.

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