# The Linear Systems Approach to Linear Rational Expectations Models

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# The Linear Systems Approach to Linear Rational Expectations Models*

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**Abstract**

This paper considers linear rational expectations models from the linear systems point of view. Using a generalization of the Wiener-Hopf factorization, the linear systems approach is able to furnish very simple conditions for existence and uniqueness of both particular and generic linear rational expectations models. To illustrate the applicability of this approach, the paper characterizes the structure of stationary and cointegrated solutions, including a generalization of Granger's representation theorem.

JEL Classification: C32, C51, C62, C63, C65.

Keywords: Rational expectations, linear systems, Wiener-Hopf factorization, vector autoregressive processes, stationarity, cointegration.

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# 1 Introduction

The linear rational expectations model (LREM) is the hallmark of modern macroeconomics and finance. The distinct feature of LREMs is that unlike classical linear systems, where the state of the system depends only on past and present values of the state and an exogenous process, the state in LREMs additionally depends on information used to formulate expectations about the future of the state. The objective of this paper is to situate the theory of LREMs within the framework of linear systems theory. It will be seen that, in addition to providing firm mathematical foundations for LREMs, linear systems theory provides a wide array of methods for tackling problems in LREM theory including existence, uniqueness, and the structure of stationary and cointegrated solutions.

To be sure, linear system theory has had important applications in a number of studies in the LREM literature. Examples include Wiener-Kolmogorov prediction, which appears in Hansen & Sargent (1980, 1981) and Whiteman (1983), and the Smith canonical form, which features in Whiteman (1983), Broze et al. (1995), Funovits (2014), and Tan & Walker (2015). However, this paper makes a forceful point that the most appropriate linear systems approach to LREM analysis is through a generalization of Wiener-Hopf factorization (WHF). WHF has had applications in filtration (Anderson & Moore, 1979), stability analysis (Desoer & Vidyasagar, 2009), and optimal control (Youla, Bongiorno & Jabr, 1976; Youla, Jabr & Bongiorno, 1976), among many other areas in linear systems theory, and has been used by Onatski (2006) to obtain conditions for existence and uniqueness of stable solutions to LREMs, both particular and generic.[1] This factorization takes as inputs a suitably well-behaved matrix function (e.g. a matrix of rational functions) and a suitably well-behaved contour (e.g. the unit circle). Existence results for WHF generally require the matrix function to be bounded and non-singular on the contour (Gohberg & Krein, 1960; Gohberg & Fel'dman, 1974; Gohberg et al., 2003). This implies that WHF cannot be employed in the context of unit roots and, therefore, cannot be applied to a wide variety of macroeconomic and financial models. This paper proposes a generalization whereby one takes the WHF with respect to an infinitesimally smaller contour that avoids any zeros and poles on the contour of interest. This factorization is termed an Inner-Limit Wiener-Hopf Factorization (ILWHF) and is shown to exist even when the matrix function is unbounded and/or singular on the contour of interest.

[1] A precursor to Onatski's paper in economics is the paper by Whiteman (1985), which uses WHF to solve an optimal control problem.

With this generalization in hand, the paper proceeds to provide existence and uniqueness results for both particular and generic LREMs, generalizing the results of Onatski (2006). The approach is closest in scope and generality to Sims (2002) in that it allows for stationary as well as non-stationary solutions and explosive solutions with heterogeneous growth rates. However, the paper takes great pains to rigorously define the solution space, the solution concept, as well as existence and uniqueness. It is demonstrated that the linear systems approach yields the simplest and most direct solution to LREMs in the literature. Moreover, the approach clarifies a number of ambiguities concerning non-uniqueness and the role played by information.

In order to demonstrate the power of the linear systems approach to LREMs, the paper describes the structure of LREM solutions under typical empirical assumptions. First, the paper describes the implications of rational expectations for the correlation structure of unique stationary solutions, extending classical results surveyed in Reinsel (2003). Then the paper considers the implications for cointegration, providing conditions for the existence of cointegration as well as a representation theorem that generalizes Granger's representation theorem (Engle & Granger, 1987) to LREMs. The results generalize the treatments given in Broze et al. (1990), Binder & Pesaran (1995), and Juselius (2008). Importantly, these applications would have been prohibitively difficult to undertake under any other framework for analysing LREMs.

The paper is organized as follows. Section 2 introduces the LREM and motivating examples used throughout the paper. Section 3 introduces the ILWHF and develops its properties. Section 4 discusses existence and uniqueness of LREM solutions. Section 5 discusses the implications of the linear systems approach for empirical models. Section 6 is the conclusion to the paper. Section 7 provides the proofs of the results. The online supplementary material at Cambridge Journals Online (`journals.cambridge.org/ect`) provides additional results and proofs.

## 2 Linear Rational Expectations Models

LREMs describe the behaviour of economic entities (e.g. households and firms) in response to observed and expected values of endogenous variables (e.g. prices and production levels) as well as exogenous variables (e.g. government policy and technology). These relationships are encoded into a formal LREM as

$$M_{-q}E_tX_{t+q} + \cdots + M_{-1}E_tX_{t+1} + M_0X_t + M_1X_{t-1} + \cdots + M_pX_{t-p} = \varepsilon_t, \quad t \geq 0. \qquad (1)$$

Equation (1) is to be understood as a relationship between the vector $X_t$, its past values $(X_{t-1}, \ldots, X_{t-p})$, its expected values $(E_tX_{t+1}, \ldots, E_tX_{t+q})$, and exogenous variables $\varepsilon_t$ for each $t \geq 0$.[2] It is considered formal because we have not yet defined existence, uniqueness, or even the meaning of the expected values. To each formal LREM of the form (1) we will associate a Laurent polynomial $M(z) = \sum_{i=-q}^{p} M_i z^i$.

An important subclass of (1) is the class of linear (or linearised) dynamic stochastic general equilibrium models, where the structural equations are obtained from an underlying dynamic optimization problem. Another important subclass is the set of models with $M_i = 0$ for $i < 0$, i.e. the set of structural VAR processes. Since $\varepsilon$ can itself have a moving average representation, it also includes the set of all structural VARMA processes.

Our task in this paper will be to provide a framework based on linear system theory for the analysis of LREMs of the form (1). In order to do that, we will refer to the following classical examples for illustration.

**Example 2.1.** A variant of the Cagan (1956) model relates the logarithm of the price level, $X$, to its expected value one period ahead and the money supply, $\varepsilon$, according to

$$aE_tX_{t+1} + bX_t = \varepsilon_t, \qquad t \geq 0.$$

Here, $M(z) = az^{-1} + b$. □

**Example 2.2.** The first-order autoregressive model,

$$bX_t + cX_{t-1} = \varepsilon_t, \qquad t \geq 0,$$

is also nested in the class of LREMs. Here $M(z) = b + cz$. □

**Example 2.3.** In the Hansen & Sargent (1980) model, the optimal level of employment of a factor of production, $X$, is related to exogenous economic forces, $\varepsilon$, by the LREM,

$$aE_tX_{t+1} + bX_t + cX_{t-1} = \varepsilon_t, \qquad t \geq 0.$$

Here, $M(z) = az^{-1} + b + cz$. □

[2] Lagged expectations (i.e. terms of the form $E_{t-i}X_{t-i+j}$ for $i, j \geq 1$) are easily fit into this framework by expanding the state (Binder & Pesaran, 1995).

**Example 2.4.** A variant of the Hall (1978) model has consumption, $X_1$, and bond holdings, $X_2$, determined by income, $\varepsilon_2$, according to the system

$$\begin{aligned} X_{1,t} &= E_t X_{1,t+1}, \\ X_{1,t} + X_{2,t} &= R X_{2,t-1} + \varepsilon_{2,t}, \end{aligned} \qquad t \geq 0.$$

Here, $M(z) = \left[\begin{smallmatrix} z^{-1}-1 & 0 \\ 1 & 1-Rz \end{smallmatrix}\right]$. □

Onatski (2006) was the first to notice the resemblance of LREMs to Wiener-Hopf equations (Gohberg & Fel'dman, 1974). He found that the principal technique in this literature, the WHF, could also be used to solve LREMs. However, a WHF of $M(z)$ exists if and only if it is bounded and non-singular on the unit circle. This condition is easily violated in Examples 2.1-2.3 if $M(z)$ is zero for any $z$ on the unit circle. It is also violated in Example 2.4 because $M(1) = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1-R \end{smallmatrix}\right]$ is singular. Thus, in order to generalize Onatski's method to allow for unit roots, the WHF must be generalized. This is taken up in the next section.

# 3 The Inner-Limit Wiener-Hopf Factorization

Linear system theory relies on a number of key factorizations (e.g. the Hermite form and the Smith-McMillan form (Hannan & Deistler, 2012)). The most natural one for LREMs, however, is the ILWHF. Here we develop its properties and its relationship to WHF.

**Definition 3.1.** $\mathbb{R}[z]$ is the set of polynomials in $z$ with real coefficients. $\mathbb{R}^{n\times m}[z]$ is the set of $n \times m$ matrices whose elements are in $\mathbb{R}[z]$. For $M(z) \in \mathbb{R}^{n\times m}[z]$, $\deg(M(z))$ is the highest power of $z$ that appears in $M(z)$. $M(z) \in \mathbb{R}^{n\times n}[z]$ is said to be unimodular if $\det(M(z))$ is a non-zero constant. $\mathbb{R}(z)$ is the set of ratios of elements of $\mathbb{R}[z]$ with no common factors. $\mathbb{R}^{n\times m}(z)$ is the set of $n \times m$ matrices whose elements are in $\mathbb{R}(z)$. $M(z) \in \mathbb{R}^{n\times n}(z)$ is said to be non-singular if $\det(M(z))$ is not identically zero. For non-negative integers $p$ and $q$, the set of Laurent matrix polynomials, $M(z) = \sum_{i=-q}^{p} M_i z^i \in \mathbb{R}^{n\times n}(z)$, is denoted by $\mathbb{R}^{n\times n}_{pq}(z)$. □

Recall that a ratio of two polynomials with no common factors of degrees $k$ and $m$ respectively has $k$ zeros and $m$ poles in $\mathbb{C}$; if $k > m$ it has a pole at infinity; and if $k < m$ it has a zero at infinity (Ahlfors, 1979, Section 2.1.4). We will need to define zeros and poles for non-singular rational matrix functions. In that case, we will rely on the following definition.

**Definition 3.2.** Let $M(z) \in \mathbb{R}^{n\times n}(z)$ be non-singular and $z_0 \in \mathbb{C} \cup \{\infty\}$. We say that $M(z)$

has a pole at $z_0$ if some element of $M(z)$ has a pole at $z_0$. We say that $M(z)$ has a zero at $z_0$ if $M^{-1}(z)$ has a pole at $z_0$. □

It is easily shown that this definition is equivalent to the standard convention in the linear systems literature (Kailath, 1980, Section 6.5.3), which applies more generally to non-square and possibly singular rational matrix functions. A useful rule that follows from the definition is that for a non-singular $M(z) \in \mathbb{R}^{n\times n}(z)$, if $z_0 \in \mathbb{C} \cup \{\infty\}$ is not a pole, then it is a zero if and only if $\det(M(z_0)) = 0$. In particular, if $M(z) \in \mathbb{R}^{n\times n}[z]$ is non-singular, we have the familiar result that $z_0 \in \mathbb{C}$ is a zero if and only if $\det(M(z_0)) = 0$.

**Example 3.1.** $M_1(z) = \begin{bmatrix} 1 & \frac{z^2}{z-1} \\ \frac{1}{z} & 1 \end{bmatrix}$ has poles at $\{0, 1, \infty\}$ and, since $M_1^{-1}(z) = \begin{bmatrix} 1-z & z^2 \\ \frac{z-1}{z} & 1-z \end{bmatrix}$, $M_1(z)$ has zeros at $\{0, \infty\}$. On the other hand, $M_2(z) = \begin{bmatrix} 1 & z^2 \\ 0 & z \end{bmatrix}$ has a pole at $\infty$ and, since $M_2^{-1}(z) = \begin{bmatrix} 1 & -z \\ 0 & \frac{1}{z} \end{bmatrix}$, $M_2(z)$ has zeros at $\{0, \infty\}$; the finite zero can also be read from $\det(M_2(z)) = z$. □

Given the definitions above, we can proceed to the basic mathematical ideas that drive all of the results of this paper.

**Definition 3.3.** Let $M(z) \in \mathbb{R}^{n\times n}(z)$ be non-singular and $\rho > 0$. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $\mathbb{D}^c = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$. $M(z) = M_f(z)M_0(z)M_b(z)$ is a Wiener-Hopf factorization (WHF) relative to $\rho\mathbb{T}$ if

(i) $M_f(z) \in \mathbb{R}^{n\times n}(z)$ has no zeros or poles in $\rho\mathbb{D}^c$.

(ii) $M_0(z) = \mathrm{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n})$, where $\kappa_1 \geq \cdots \geq \kappa_n$ are integers.[3]

(iii) $M_b(z) \in \mathbb{R}^{n\times n}(z)$ has no zeros or poles in $\rho\overline{\mathbb{D}}$.

If (iii) is weakened to:

(iii)′ $M_b(z) \in \mathbb{R}^{n\times n}(z)$ has no zeros or poles in $\rho\mathbb{D}$.

we obtain an Inner-Limit Wiener-Hopf factorization (ILWHF) of $M(z)$ relative to $\rho\mathbb{T}$. In either case, we refer to $M_f(z)$, $M_0(z)$, and $M_b(z)$ as the forward, null, and backward components of $M(z)$ respectively, while the integers $\{\kappa_1, \ldots, \kappa_n\}$ are called partial indices.[4] □

Clearly, every WHF relative to $\rho\mathbb{T}$ is an ILWHF relative to $\rho\mathbb{T}$ but the reverse inclusion does not hold as we will see shortly. For a non-singular $M(z) \in \mathbb{R}^{n\times n}(z)$, a WHF relative to

[3] $\mathrm{diag}(a_1, \ldots, a_n)$ is the $n \times n$ matrix whose $ij$-th element is $a_i$ if $i = j$ and 0 if $i \neq j$.

[4] The WHF literature uses the non-mnemonic notation $M_-(z)$ and $M_+(z)$ for $M_f(z)$ and $M_b(z)$ respectively. The reason for our change of notation will become apparent in the next section.

$\rho\mathbb{T}$ exists if and only if it has no zeros or poles on $\rho\mathbb{T}$ (Gohberg et al., 2003, Theorem 1.6). We obtain conditions for existence of ILWHF below.

It is important to note that, like the WHF, the ILWHF can be defined for a non-singular rational matrix function relative to the more general class of curves in $\mathbb{C}$ that are homeomorphic to $\mathbb{T}$ (Gohberg et al., 2003, p. 3). However, we restrict attention to the class of curves $\rho\mathbb{T}$ because: (i) most of this paper only requires factorization relative to $\mathbb{T}$, (ii) finding exponentially growing solutions to LREMs requires factorizing relative to $\rho\mathbb{T}$ with $\rho < 1$, and (iii) allowing $\rho$ to vary makes it possible to understand the relationship between ILWHF and WHF.

**Example 3.2.** Consider $M(z)$ from Example 2.1 with $ab \neq 0$. We can find the following ILWHFs relative to $\rho\mathbb{T}$,

$$\begin{array}{llll} M_f(z) = az^{-1} + b & M_0(z) = 1, & M_b(z) = 1, & \text{if } \rho > |a/b| \\ M_f(z) = 1, & M_0(z) = z^{-1}, & M_b(z) = a + bz, & \text{if } \rho \leq |a/b|. \end{array}$$

If $\rho = |a/b|$, $M(z)$ has no WHF relative to $\rho\mathbb{T}$. □

**Example 3.3.** Consider $M(z)$ from Example 2.2 with $bc \neq 0$. We can find the following ILWHFs relative to $\rho\mathbb{T}$,

$$\begin{array}{llll} M_f(z) = bz^{-1} + c & M_0(z) = z, & M_b(z) = 1, & \text{if } \rho > |b/c| \\ M_f(z) = 1, & M_0(z) = 1, & M_b(z) = b + cz, & \text{if } \rho \leq |b/c|. \end{array}$$

If $\rho = |b/c|$, $M(z)$ has no WHF relative to $\rho\mathbb{T}$. □

**Example 3.4.** Consider $M(z)$ from Example 2.3 with $ac \neq 0$, and write it as $M(z) = cz^{-1}(z - \zeta_1)(z - \zeta_2)$. Then we can find the following ILWHFs relative to $\rho\mathbb{T}$.

$$\begin{array}{llll} M_f(z) = 1, & M_0(z) = z^{-1}, & M_b(z) = a + bz + bz^2, & \text{if } \rho \leq |\zeta_1|, |\zeta_2| \\ M_f(z) = 1 - \zeta_1 z^{-1}, & M_0(z) = 1, & M_b(z) = c(z - \zeta_2), & \text{if } |\zeta_1| < \rho \leq |\zeta_2| \\ M_f(z) = az^{-2} + bz^{-1} + c, & M_0(z) = z, & M_b(z) = 1, & \text{if } |\zeta_1|, |\zeta_2| < \rho. \end{array}$$

If $\rho = |\zeta_1|$ or $\rho = |\zeta_2|$, then $M(z)$ has no WHF relative to $\rho\mathbb{T}$. □

**Example 3.5.** Consider $M(z)$ from Example 2.4. Then we can find the following ILWHFs relative to $\rho\mathbb{T}$

$$\begin{aligned}
M_f(z) &= \begin{bmatrix} 0 & z^{-1}-1 \\ z^{-1}-R & 1 \end{bmatrix}, \quad M_0(z) = \left[\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad M_b(z) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \quad \text{if } |R^{-1}|, 1 < \rho \\
M_f(z) &= \left[\begin{smallmatrix} z^{-1}-1 & R-1 \\ 1 & 1 \end{smallmatrix}\right], \quad M_0(z) = I_2, \quad M_b(z) = \begin{bmatrix} 1 & -(R-1)z \\ 0 & 1-z \end{bmatrix}, \quad \text{if } |R^{-1}| < \rho \le 1 \\
M_f(z) &= \left[\begin{smallmatrix} z^{-1}-1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \quad M_0(z) = I_2, \quad M_b(z) = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1-Rz \end{smallmatrix}\right], \quad \text{if } 1 < \rho \le |R^{-1}| \\
M_f(z) &= \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \quad M_0(z) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & z^{-1} \end{smallmatrix}\right], \quad M_b(z) = \left[\begin{smallmatrix} 1 & 1-Rz \\ 1-z & 0 \end{smallmatrix}\right], \quad \text{if } \rho \le |R^{-1}|, 1.
\end{aligned}$$

It is easily checked that the last two factorizations are the relevant ones when $R = 0$. $M(z)$ has no WHF whenever $\rho = 1$ or $\rho = |R^{-1}|$. $\square$

Our discussion so far suggests that ILWHF is a strict generalization of WHF. However, a more accurate characterization of the relationship between the two is given in the next result, which also explains where the "inner-limit" part of ILWHF comes from.

**Proposition 3.1.** Let $M(z) \in \mathbb{R}^{n \times n}(z)$ be non-singular and $\rho > 0$. Let $r\mathbb{T} \subset \rho\mathbb{D}$ encircle all the zeros and poles of $M(z)$ that are in $\rho\mathbb{D}$ and set $N(z) = M((r/\rho)z)$.

(i) $M(z) = M_f(z)M_0(z)M_b(z)$ is a WHF relative to $r\mathbb{T}$ if and only if it is also an ILWHF relative to $\rho\mathbb{T}$.

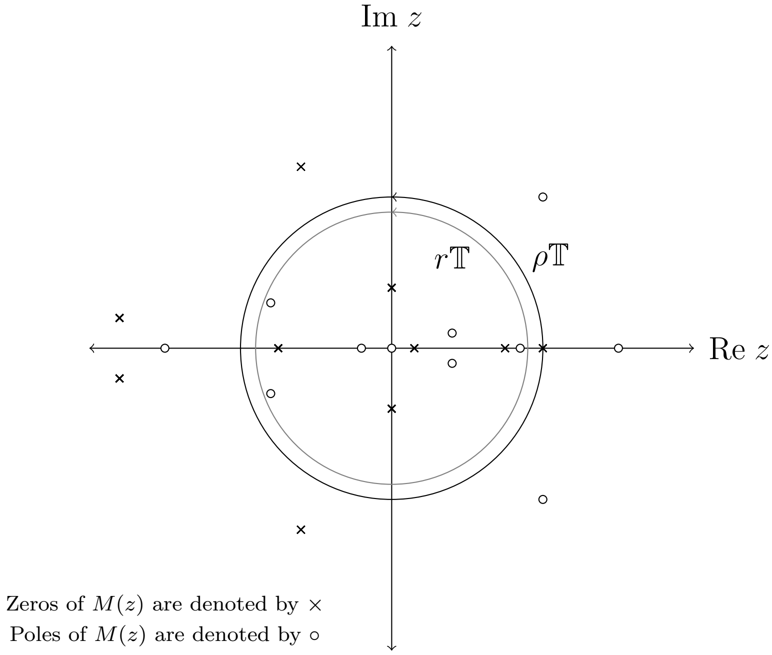
(ii) If $N(z) = N_f(z)N_0(z)N_b(z)$ is a WHF relative to $\rho\mathbb{T}$, then with $M_f(z) = N_f((\rho/r)z)N_0(\rho/r)$, $M_0(z) = N_0(z)$, and $M_b(z) = N_b((\rho/r)z)$, $M(z) = M_f(z)M_0(z)M_b(z)$ is an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$. Conversely, if $M(z) = M_f(z)M_0(z)M_b(z)$ is an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$, then with $N_f(z) = M_f((r/\rho)z)M_0(r/\rho)$, $N_0(z) = M_0(z)$, and $N_b(z) = M_b((r/\rho)z)$, $N(z) = N_f(z)N_0(z)N_b(z)$ is a WHF relative to $\rho\mathbb{T}$.

It follows from Proposition 3.1 (i) that, given any sequence $\rho_k \uparrow \rho$, for $k$ large enough $\rho_k$ eventually exceeds $r$ and then a WHF relative $\rho_k\mathbb{T}$ is an ILWHF relative to $\rho\mathbb{T}$; hence the "inner-limit" part of ILWHF.[5] Proposition 3.1 (ii) exploits the geometry of $\mathbb{T}$ to obtain an alternative derivation that amounts to a preliminary stretching of the complex plane that pushes any zeros or poles on $\rho\mathbb{T}$ outwards without letting any zeros or poles out of $\rho\mathbb{D}$, then obtaining the WHF relative to $\rho\mathbb{T}$, then contracting the complex plane to undo the effect of stretching. The $r$ that appears in Proposition 3.1 is illustrated in Figure 1.

Proposition 3.1 implies that if one knows how to compute WHFs, then one can automatically compute ILWHF by a judicious contraction of the contour of interest or expansion of the

[5]The generalization of this result from $\rho\mathbb{T}$ to the class of curves in $\mathbb{C}$ homeomorphic to $\mathbb{T}$ is a straightforward application of the Jordan-Schoenflies Theorem.

Figure 1: Contour of the Inner-Limit Wiener-Hopf Factorization.




complex plane. Because every WHF is an ILWHF, Proposition 3.1 is best illustrated when a WHF fails to exist relative to the contour of interest.

**Example 3.6.** Consider Example 3.4 when either $\rho = |\zeta_1|$ or $\rho = |\zeta_2|$ so a WHF relative to $\rho\mathbb{T}$ fails to exist. There are two cases to consider:

(i) If $0 < \rho = |\zeta_1| \leq |\zeta_2|$, then Proposition 3.1 (i) implies that any WHF relative to $r\mathbb{T}$ for $0 < r < \rho$ is also an ILWHF relative to $\rho\mathbb{T}$. For example, $M_f(z) = 1$, $M_0(z) = z^{-1}$, $M_b(z) = a + bz + cz^2$ is a WHF relative to $r\mathbb{T}$ and an ILWHF relative to $\rho\mathbb{T}$.

For $0 < r < \rho$, $N(z) = M((r/\rho)z)$ has a WHF relative to $\rho\mathbb{T}$, which can be used to construct an ILWHF for $M(z)$ relative to $\rho\mathbb{T}$. For example, a WHF of $N(z)$ relative to $\rho\mathbb{T}$ is given by $N_f(z) = \rho/r$, $N_0(z) = z^{-1}$, $N_b(z) = a + b(r/\rho)z + c(r/\rho)^2 z^2$; then using the formulas in Proposition 3.1 (ii), an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ is found as $M_f(z) = 1$, $M_0(z) = z^{-1}$, $M_b(z) = a + bz + cz^2$.

(ii) If $|\zeta_1| < \rho = |\zeta_2|$, then Proposition 3.1 (i) implies that any WHF relative to $r\mathbb{T}$ for $|\zeta_1| < r < \rho$ is also an ILWHF relative to $\rho\mathbb{T}$. For example, $M_f(z) = 1 - \zeta_1 z^{-1}$, $M_0(z) = 1$, $M_b(z) = c(z - \zeta_2)$ is a WHF relative to $r\mathbb{T}$ and an ILWHF relative to $\rho\mathbb{T}$. Any WHF relative to $r\mathbb{T}$ with $0 < r < |\zeta_1|$ cannot be an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ because the WHF's backward component would have a zero at $\zeta_1 \in \rho\mathbb{D}$. For example, a WHF relative to $r\mathbb{T}$ with $0 < r < |\zeta_1|$ is given by $M_f(z) = 1$, $M_0(z) = z^{-1}$,

$M_b(z) = a + bz + cz^2$ and this cannot be an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ because $M_b(z)$ has a zero at $\zeta_1 \in \rho\mathbb{D}$.

For $|\zeta_1| < r < \rho$, $N(z) = M((r/\rho)z)$ has a WHF relative to $\rho\mathbb{T}$, which can be used to construct an ILWHF for $M(z)$ relative to $\rho\mathbb{T}$. For example, a WHF of $N(z)$ relative to $\rho\mathbb{T}$ is given by $N_f(z) = (1 - \zeta_1(\rho/r)z^{-1})$, $N_0(z) = 1$, $N_b(z) = c(r/\rho)(z - \zeta_2(\rho/r))$; then using the formulas in Proposition 3.1 (ii), an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ is found as $M_f(z) = 1 - \zeta_1 z^{-1}$, $M_0(z) = 1$, $M_b(z) = c(z - \zeta_2)$. Employing Proposition 3.1 (ii) with $0 < r < |\zeta_1|$ does not produce an ILWHF for $M(z)$ relative to $\rho\mathbb{T}$ because the backward component in the WHF of $N(z)$ would have a zero at $(\rho/r)\zeta_1$, which would translate to a zero of $M_b(z)$ at $\zeta_1 \in \rho\mathbb{D}$. For example, a WHF of $N(z)$ relative to $\rho\mathbb{T}$ is given by $N_f(z) = \rho/r$, $N_0(z) = z^{-1}$, $N_b(z) = a + b(r/\rho)z + c(r/\rho)^2 z^2$ and using the formulas in Proposition 3.1 (ii), would suggest an ILWHF of $M(z)$ relative to $\rho\mathbb{T}$ with $M_f(z) = 1$, $M_0(z) = z^{-1}$, $M_b(z) = a + bz + cz^2$. But this is incorrect because $M_b(z)$ has a zero at $\zeta_1 \in \rho\mathbb{D}$. □

The most important application of Proposition 3.1 is in allowing us to derive existence and uniqueness results for ILWHF from the analogous results for WHF.

**Theorem 3.1.** Let $M(z) \in \mathbb{R}^{n\times n}(z)$ be non-singular and $\rho > 0$.

(i) There exists an ILWHF $M(z) = M_f(z)M_0(z)M_b(z)$ relative to $\rho\mathbb{T}$.

(ii) The partial indices of $M(z)$ in any ILWHF relative to $\rho\mathbb{T}$ are unique.

Theorem 3.1 states that ILWHFs exist under the weakest possible assumptions and the partial indices are uniquely defined. Section A of the online supplement provides more details on existence and uniqueness of ILWHF. It emerges, in particular, that the forward and backward components are unique only up to a unimodular transformation of a particular form; when the partial indices are all zero, the forward and backward components are unique up to multiplication by a constant invertible matrix, so we can choose a unique ILWHF by setting $M_f(\infty) = I_n$ or $M_b(0) = I_n$.[6]

The class of LREMs we consider (1) entails factorizing Laurent matrix polynomials. In this case, ILWHFs take a particularly simple form.

**Theorem 3.2.** Let $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ be non-singular and $\rho > 0$, then $M(z)$ has an ILWHF $M_f(z)M_0(z)M_b(z)$ relative to $\rho\mathbb{T}$ if and only if:

[6] $I_n$ is the identity matrix of dimension $n \times n$.

(i) $M_f(z^{-1}) \in \mathbb{R}^{n\times n}[z]$ has no zeros in $\rho^{-1}\overline{\mathbb{D}}$.

(ii) $M_0(z) = \operatorname{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n})$, where $\kappa_1 \geq \cdots \geq \kappa_n$ are integers.

(iii) $M_b(z) \in \mathbb{R}^{n\times n}[z]$ has no zeros in $\rho\mathbb{D}$.

Moreover $p \geq \kappa_1 \geq \cdots \geq \kappa_n \geq -q$, $\deg(M_f(z^{-1})) \leq p+q$, and $\deg(M_b(z)) \leq p+q$. If the partial indices are all zero, then $\deg(M_f(z^{-1})) \leq q$ and $\deg(M_b(z)) \leq p$.

Theorem 3.2 states that, for a non-singular Laurent matrix polynomial, the forward component is a matrix polynomial in $z^{-1}$, while the backward component is a matrix polynomial in $z$.[7] Section C of the online supplement provides an algorithm for computing the ILWHF of a non-singular Laurent matrix polynomial relative to $\rho\mathbb{T}$.

# 4 Existence and Uniqueness of Solutions to LREMs

Having developed the mathematical machinery necessary to study LREMs, we now proceed to the solution of these models. We first derive some preliminary results necessary for the construction of solutions. We then proceed to discuss existence and uniqueness. The role of information is strongly emphasized. Finally, the section closes with a discussion of solutions that exhibit exponential growth.

## 4.1 The Solution Space

Now, in order to discuss existence and uniqueness, it is necessary to restrict the solution space and the space of exogenous processes (Pesaran, 1987, Section 5.3.2).

**Definition 4.1.** Given a probability space $(\Omega, \mathscr{A}, P)$, let $\mathcal{L}^1(\Omega, \mathscr{A}, P)$ be the set of random variables $Z$ defined on $\Omega$ with finite expected values, $E(Z) = \int_\Omega Z(\omega)P(d\omega)$. The set of $n$-dimensional sub-exponential processes, $\mathcal{S}^n(\Omega, \mathscr{A}, P)$, is defined as the set of stochastic processes $X = \left\{X_t = (X_{1t}, \ldots, X_{nt})' : X_{it} \in \mathcal{L}^1(\Omega, \mathscr{A}, P), i = 1, \ldots, n, t \in \mathbb{Z}\right\}$, such that for any $0 \leq \theta < 1$, $\lim_{t\to\infty} \theta^t E\|X_t\| = 0$, where $\|\cdot\|$ is the Euclidean norm. $X, \hat{X} \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ are said to be indistinguishable if $P\left(\hat{X}_t = X_t\right) = 1$ for all $t \in \mathbb{Z}$. When there is no danger of confusion, we will drop the reference to the probability space and simply write $\mathcal{L}^1$ and $\mathcal{S}^n$. $\square$

Our motivation for the class of sub-exponential processes is both empirical and mathematical. Empirically, the set of sub-exponential processes includes large classes of stochastic

[7] Note that $M_f(z^{-1})$ has zeros in $\rho^{-1}\overline{\mathbb{D}} = (\rho\mathbb{D}^c)^{-1}$ if and only if $M_f(z)$ has zeros in $\rho\mathbb{D}^c$.

processes of practical importance such as weakly stationary processes as well as stable linear processes in $\mathcal{L}^1$ and unstable linear processes in $\mathcal{L}^1$ with zeros on $\mathbb{T}$ (see Example 4.2 below). It also includes trigonometric trends, polynomial trends, dummies, and their interactions. Note that it is not possible to relax the condition of membership in $\mathcal{L}^1$ in Definition 4.1 as this is required in order to be able to take conditional expectations (Williams, 1991, Definition 9.2); however, only the first moment is required to exist and so $\mathcal{S}^n$ also includes processes that exhibit heavy tails for example. Exponentially increasing processes such as explosive linear processes are excluded from $\mathcal{S}^n$; however, we discuss solutions to LREMs in this class of processes later in this section.

The mathematical advantage of $\mathcal{S}^n$ is that it is closed under all operations necessary for the study of LREMs. It is easily verified that

$$(X_1', X_2')' = \{(X_{1t}', X_{2t}')' : t \in \mathbb{Z}\} \in \mathcal{S}^{n_1+n_2} \text{ if and only if } X_1 \in \mathcal{S}^{n_1}, X_2 \in \mathcal{S}^{n_2}$$

so that sub-exponential processes can be combined to form other sub-exponential processes or extracted from other sub-exponential processes. Clearly, $\mathcal{S}^n$ is closed under the operations

$$aX_1 = \{aX_{1t} : t \in \mathbb{Z}\}, \qquad X_1 + X_2 = \{X_{1t} + X_{2t} : t \in \mathbb{Z}\},$$

for $a \in \mathbb{R}$ and $X_1, X_2 \in \mathcal{S}^n$. This implies that

$$MX = \{MX_t : t \in \mathbb{Z}\} \in \mathcal{S}^k \text{ if } X \in \mathcal{S}^n \text{ and } M \in \mathbb{R}^{k\times n}.$$

The backward shift operator is also well defined

$$LX = \{LX_t = X_{t-1} : t \in \mathbb{Z}\} \in \mathcal{S}^n \text{ for } X \in \mathcal{S}^n.$$

The operator that results from $p \geq 1$ applications of $L$ is denoted by $L^p$. Analogously, the forward shift operator is well defined

$$L^{-1}X = \{L^{-1}X_t = X_{t+1} : t \in \mathbb{Z}\} \in \mathcal{S}^n \text{ for } X \in \mathcal{S}^n.$$

We denote the operator that results from $q \geq 1$ applications of $L^{-1}$ as $L^{-q}$. The operator $L^0$ will be understood to be the identity map on $\mathcal{S}^n$. It follows that whenever $M(z) = \sum_{i=-q}^{p} M_i z^i \in \mathbb{R}_{pq}^{n\times n}(z)$, then

$$M(L)X = \left\{ M(L)X_t = \sum_{i=-q}^{p} M_i L^i X_t = \sum_{i=-q}^{p} M_i X_{t-i} : t \in \mathbb{Z} \right\} \in \mathcal{S}^n \text{ for } X \in \mathcal{S}^n.$$

Clearly, if $M(z), N(z) \in \mathbb{R}^{n\times n}_{pq}(z)$, then $M(L)(N(L)X) = (M(L)N(L))X$ so associativity holds for these types of operators. We will also be interested in applying certain operators composed of infinite weighted sums of forward shift operators; these operators are studied in the following result.

**Lemma 4.1.** Given a probability space $(\Omega, \mathscr{A}, P)$, let $Y \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$, let $\mathscr{I} = \{\mathscr{I}_t \subset \mathscr{A} : t \in \mathbb{Z}\}$ be a filtration, let $N(z) \in \mathbb{R}^{n\times n}(z)$ be non-singular and have no poles in $\mathbb{D}^c$, and define

$$N(L)Y = \left\{ N(L)Y_t = \sum_{i=0}^{\infty} N_i L^{-i} Y_t = \sum_{i=0}^{\infty} N_i Y_{t+i} : t \in \mathbb{Z} \right\},$$

where $\sum_{i=0}^{\infty} N_i z^{-i}$ is the Laurent series expansion of $N(z)$ in $\mathbb{D}^c$.

(i) $N(L)Y \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$.

(ii) If $M(z) \in \mathbb{R}^{n\times n}(z)$ also has no poles in $\mathbb{D}^c$, then $M(L)(N(L)Y_t) = (M(L)N(L))Y_t$ a.s. for all $t \in \mathbb{Z}$ and $M(L)N(L)Y \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$.

(iii) $\{E(N(L)Y_t|\mathscr{I}_t) : t \in \mathbb{Z}\} \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ and $E(N(L)Y_t|\mathscr{I}_t) = \sum_{i=0}^{\infty} N_i E(Y_{t+i}|\mathscr{I}_t)$ a.s. for all $t \in \mathbb{Z}$.[8]

Lemma 4.1 shows that if $N(z) \in \mathbb{R}^{n\times n}(z)$ is non-singular and has no poles in $\mathbb{D}^c$, then $N(L) = \sum_{i=0}^{\infty} N_i L^{-i}$ is well-defined on $\mathcal{S}^n$, leaves $\mathcal{S}^n$ invariant, is associative with other such operators, and interacts well with the conditional expectations operator. Lemma 4.1 plays a fundamental role in the solution of LREMs as it applies to (inverses of) forward components of ILWHFs relative to $\mathbb{T}$ and determines the dependence of the solution on current and expected values of the exogenous process. The “almost sure” ambiguities that appear in Lemma 4.1 (and in the subsequent analysis) come from two sources: (i) conditional expectations are defined only almost surely and that is “something one has to live with in general” (Williams, 1991, p. 85) and (ii) the asymptotic behaviour of $X \in \mathcal{S}^n$ is determined only in expectation, thus any statement about its asymptotic behaviour can hold at most almost surely.

**Example 4.1.** Consider Example 3.2 with $|a/b| < \rho = 1$, then $M_f^{-1}(z) = \frac{1}{az^{-1}+b}$ has no poles in $\mathbb{D}^c$ and has the Laurent series expansion $\sum_{i=0}^{\infty}(-a/b)^i b^{-1} z^{-i}$ in $\mathbb{D}^c$. Let $\varepsilon \in \mathcal{S}^1$ and suppose $\{\mathscr{I}_t : t \in \mathbb{Z}\}$ is a filtration. Then Lemma 4.1 (i) implies that $M_f^{-1}(L)\varepsilon = \left\{\sum_{i=0}^{\infty}(-a/b)^i b^{-1}\varepsilon_{t+i} : t \in \mathbb{Z}\right\} \in \mathcal{S}^1$. Lemma 4.1 (iii) also implies that $\left\{E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) = \sum_{i=0}^{\infty}(-a/b)^i b^{-1} E(\varepsilon_{t+i}|\mathscr{I}_t) : t \in \mathbb{Z}\right\} \in \mathcal{S}^1$. □

[8] The abbreviation “a.s.” stands for “almost surely.”

The final mathematical advantage of $\mathcal{S}^n$ we will consider is closedness with respect to non-explosive autoregressions driven by elements of $\mathcal{S}^n$ and $\mathcal{L}^1$ initial conditions.

**Lemma 4.2.** Given a probability space $(\Omega, \mathscr{A}, P)$, suppose $Y \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ and the initial conditions $\left\{\tilde{X}_t = (\tilde{X}_{1t}, \ldots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathscr{A}, P), i = 1, \ldots, n, t < 0\right\}$ are given. Let $N(z) \in \mathbb{R}^{n\times n}[z]$ be non-singular and have no zeros in $\mathbb{D}$.

(i) There exists $X \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ that solves

$$\begin{aligned} X_t &= \tilde{X}_t \quad \text{a.s.} \quad t < 0, \\ N(L)X_t &= Y_t \quad \text{a.s.} \quad t \geq 0. \end{aligned}$$

(ii) If $\hat{X} \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ is any other solution, then $X$ and $\hat{X}$ are indistinguishable.

Lemma 4.2 implies that $\mathcal{S}^n$ includes stable linear processes in $\mathcal{L}^1$ as well as unstable linear processes in $\mathcal{L}^1$ with zeros on $\mathbb{T}$ (e.g. unit root and seasonally integrated processes). Lemma 4.2 will play a fundamental role in the solution of LREMs as it will apply to backward components of ILWHFs relative to $\mathbb{T}$ and will determine the dependence of the solution on its past.

**Example 4.2.** Consider Example 3.3 with $|c/b| \leq \rho = 1$, then $M_b(z) = b + cz$ has no zeros in $\mathbb{D}$. Let $\varepsilon \in \mathcal{S}^1$ and suppose $\{\mathscr{I}_t : t \in \mathbb{Z}\}$ is a filtration. Then Lemma 4.2 implies that for any set of initial conditions $\left\{\tilde{X}_t \in \mathcal{L}^1 : t < 0\right\}$, the process $X$ defined by $X_t = \tilde{X}_t$ for $t < 0$ and $X_t = (-c/b)^{t+1}\tilde{X}_{-1} + \sum_{j=0}^{t}(-c/b)^j b^{-1}\varepsilon_{t-j}$ for $t \geq 0$ is an element of $\mathcal{S}^1$. This holds even in the presence of a unit root (i.e. when $|c/b| = 1$). Note that when $|c/b| < 1$ and $\varepsilon$ is weakly stationary, the choice of initial conditions $\left\{\tilde{X}_t = \sum_{j=0}^{\infty}(-c/b)^j b^{-1}\varepsilon_{t-j} : t < 0\right\}$ makes $X$ weakly stationary (Hannan & Deistler, 2012, p. 11). $\square$

## 4.2 Existence and Uniqueness

Having identified the appropriate solution space for our problem, the next order of business is to assign meaning to existence and uniqueness of solutions to (1).

**Definition 4.2.** Let $(\Omega, \mathscr{A}, P)$ be a given probability space. Given $\varepsilon \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$, initial conditions $\left\{\tilde{X}_t = (\tilde{X}_{1t}, \ldots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathscr{A}, P), i = 1, \ldots, n, t < 0\right\}$, and $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$, a solution to (1) is a pair $(X, \mathscr{I})$ such that:

(i) $\mathscr{I} = \{\mathscr{I}_t \subset \mathscr{A} : t \in \mathbb{Z}\}$ is a filtration satisfying $\sigma(\tilde{X}_s : s \leq t) \subseteq \mathscr{I}_t$ for all $t < 0$ and $\varepsilon_t \in \mathrm{m}\mathscr{I}_t$ for all $t \geq 0$.[9]

[9]For a $\sigma$-algebra $\mathscr{F}$, $\mathrm{m}\mathscr{F}$ is the set of finite dimensional random vectors measurable with respect to $\mathscr{F}$. For a

(ii) $X \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ is adapted to $\mathscr{I}$.

(iii) $X_t = \tilde{X}_t$ a.s. for all $t < 0$.

(iv) $E(M(L)X_t|\mathscr{I}_t) = \varepsilon_t$ a.s. for all $t \geq 0$.

For a given filtration, $\mathscr{I}$, a solution $(X, \mathscr{I})$ is said to be unique if for any other solution $(\hat{X}, \mathscr{I})$, $X$ and $\hat{X}$ are indistinguishable. A solution $(X, \mathscr{I})$ is said to be fundamental if $\mathscr{I} = \left\{\mathscr{I}_t = \sigma(\varepsilon_s : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z}\right\}$.[10] □

Similar to martingale theory (Williams, 1991), the solution involves the specification of a filtration.[11] Condition (i) requires the filtration to contain the initial conditions and exogenous variables, i.e. the fundamental economic forces at play. Condition (ii) is a causality condition (see e.g. pp. 4-5 of Hannan & Deistler (2012)) requiring $X$ to be a sub-exponential process determined by the information available at hand. Condition (iii) requires the solution to satisfy whatever initial conditions are specified. Finally, condition (iv) requires the solution to satisfy the structural relationships specified in (1), where the formal terms $E_t X_{t+i}$ are now substituted by $E(X_{t+i}|\mathscr{I}_t)$. Note that the filtration of fundamental solutions is the smallest for which a solution to the LREM may exist.

With the notions of existence and uniqueness made explicit, we are now in a position to derive conditions for existence and uniqueness of solutions to LREMs.

**Theorem 4.1.** Let $(\Omega, \mathscr{A}, P)$ be a given probability space. Given $\varepsilon \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$, initial conditions $\left\{\tilde{X}_t = (\tilde{X}_{1t}, \ldots, \tilde{X}_{nt})' : \tilde{X}_{it} \in \mathcal{L}^1(\Omega, \mathscr{A}, P), i = 1, \ldots, n, t < 0\right\}$, a non-singular $M(z) \in \mathbb{R}^{n \times n}_{pq}(z)$, and filtration $\mathscr{I}$ that satisfies Definition 4.2 (i), if $M(z)$ has an ILWHF relative to $\mathbb{T}$, $M_f(z)M_0(z)M_b(z)$, with partial indices $\kappa_1 \geq \cdots \geq \kappa_n$, then the following holds:

(i) If the partial indices of $M(z)$ are all zero, then there exists a unique solution to (1) $(X, \mathscr{I})$ generated recursively as

$$\begin{aligned} X_t &= \tilde{X}_t, & t &< 0, \\ M_b(L)X_t &= E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t), & t &\geq 0. \end{aligned} \tag{2}$$

(ii) If the partial indices of $M(z)$ are non-positive and $k$ are negative, then for every set of additional initial conditions $\{M_{b,i\cdot}(0)X_t \in \mathrm{m}\mathscr{I}_t : n - k < i \leq n, 0 \leq t < |\kappa_i|\} \subset$

collection of finite dimensional random vectors $\{Z_i, i \in I\}$, $\sigma(Z_i : i \in I)$ is the smallest $\sigma$-algebra with respect to which every $Z_i$ is measurable.

[10] For $\mathscr{F}, \mathscr{G} \subset \mathscr{A}$, $\mathscr{F} \vee \mathscr{G}$ is the $\sigma$-algebra generated by $\mathscr{F} \cup \mathscr{G}$.

[11] A related generalization of martingales is the set of processes known as harnesses (Williams, 1991, Section 15.10).

$\mathcal{L}^1(\Omega, \mathscr{A}, P)$, where $M_{b,i\cdot}(0)$ is the $i$-th row of $M_b(0)$, and every $\nu \in \mathcal{S}^k(\Omega, \mathscr{A}, P)$ adapted to $\mathscr{I}$ and satisfying $E(M_0(L)S\nu_t|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ for $S = \left[\begin{smallmatrix} 0 \\ I_k \end{smallmatrix}\right]$, there exists a solution to (1) $(X, \mathscr{I})$ generated recursively as

$$\begin{aligned} X_t &= \tilde{X}_t, & t &< 0, \\ M_0(L)M_b(L)X_t &= E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) + M_0(L)S\nu_t, & t &\geq 0. \end{aligned} \tag{3}$$

(iii) If any partial index is positive, there exists an exogenous process and/or a set of initial conditions for which there is no solution to (1).

The assumptions of Theorem 4.1 are quite weak relative to the literature. Theorem 4.1 does not require $\varepsilon$ to have a Wold decomposition (Whiteman, 1983; Tan & Walker, 2015), invertibility of $M_0$ (Broze et al., 1985, 1995; Binder & Pesaran, 1995, 1997), or a priori knowledge of the predetermined variables (Blanchard & Kahn, 1980). The result of Onatski (2006) is not nested above because he allows $M(z)$ to be non-rational. However, when restricting attention to Laurent polynomials, the conditions for existence and uniqueness in Theorem 4.1 generalize those found in Onatski (2006) because they allow for zeros on $\mathbb{T}$.

If all the partial indices are zero, then there exists a unique solution (2) representable as an autoregressive process driven by current and expected values of $\varepsilon$. In this representation, the forward component determines the dependence on current and expected values of $\varepsilon$, while the backward component determines the dependence on the past values of the solution.

If all the partial indices are non-positive and $k$ are negative, there exists a solution determined up to an arbitrary stochastic process $\nu \in \mathcal{S}^k$ that affects the solution through

$$M_0(L)S\nu_t = (\underbrace{0\ ,\ \ldots\ ,\ 0}_{n-k \text{ elements}}, \nu_{1,t+|\kappa_{n-k+1}|}, \ldots, \nu_{k,t+|\kappa_n|})'$$

as well as arbitrary values of certain linear combinations of $X_0, \ldots, X_{|\kappa_n|-1}$. The fact that $E(M_0(L)S\nu_t|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$, along with (3), implies that

$$E(M_0(L)M_b(L)X_t|\mathscr{I}_t) = M_0(L)M_b(L)X_t - M_0(L)S\nu_t \qquad \text{a.s.} \qquad t \geq 0.$$

Thus, $\nu$ affects an arbitrary modification to the time-$t$ expectations of certain linear combinations of $X_{t+1}, \ldots, X_{t+|\kappa_n|}$ for $t \geq 0$. Therefore, the economic entities modelled in (1) can hold beliefs about the future that are completely ungrounded in the fundamental economic forces of the system (i.e. $\varepsilon$ and the initial conditions). The stochastic process $\nu$ is known as a sunspot in the macroeconomic literature (Farmer, 1999). On the other hand, the indeterminacy of the initial values of some linear combinations of $X$ is often overlooked in the literature (e.g. Lubik & Schorfheide (2003) never mentions it). That is because most treatments transform (3) to obtain the representation

$$M_b(L)X_t = M_0^{-1}(L)E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) + S\nu_t, \qquad t \geq |\kappa_n|,$$

which masks this additional indeterminacy in $X$. Proper accounting of the structural equations shows that this representation holds only for $t \geq |\kappa_n|$ and the correct representation for $t \geq 0$ is (3). Notice that, when there is no uncertainty (i.e. when $\mathscr{I} = \{\mathscr{I}_t = \mathscr{A} : t \in \mathbb{Z}\}$), $\nu$ does not enter into (3) although the indeterminacy of the initial values of $X$ remains. Note finally, that although Theorem 4.1 (ii) is phrased as a description of a certain class of solutions to (1), it is in fact a description of the general structure of every solution $(X, \mathscr{I})$ when the partial indices are non-positive. To see this, let $(X, \mathscr{I})$ be a solution to (1), then simply define $\nu$ as

$$M_0(L)S\nu_t = M_0(L)M_b(L)X_t - E(M_0(L)M_b(L)X_t|\mathscr{I}_t) \qquad t \geq 0.$$

Finally, if any partial index is positive there is no solution in general, in the sense that one can always find exogenous processes and/or initial conditions that violate the structural equations. In fact, the proof of Theorem 4.1 (iii) and the examples below make clear that existence can only hold under very unnatural conditions where the exogenous process and/or initial conditions are restricted.

The solution concept advanced in Theorem 4.1 is a straightforward generalization to the multivariate setting of the univariate device of factorizing an LREM into a part to solve forwards and a part to solve backwards. Multivariate extensions of univariate ideas invariably involve diagonalization, and this leads directly to the ILWHF utilized in Theorem 4.1. In fact, vestiges of this trick appear in every single solution method in the LREM literature. Thus, an ILWHF is obtained implicitly in every single solution method in the literature. Note that the linear systems approach allows the researcher to obtain the VAR representations (2) and (3) directly without having to go through any rearrangement as in Klein (2000) and Sims (2002). The representations are, moreover, clearly the simplest and most compact in the literature.

**Example 4.3.** Consider Example 2.1 with $ab \neq 0$ and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. Theorem 4.1 and the ILWHFs computed in Example 3.2 imply that there are two possibilities to consider:

(i) If $|a/b| < 1$, the unique solution is $(X, \mathscr{I})$ with

$$
\begin{aligned}
X_t &= \tilde{X}_t, && t < 0, \\
X_t &= \sum_{i=0}^{\infty} (-a/b)^i b^{-1} E(\varepsilon_{t+i}|\mathscr{I}_t), && t \geq 0.
\end{aligned}
$$

(ii) When $|a/b| \geq 1$, then for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1}|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap \mathrm{m}\mathscr{I}_0$, there is a solution $(X, \mathscr{I})$ with

$$
\begin{aligned}
X_t &= \tilde{X}_t, && t < 0, \\
X_t &= X_0, && t = 0, \\
X_t &= -(b/a) X_{t-1} + a^{-1}\varepsilon_{t-1} + \nu_t, && t \geq 1.
\end{aligned}
$$

Note that the $t = 0$ structural equation is used to determine $X_1$ and so $X_0$ is left indeterminate. □

**Example 4.4.** Consider Example 2.2 with $bc \neq 0$ and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. Theorem 4.1 and the ILWHFs computed in Example 3.3 imply that there are two possibilities to consider:

(i) If $|b/c| \geq 1$, the unique solution is $(X, \mathscr{I})$ with

$$
\begin{aligned}
X_t &= \tilde{X}_t, && t < 0, \\
X_t &= -(c/b) X_{t-1} + b^{-1}\varepsilon_t, && t \geq 0.
\end{aligned}
$$

This solution has already been derived in Example 4.2.

(ii) When $|b/c| < 1$, there can be no solution in general. To see this, suppose $(X, \mathscr{I})$ is a solution, then, by Lemma 4.1, applying the operator $E(M_f^{-1}(L)(\cdot)|\mathscr{I}_t)$ to both sides of Definition 4.2 (iv) yields $X_{t-1} = E\left((bL^{-1} + c)^{-1}\varepsilon_t|\mathscr{I}_t\right)$ a.s. for all $t \geq 0$ because $M_0(z) = z$. However, the $t = 0$ equation cannot be ensured to hold. □

**Example 4.5.** Consider the setting of Example 2.3 with $ac \neq 0$ and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. Theorem 4.1 and the ILWHFs computed in Example 3.4 imply that there are three possibilities to consider:

(i) If $|\zeta_1| < 1 \leq |\zeta_2|$, the unique solution $(X, \mathscr{I})$ has

$$
\begin{aligned}
X_t &= \tilde{X}_t, && t < 0, \\
X_t &= \zeta_2^{-1} X_{t-1} - (c\zeta_2)^{-1} E\left( \sum_{i=0}^{\infty} \zeta_1^i \varepsilon_{t+i} \,\middle|\, \mathscr{I}_t \right), && t \geq 0.
\end{aligned}
$$

(ii) If $|\zeta_1|, |\zeta_2| \geq 1$, then for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1}|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap \mathrm{m}\mathscr{I}_0$, there is a solution $(X, \mathscr{I})$ with

$$\begin{aligned} X_t &= \tilde{X}_t, && t < 0, \\ X_t &= X_0, && t = 0, \\ X_t &= -(b/a)X_{t-1} - (c/a)X_{t-2} + a^{-1}\varepsilon_{t-1} + \nu_t, && t \geq 1. \end{aligned}$$

Note that the $t = 0$ structural equation is used to determine $X_1$ and so $X_0$ is left indeterminate.

(iii) If $|\zeta_1|, |\zeta_2| < 1$, there can be no solution in general. To see this, suppose $(X, \mathscr{I})$ is a solution, then, by Lemma 4.1, applying the operator $E(M_f^{-1}(L)(\cdot)|\mathscr{I}_t)$ to both sides of Definition 4.2 (iv) yields $X_{t-1} = E\left((aL^{-2} + bL^{-1} + c)^{-1}\varepsilon_t|\mathscr{I}_t\right)$ a.s. for all $t \geq 0$ because $M_0(z) = z$. However, the $t = 0$ equation cannot be ensured to hold. $\square$

**Example 4.6.** Consider the setting of Example 2.4 and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. By Theorem 4.1 and the ILWHFs calculated in Example 3.5, there are two possibilities to consider:

(i) If $|R| > 1$, the unique solution is given by $(X, \mathscr{I})$ with

$$\begin{aligned} X_t &= \tilde{X}_t && t < 0. \\ X_t &= \left[\begin{smallmatrix} 0 & R-1 \\ 0 & 1 \end{smallmatrix}\right] X_{t-1} + E\left(\left[\begin{smallmatrix} R-1 \\ 1-L^{-1} \end{smallmatrix}\right] \sum_{i=0}^{\infty} R^{-1-i}\varepsilon_{2,t+i} \middle| \mathscr{I}_t\right) && t \geq 0. \end{aligned}$$

(ii) If $|R| \leq 1$, then for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1}|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_{1,0} \in \mathcal{L}^1 \cap \mathrm{m}\mathscr{I}_0$, there is a solution $(X, \mathscr{I})$ with

$$\begin{aligned} X_t &= \tilde{X}_t, && t < 0, \\ \begin{bmatrix} L^{-1}-1 & 0 \\ 1 & 1-RL \end{bmatrix} X_t &= \left[\begin{smallmatrix} \nu_{t+1} \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ \varepsilon_{2t} \end{smallmatrix}\right], && t \geq 0. \end{aligned}$$

Note that the $t = 0$ equations are used to determine $X_{1,1}$ and $X_{2,0}$ and so $X_{1,0}$ is left indeterminate. $\square$

The role played by $\mathscr{I}$ is non-trivial and does not seem to have garnered sufficient attention in the literature. To see how it can make a significant difference to the solution, consider the following example.

**Example 4.7.** Consider Example 2.1 with $ab \neq 0$ and let the initial conditions satisfy the conditions of Definition 4.2. Let $|a/b| < 1$ and set $\varepsilon = m_1 + m_2$, where $m_1, m_2 \in \mathcal{S}^1$ are i.i.d. and independent of each other. Now define

$$\mathscr{I}_1 = \left\{ \mathscr{I}_{1t} = \sigma(\varepsilon_s : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z} \right\}$$

$$\mathscr{I}_2 = \{ \mathscr{I}_{2t} = \mathscr{I}_{1t} \vee \sigma(m_{2t} : t \in \mathbb{Z}) : t \in \mathbb{Z} \}$$

$$\mathscr{I}_3 = \{ \mathscr{I}_{3t} = \mathscr{A} : t \in \mathbb{Z} \} .$$

Thus, $\mathscr{I}_1$ is the filtration of fundamental solutions, $\mathscr{I}_2$ corresponds to the setting where, additionally, information about all current and future values of $m_{2t}$ are known, and $\mathscr{I}_3$ corresponds to the case of no uncertainty. Now for $t \geq 0$, set

$$\begin{aligned} X_{1t} &= b^{-1}(m_{1t} + m_{2t}) \\ X_{2t} &= b^{-1} m_{1t} + \sum_{i=0}^{\infty} (-a/b)^i b^{-1} m_{2t+i} \\ X_{3t} &= \sum_{i=0}^{\infty} (-a/b)^i b^{-1} (m_{1t+i} + m_{2t+i}) \end{aligned}$$

and $X_{it} = \tilde{X}_t$ for $t < 0$ and $i = 1, 2, 3$. Then we have three completely different solutions $(X_1, \mathscr{I}_1)$, $(X_2, \mathscr{I}_2)$, and $(X_3, \mathscr{I}_3)$ each of which is unique. □

Of course, if irrelevant information is added to the filtration, it is reasonable to expect it to have no effect on the solution.

**Example 4.8.** Consider the setting of Example 4.7 and let $m_3 \in \mathcal{S}^1$ be independent of $m_1$ and $m_2$. If

$$\mathscr{I}_4 = \{ \mathscr{I}_{4t} = \mathscr{I}_{1t} \vee \sigma(m_{3s} : 0 \leq s \leq t) : t \in \mathbb{Z} \} ,$$

then $\mathscr{I}_4$ corresponds to the setting where the filtration of the fundamental solution is augmented with irrelevant information. Then with $X_t = b^{-1}\varepsilon_t$ for $t \geq 0$ and $X_t = \tilde{X}_t$ for $t < 0$, we have that $(X, \mathscr{I}_1)$ and $(X, \mathscr{I}_4)$ are unique solutions to the LREM. □

The key idea in the examples above is that filtrations factor into equivalence classes according to how well they predict $M_f^{-1}(L)\varepsilon$. The next corollary follows directly from (2).

**Corollary 4.1.** Under the assumptions of Theorem 4.1, if the partial indices of $M(z)$ are all zero and $(X_1, \mathscr{I}_1)$ and $(X_2, \mathscr{I}_2)$ are solutions to (1), then $X_1$ and $X_2$ are indistinguishable if and only if $E(M_f^{-1}(L)\varepsilon_t | \mathscr{I}_{1t}) = E(M_f^{-1}(L)\varepsilon_t | \mathscr{I}_{2t})$ a.s. for all $t \geq 0$.

Corollary 4.1 defines an equivalence relationship between filtrations of solutions to (1) when the partial indices are all zero: $\mathscr{I}_1$ and $\mathscr{I}_2$ are equivalent if and only if both produce a.s. the same predictions of $M_f^{-1}(L)\varepsilon$, in which case they produce indistinguishable solutions. Recalling that the filtration of fundamental solutions is the smallest for which a solution to (1) may exist, the equivalence class of the filtration of fundamental solutions is the set of filtrations that fail to Granger-cause $M_f^{-1}(L)\varepsilon$ at all horizons. Conversely, if we maintain that the partial indices are non-positive, then if $(X_1, \mathscr{I}_1)$ and $(X_2, \mathscr{I}_2)$ are two solutions to (1) such that $\mathscr{I}_1$ and $\mathscr{I}_2$ give a.s. the same predictions of $M_f^{-1}(L)\varepsilon$, and $X_1$ and $X_2$ are not indistinguishable, then Corollary 4.1 implies that at least one of the partial indices must be negative. In words, irrelevant additional information leads to no change in the solution to an LREM if the partial indices are all zero and can lead to sunspot solutions only if the partial indices are non-positive and some are negative.

## 4.3 Generic Existence and Uniqueness

So far, we have considered the existence and uniqueness of solutions to particular LREMs. We now turn our attention to existence and uniqueness of solutions to generic LREMs. Onatski (2006) proved a general result that specializes in our context to the fact that a generic LREM (1) with $\det(M(z)) \neq 0$ for $z \in \mathbb{T}$ satisfies existence and uniqueness, existence but non-uniqueness, or non-existence, according to whether $\det(M(z))$ winds around the origin zero, a negative, or a positive number of times respectively as $\mathbb{T}$ is traversed counter-clockwise. However, the winding number is not defined when $\det(M(z))$ has a zero on $\mathbb{T}$. We now show how this result can be extended.[12]

**Theorem 4.2.** Suppose the initial conditions, $\mathscr{I}$, and $\varepsilon$ are as in Theorem 4.1. Let $r$ be as in Proposition 3.1. Then, for a generic non-singular $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ we have existence and uniqueness, existence but non-uniqueness, or non-existence, according to whether $\det(M(z))$ winds around the origin zero, a negative, or a positive number of times respectively as $r\mathbb{T}$ is traversed counter-clockwise.

**Corollary 4.2.** Under the assumptions of Theorem 4.2, for a generic non-singular $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ we have existence and uniqueness, existence but non-uniqueness, or non-existence,

[12] Readers familiar with the stability theory of linear systems will see similarity to the Nyquist stability criterion. Indeed, both results rely on the argument principle in complex analysis (Ahlfors, 1979, Section 5.2).

according to whether $n_Z - n_P$ is zero, negative, or positive, where $n_Z$ and $n_P$ are the number of zeros and poles of $\det(M(z))$ (counting multiplicity) that are inside $\mathbb{D}$ respectively.

**Example 4.9.** Consider the setting of Example 4.6, then $\det(M(z)) = (z^{-1} - 1)(1 - Rz)$. Onatski's original winding number index cannot be calculated for this system. However, $\det(M(z))$ has zeros at $\{1, R^{-1}\}$ and poles at $\{0, \infty\}$. Thus, Corollary 4.2 correctly predicts existence and uniqueness when $|R| > 1$ and existence but non-uniqueness when $|R| \leq 1$. $\square$

Strictly speaking, Example 4.9 is a misapplication of Theorem 4.2 and Corollary 4.2. These results should be applied only when all parameters are free of restrictions. If $M(z)$ is restricted in any way, there is no guarantee that the criteria will work correctly, as both Onatski (2006) and Sims (2007) have warned.

**Example 4.10.** Consider the LREM with $M(z) = \left[\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right]$. Then $\det(M(z)) = 1$, which has $n_Z = n_P = 0$, thus Corollary 4.2 suggests existence and uniqueness. By Theorem 4.1 (iii), this is incorrect because $M(z)$ has a positive partial index. To see that this system is non-generic, consider a perturbation of the form $M_\epsilon(z) = \left[\begin{smallmatrix} z & \epsilon \\ 0 & z^{-1} \end{smallmatrix}\right]$. It is easily checked that $M_\epsilon(z)$ has partial indices of zeros whenever $\epsilon \neq 0$ and since $\det(M_\epsilon(z)) = 1$, Corollary 4.2 now provides the correct answer of existence and uniqueness. $\square$

A full characterization of the generic subset of non-singular elements of $\mathbb{R}^{n\times n}_{pq}(z)$ to which Theorem 4.2 and Corollary 4.2 apply is available in Section A.3 of the online supplement.

## 4.4 Exponentially Growing Solutions

We close this section with a generalization of existence and uniqueness to spaces beyond $\mathcal{S}^n$. Theoretical considerations sometime warrant constructing solutions that exhibit exponential growth (Blanchard & Fischer, 1989, Chapter 5). While it is possible to give a general formal treatment for solutions outside of $\mathcal{S}^n$ that parallels the treatment in Subsection 4.2, including it here would feel too repetitive, especially considering that such solutions are often not the object of interest in empirical work. Thus, we opt for a slightly less formal treatment that conveys the main message.

Finding exponentially growing solutions in the univariate case is as simple as computing the ILWHF relative to $\rho\mathbb{T}$ with $0 < \rho < 1$.

**Example 4.11.** Consider the setting of Example 2.3 with $ac \neq 0$ and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. Suppose we would like to obtain $\mathcal{L}^1$ solutions that exhibit a growth rate of up to $\rho^{-1} > 1$ in the sense that $\lim_{t\to\infty} \theta^t E|X_t| = 0$ for $0 \leq \theta < \rho$. Utilizing the ILWHFs computed in Example 3.4, there are three cases to consider:

(i) If $|\zeta_1| < \rho \leq |\zeta_2|$, there exists a process-filtration pair $(X, \mathscr{I})$ generated recursively as

$$\begin{aligned} X_t &= \tilde{X}_t, && t < 0, \\ X_t &= \zeta_2^{-1} X_{t-1} - (c\zeta_2)^{-1} E\left( \sum_{i=0}^{\infty} \zeta_1^i \varepsilon_{t+i} \middle| \mathscr{I}_t \right), && t \geq 0, \end{aligned}$$

which satisfies all of the conditions of Definition 4.2, except that $X \notin \mathcal{S}^1$. Following arguments that parallel those used to prove Theorem 4.1 (i), it is easy to show that this is the unique solution in the class of $\mathcal{L}^1$ processes that exhibit exponential growth rate of up to $\rho^{-1}$.

(ii) If $|\zeta_1|, |\zeta_2| \geq \rho$, then, following arguments that parallel those used to prove Theorem 4.1 (ii), for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1}|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ and any $X_0 \in \mathcal{L}^1 \cap \mathrm{m}\mathscr{I}_0$, there is a solution $(X, \mathscr{I})$ with

$$\begin{aligned} X_t &= \tilde{X}_t, && t < 0, \\ X_t &= X_0, && t = 0, \\ X_t &= -(b/a)X_{t-1} - (c/a)X_{t-2} + a^{-1}\varepsilon_{t-1} + \nu_t, && t \geq 1, \end{aligned}$$

which satisfies all of the conditions of Definition 4.2, except that $X \notin \mathcal{S}^1$.

(iii) If $|\zeta_1|, |\zeta_2| < \rho$, then, following arguments that parallel those used to prove Theorem 4.1 (iii), there is no solution in general. □

Example 4.11 highlights the key insight to finding exponentially growing solutions to LREMs: such solutions are characterized by an autoregressive representation for $X$ driven by current and expected values of $\varepsilon$, where the matrix polynomial associated with the autoregressive part has a zero in $\mathbb{D}\backslash\{0\}$.[13] Therefore, solutions to (1) that exhibit exponential

[13] The point $z = 0$ is excluded because, otherwise, there is no causal solution. For example, if the polynomial associated by an autoregressive model is $M(z) = b + cz$ and is non-singular, then $M(z)$ has a zero at $z = 0$ if and only if $b = 0$. When $b = 0$, the autoregressive model it describes cannot possibly generate a causal solution.

growth are possible if and only if $M(z)$ has zeros in $\mathbb{D}\backslash\{0\}$. It follows that a researcher who wishes to model an economic system that exhibits exponential growth must parametrize $M(z)$ so that it has zeros in $\mathbb{D}\backslash\{0\}$.

To produce these exponentially growing solutions in the multivariate case, one may proceeds as follows. Let $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ be non-singular and factorize it as $M(z) = \hat{M}(z)G(z)$, where $\hat{M}(z)$ is a square Laurent matrix polynomial and $G(z)$ is a matrix polynomial with all its zeros in $\mathbb{D}\backslash\{0\}$.[14] $G(z)$ may contain some or all of the zeros of $M(z)$ in $\mathbb{D}\backslash\{0\}$. We then obtain the ILWHF of $\hat{M}(z)$ relative to $\mathbb{T}$ as $\hat{M}_f(z)\hat{M}_0(z)\hat{M}_b(z)$. Given initial conditions, $\mathscr{I}$, and $\varepsilon$ that satisfy the conditions of Definition 4.2 and if all the partial indices of $\hat{M}(z)$ are non-positive, we may solve (1) in two steps. First, for a given filtration that satisfies Definition 4.2 (i), we obtain the solution $(\hat{X}, \mathscr{I})$,

$$
\begin{aligned}
\hat{X}_t &= G(L)\tilde{X}_t, && t < 0,\\
\hat{M}_0(L)\hat{M}_b(L)\hat{X}_t &= E(\hat{M}_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) + \hat{M}_0(L)S\nu_t, && t \geq 0.
\end{aligned}
$$

where $S$ and $\nu$ are as in Theorem 4.1 (ii). Next, we solve for $X$ recursively in

$$
\begin{aligned}
X_t &= \tilde{X}_t, && t < 0,\\
G(L)X_t &= \hat{X}_t, && t \geq 0.
\end{aligned}
$$

Thus, while $\hat{X} \in \mathcal{S}^n$ by Theorem 4.1, $X \notin \mathcal{S}^n$ in general. The pair $(X, \mathscr{I})$ satisfies all of the conditions for an LREM solution in Definition 4.2, except membership in $\mathcal{S}^n$.

**Example 4.12.** Consider the setting of Example 2.4 and let the initial conditions, $\mathscr{I}$, and $\varepsilon$ satisfy the conditions of Definition 4.2. Since $M(z)$ has zeros at $\{1, R^{-1}\}$, the model can exhibit exponentially growing solutions if and only if $|R| > 1$. If $|R| > 1$ and we are interested in solutions that exhibit a growth rate of no more than $\rho^{-1} \geq |R|$, we may write $M(z) = \hat{M}(z)G(z)$ with $\hat{M}(z) = \left[\begin{smallmatrix} z^{-1}-1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $G(z) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1-Rz \end{smallmatrix}\right]$. An ILWHF of $\hat{M}(z)$ relative to $\mathbb{T}$ is $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & z^{-1} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 1-z & 0 \end{smallmatrix}\right]$. It follows that for any $\nu \in \mathcal{S}^1$ satisfying $E(\nu_{t+1}|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$ and any $\hat{X}_{1,0} \in \mathcal{L}^1 \cap \mathrm{m}\mathscr{I}_0$,

$$
\begin{aligned}
\hat{X}_t &= \left[\begin{smallmatrix} \tilde{X}_{1,t} \\ \tilde{X}_{2,t} - R\tilde{X}_{2,t-1} \end{smallmatrix}\right], && t < 0,\\
\left[\begin{smallmatrix} L^{-1}-1 & 0 \\ 1 & 1 \end{smallmatrix}\right]\hat{X}_t &= \left[\begin{smallmatrix} \nu_{t+1} \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ \varepsilon_{2t} \end{smallmatrix}\right], && t \geq 0.
\end{aligned}
$$

[14] Such factorizations are standard in the linear systems literature. One may extract $G(z)$ from $z^q M(z)$ zero-by-zero as in p. 40 of Hannan & Deistler (2012) or using the Smith canonical form (see the proof of Theorem A.1 in the online supplement).

Thus, $\hat{X}_{1,t} = \hat{X}_{1,0} + \sum_{s=1}^{t} \nu_s$ and $\hat{X}_{2,t} = -\hat{X}_{1,0} - \sum_{s=1}^{t} \nu_s + \varepsilon_{2t}$ for $t \geq 0$. We can then solve for $X$ from

$$X_t = \tilde{X}_t, \qquad t < 0,$$

$$X_t = \left[\begin{smallmatrix} 0 & 0 \\ 0 & R \end{smallmatrix}\right] X_{t-1} + \hat{X}_t, \qquad t \geq 0.$$

Thus, $X_{1,t} = \hat{X}_{1,0} + \sum_{s=1}^{t} \nu_s$ and $X_{2,t} = RX_{2,t-1} - \hat{X}_{1,0} - \sum_{s=1}^{t} \nu_s + \varepsilon_{2,t}$ for $t \geq 0$. Thus, in any solution of the LREM that exhibits exponential growth, it is bond holdings that experience the growth, while consumption continues to exhibit its random walk behaviour. □

A couple of comments are in order here. First, it is clear that the method above allows us to find any and all exponentially growing solutions to the LREM, including the cases discussed in Sims (2002) where one imposes exponential growth restrictions on certain linear combinations of $X$. Second, the logic above can be used to extract any set of non-zero zeros of $M(z)$ into $G(z)$ and not just the ones that are in $\mathbb{D}$. In particular, when $G(z)$ extracts a set of zeros of $M(z)$ in $\mathbb{D}^c$, the resulting solution is the same as in Theorem 4.1. Thus, if one insists on using the WHF instead of the ILWHF, then the method above could be used to extract all the zeros on $\mathbb{T}$ in $G(z)$ before applying the WHF. If $G(z)$ extracts all the non-zero zeros of $M(z)$, we obtain the solution studied in Broze et al. (1995).

# 5 Empirical LREMs

This section considers the general structure of solutions to LREMs typically found in empirical econometrics.

**Example 5.1.** Let $A(z), B(z) \in \mathbb{R}^{n\times n}[z]$ and $N(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ and consider the LREM with

$$M(z) = \begin{bmatrix} I_n & 0 & 0 \\ -B(z) & A(z) & 0 \\ 0 & -I_n & N(z) \end{bmatrix}.$$

Letting $\eta$ be an $n$-dimensional i.i.d. sequence of zero mean and positive definite covariance matrix, setting $\varepsilon = (\eta', 0, \ldots, 0)'$, and setting $\mathscr{I}$ to the filtration of fundamental solutions, we obtain the general structure of most LREMs in the empirical literature. Here, $\eta$ drives a VARMA process of exogenous disturbance (e.g. shifts in technology, monetary policy, etc.) that enter into the last block of equations that determines the endogenous variables of interest (e.g. consumption, output, etc.). If $A(z)$ is non-singular and has no zeros in $\mathbb{D}$ and $N(z)$ is non-singular with ILWHF relative to $\mathbb{T}$ with zero partial indices, then it is easily shown that $M(z)$ will be non-singular with an ILWHF relative to $\mathbb{T}$ with zero partial indices. □

Abstracting from some of the unnecessary structure in Example 5.1, we will work with the following assumption.

**Assumptions 5.1.** For a given probability space $(\Omega, \mathscr{A}, P)$, $\varepsilon \in \mathcal{S}^n(\Omega, \mathscr{A}, P)$ is an i.i.d. sequence of zero mean and covariance matrix $\Sigma$ and $\{\tilde{X}_t : t < 0\} \subset \mathcal{L}^1(\Omega, \mathscr{A}, P)$. $\mathscr{I} = \Big\{\mathscr{I}_t = \sigma(\varepsilon_s : 0 \leq s \leq t) \vee \sigma(\tilde{X}_s : s \leq \min\{t, -1\}) : t \in \mathbb{Z}\Big\}$. $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$ is non-singular and its ILWHF relative to $\mathbb{T}$ has zero partial indices.

Under Assumptions 5.1, $M(z)$ has an ILWHF relative to $\mathbb{T}$ of the form $M(z) = M_f(z)M_b(z)$ and Theorem 4.1 then implies that there exists a unique solution to (1) satisfying

$$\begin{aligned} X_t &= \tilde{X}_t, \quad t < 0, \\ M_b(L)X_t &= \varepsilon_t, \quad t \geq 0, \end{aligned} \tag{4}$$

where we have further taken $M_f(\infty) = I_n$. By Theorem 3.2, $M_b(z) \in \mathbb{R}^{n\times n}[z]$ and is of degree at most $p$. The stability of this solution can be read directly from $M(z)$.

**Proposition 5.1.** Under Assumptions 5.1, the solution to the LREM (1) is stable if and only if $M(z)$ has no zeros on $\mathbb{T}$.

It follows from Proposition 5.1 that a researcher who wishes the model to accommodate integration and/or seasonal integration, must allow the parameterization of $M(z)$ to yield zeros on $\mathbb{T}$.

We now consider separately the cases of stable and unstable solutions.

## 5.1 Stationary Solutions

Under Assumptions 5.1, if $M(z)$ has no zeros on $\mathbb{T}$ then, by Proposition 5.1, $M_b(z)$ has no zeros on $\overline{\mathbb{D}}$ and therefore $M_b^{-1}(z)$ has no poles in $\overline{\mathbb{D}}$. If we now set the initial conditions to $\tilde{X}_t = M_b^{-1}(L)\varepsilon_t$ for $t < 0$, where $M_b^{-1}(L)\varepsilon_t$ is defined in the classical sense (e.g. Section 1.2 of Hannan & Deistler (2012)), then $\mathscr{I}_t = \sigma(\varepsilon_s : s \leq t)$ for all $t \in \mathbb{Z}$ and the solution $(X, \mathscr{I})$ to the LREM is stationary and satisfies

$$X_t = M_b^{-1}(L)\varepsilon_t, \quad \text{a.s.} \quad t \in \mathbb{Z}.$$

The spectral density matrix of the process is then immediately given as, $M_b^{-1}(e^{i\lambda})\Sigma M_b^{-1}(e^{-i\lambda})$. As is well known in the linear systems literature, the structure of this stationary linear process is determined by the relationship between

$$F_{t+1|t} = \begin{bmatrix} E(X_{t+1}|\mathscr{I}_t) \\ E(X_{t+2}|\mathscr{I}_t) \\ E(X_{t+3}|\mathscr{I}_t) \\ \vdots \end{bmatrix} \qquad \text{and} \qquad P_t = \begin{bmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \end{bmatrix}.$$

See for example Chapter 3 of Reinsel (2003) and Chapter 2 of Hannan & Deistler (2012). We can gain more insight about this relationship by considering the infinite set of equations $E(M(L)X_s|\mathscr{I}_t) = 0$ for $s > t \in \mathbb{Z}$.[15] These may be arranged as

$$\underbrace{\begin{bmatrix} M_0 & M_{-1} & M_{-2} & \cdots \\ M_1 & M_0 & M_{-1} & \cdots \\ M_2 & M_1 & M_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Theta} \underbrace{\begin{bmatrix} E(X_{t+1}|\mathscr{I}_t) \\ E(X_{t+2}|\mathscr{I}_t) \\ E(X_{t+3}|\mathscr{I}_t) \\ \vdots \end{bmatrix}}_{F_{t+1|t}} + \underbrace{\begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \cdots \\ M_3 & M_4 & M_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} X_t \\ X_{t-1} \\ X_{t-2} \\ \vdots \end{bmatrix}}_{P_t} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{a.s. } t \in \mathbb{Z}. \tag{5}$$

Remarkably, this set of equations is sufficient for determining $F_{t+1|t}$ from $P_t$.

**Proposition 5.2.** Under Assumptions 5.1, if $M(z)$ has no zeros on $\mathbb{T}$, the initial conditions are set to $\tilde{X}_t = M_b^{-1}(L)\varepsilon_t$ for $t < 0$, and $(X, \mathscr{I})$ is the solution to (1), then for any $t \in \mathbb{Z}$, $F_{t+1|t}$ is a.s. the unique solution to (5) in the Banach space $l_n^\infty = \{Z = (Z_1', Z_2', \ldots)' : Z_i \in \mathbb{R}^n, \sup_{i\geq 1} \|Z_i\| < \infty\}$ and $F_{t+1|t} = -\Theta^{-1}\Psi P_t$ a.s.

Note that each equation in (5) determines a linear combination of expected values of $X$ that is predictable by some linear combination of current and past values of $X$. Thus, the equation above can be interpreted as an infinite set of subspace Granger non-causality restrictions imposed by linear rational expectations (Al-Sadoon, 2014).

Finally, it is important to note that if (5) is augmented by the equation $E(M(L)X_t|\mathscr{I}_t) = \varepsilon_t$ a.s., then solving this augmented system for $(X_t', F_{t+1|t}')'$ in terms of $P_t$ and $\varepsilon_t$ yields a solution to the LREM. This is indeed the approach of Shiller (1978) and Onatski (2006), who analyse the LREM as an infinite set of structural equations that determine current and expected values of $X$ in terms of past values of $X$ as well as current and expected values of $\varepsilon$.

[15] To see why these equations hold for all $t \in \mathbb{Z}$, apply the operator $M(L)$ to both sides of $X_s = M_b^{-1}(L)\varepsilon_s$, $s \in \mathbb{Z}$, then take the conditional expectation with respect to $\mathscr{I}_t$ for $t < s$.

## 5.2 Non-Stationary Theory

Under Assumptions 5.1, if $M(z)$ has a zero on $\mathbb{T}$, then it is well known that the Smith canonical form of $M_b(z)$ in (4) determines the range of non-stationary phenomena that the solution can exhibit (Hylleberg et al., 1990; Engle & Yoo, 1991; Schumacher, 1991; Haldrup & Salmon, 1998). Thus, the Smith canonical form of $M_b(z)$ determines whether the solution exhibits cointegration or seasonal cointegration, the order of integration, and whether the cointegration is of the polynomial type. Remarkably, the Smith canonical form of $M_b(z)$ is precisely the backward component of the Smith-McMillan canonical form of $M(z)$ (see Corollary A.1 of the online supplement). Thus, all of the non-stationary behaviour of the solution can be referred directly back to the parameterization of $M(z)$. We illustrate this idea by considering the conditions on $M(z)$ that ensure an $I(1)$ cointegrated solution. Extensions to higher orders of integration, seasonal cointegration, and polynomial cointegration are straightforward to obtain and are omitted.

The following version of Granger's classical representation theorem (Engle & Granger, 1987) extends the version stated in Theorem 4.2 of Johansen (1995).

**Proposition 5.3.** Under Assumptions 5.1, if for $z \in \mathbb{T}$, $\det(M(z)) = 0$ only at $z = 1$ and $\text{rank}(M(1)) = r < n$, then there exist $\alpha, \beta \in \mathbb{R}^{n\times r}$ of full rank such that, $M(1) = \alpha\beta'$ and the unique solution $(X, \mathscr{I})$ to (1) is also the unique solution to

$$\begin{aligned} &M^*_{-q}E_t\Delta X_{t+q} + \ldots + M^*_{-1}E_t\Delta X_{t+1} + \\ &\qquad M^*_0\Delta X_t + \alpha\beta' X_{t-1} + M^*_1\Delta X_{t-1} + \ldots + M^*_{p-1}\Delta X_{t-p+1} = \varepsilon_t, \quad t \geq 0, \end{aligned}$$

where $M^*(z) = (1-z)^{-1}(M(z) - M(1)z) \in \mathbb{R}^{n\times n}_{p-1q}(z)$ and $\Delta$ is the difference operator. A necessary and sufficient condition that $\Delta X$ and $\beta' X$ can be represented as stable linear processes driven by $\varepsilon$ is that $\det(\alpha'_\perp M^*(1)\beta_\perp) \neq 0$ or equivalently $\det\left(\alpha'_\perp \frac{dM(1)}{dz}\beta_\perp\right) \neq 0$.[16]

Under the conditions of Proposition 5.3, $M(z)$ specifies a cointegrated VAR solution to the LREM with cointegration rank $r$ and the cointegration space of the solution can be read directly from $M(z)$ as $\ker(M(1))$. These results generalize those by Binder & Pesaran (1995) and Juselius (2008), who consider a specification of the form considered in Example 5.1 and allow for unit roots only in $A(z)$ but not in $N(z)$. Thus, their results cannot apply to the consumption model (Example 2.4) or any other LREM that generates unit roots endogenously.

[16] $\alpha_\perp, \beta_\perp$ are any matrices in $\mathbb{R}^{n\times(n-r)}$ of full rank and satisfying $\alpha'_\perp\alpha = \beta'_\perp\beta = 0$.

The classical structural vector error correction model is clearly the special case of Proposition 5.3, where $M_i^* = 0$ for $i < 0$. It is easy to see that $M_i^* = 0$ for $i < 0$ if and only if $M_i = 0$ for $i < 0$. This LREM error correction expression is similar to that found in Broze et al. (1990), except that they arrange the forward terms as expectational errors rather than expected differences; they also do not provide conditions under which the order of integration is bounded by 1.

It follows from Proposition 5.3 that a researcher who wishes to allow for cointegration of a particular rank in their model must parametrize $M(z)$ so that $M(1)$ is of fixed rank. Moreover, if the researcher wishes to ensure that the order of integration is bounded by 1, they must ensure that for all admissible values of $M(z)$, $\det(\alpha_\perp' M^*(1)\beta_\perp) \neq 0$ (equiv. $\det\left(\alpha_\perp' \frac{dM(1)}{dz}\beta_\perp\right) \neq 0$). These points are illustrated in the following example.

**Example 5.2.** Consider the setting of Example 2.4. The conditions on $M(z)$ in Assumptions 5.1 require that $|R| > 1$. Since $\det(M(z)) = (z^{-1} - 1)(1 - Rz)$, the system may generate integrated solutions for any admissible value of $R$. Since $M(1) = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1-R \end{smallmatrix}\right]$, the cointegration vector, $\beta = (1, 1-R)'$, is immediately evident; we may also choose $\alpha = (0,1)'$. Since $M^*(z) = \left[\begin{smallmatrix} z^{-1} & 0 \\ 1 & 1 \end{smallmatrix}\right]$, if we choose $\alpha_\perp = (1,0)'$ and $\beta_\perp = (R-1, 1)'$, then $\alpha_\perp' M^*(1)\beta_\perp = R - 1 \neq 0$. Thus, the parameterization of the model is such that it is guaranteed not to produce orders of integration greater than 1. □

It is remarkable indeed that, like structural VARMA models, so much of the long-run behaviour of solutions to LREMs can be gleaned without having to solve the model first. However, the results above seem to exhaust all the low hanging fruit; for example, under the assumptions of Proposition 5.3, the long-run impact matrix is easily computed as $\beta_\perp(\alpha_\perp' M^*(1)\beta_\perp)^{-1}\alpha_\perp' M_f(1)$, which cannot be obtained without computing the ILWHF of $M(z)$ relative to $\mathbb{T}$ first.

# 6 Conclusion

This paper has attempted to situate LREM theory in the wider linear systems literature by providing firm mathematical foundations for the former and bringing to bear the wide arsenal of techniques from the latter. In the remainder, we discuss possible venues for future research, some of which are already part of ongoing research.

First, the present work begins a series of papers that seeks to resolve some long-standing econometric problems with LREMs. This includes exogeneity, parameterization, observational equivalence, structural identification, estimation, inference, and specification analysis.

Second, the causal meaning of structural vector autoregressions has been explored recently in a number of papers (e.g. White & Lu (2010), White et al. (2011), and White & Pettenuzzo (2014)). The framework of this paper can elucidate the causal content of LREM and is taken up in White et al. (2015).

Third, the ILWHF is easy to generalize to non-rational functions meromorphic in a neighbourhood of a curve in $\mathbb{C}$ homeomorphic to $\mathbb{T}$ as the limit with respect to sequences of contours that tend to the contour of interest from the inside using the Jordan-Schoenflies theorem. Finding the most general class of functions with respect to which such a generalization holds is an interesting question that deserves attention. Because spectral factorizations can be computed from ILWHFs (Clancey & Gohberg, 1981, Chapter 5), such a theory could potentially provide important lower-level assumptions for fractionally integrated processes.

Fourth, it is easily seen that continuous-time LREMs also utilize a Wiener-Hopf factorization albeit relative to a different contour than we considered in this paper. The theory of continuous time LREMs therefore follows almost word-for-word from the theory of this paper. However, it deserves further investigation as the mathematics of stochastic differential equations is substantially more involved than that of discrete time processes.

# 7 Proofs

*Proof of Proposition 3.1.* (i) *The "if" part.* $M_b(z)$ has no zeros or poles in $\rho\mathbb{D}$, thus it has no zeros or poles in $r\mathbb{D}$ by inclusion. On the other hand, $M_f(z)$ has no zeros or poles in $\rho\mathbb{D}^c$. If it had a zero or pole in $r\mathbb{D}^c \cap \rho\mathbb{D}$, this would translate to a zero or pole of $M(z)$ in that same region, contradicting the definition of $r$. Thus $M_f(z)$ has no zeros or poles in $\rho\mathbb{D}^c$.

*The "only if" part.* $M_f(z)$ has no zeros or poles in $r\mathbb{D}^c$, thus it has no zeros or poles in $\rho\mathbb{D}^c$ by inclusion. On the other hand, $M_b(z)$ has no zeros or poles in $r\mathbb{D}$. If it had a zero or pole in $r\mathbb{D}^c \cap \rho\mathbb{D}$, this would translate to a zero or pole of $M(z)$ in that same region, contradicting the definition of $r$. Thus $M_b(z)$ has no zeros or poles in $\rho\mathbb{D}$.

(ii) *WHF to ILWHF.* $N_f(z)$ has no zeros or poles in $\rho\mathbb{D}^c$, therefore $N_f((\rho/r)z)N_0(\rho/r)$ has no zeros or poles in $(r/\rho)\rho\mathbb{D}^c = r\mathbb{D}^c$ and, by inclusion, $N_f((\rho/r)z)N_0(\rho/r)$ has no zeros or poles in $\rho\mathbb{D}^c$. On the other hand, $N_b(z)$ has no zeros or poles in $\rho\mathbb{D}$, thus $N_b((\rho/r)z)$ has no zeros or poles in $(r/\rho)\rho\mathbb{D} = r\mathbb{D}$. If $N_b((\rho/r)z)$ had a zero or pole in $r\mathbb{D}^c \cap \rho\mathbb{D}$, this would translate to a zero or pole of $M(z)$ in the same region, contradicting the definition of $r$. Thus, $N_b((\rho/r)z)$ has no zeros or poles in $\rho\mathbb{D}$.

*ILWHF to WHF.* $M_b(z)$ has no zeros or poles in $\rho\mathbb{D}$, therefore $M_b((r/\rho)z)$ has no zeros or poles in $(\rho^2/r)\mathbb{D}$ and, since $\rho^2/r > \rho$, $M_b((r/\rho)z)$ has no zeros or poles in $\rho\mathbb{D}$. On the other hand, $M_f(z)$ has no zeros or poles in $\rho\mathbb{D}^c$, thus $M_f((r/\rho)z)M_0(r/\rho)$ has no zeros or poles in $(\rho^2/r)\mathbb{D}^c$. If $M_f((r/\rho)z)M_0(r/\rho)$ had a zero or pole in $\rho\mathbb{D}^c \cap (\rho^2/r)\mathbb{D}$, this would translate to a zero or pole of $M((r/\rho)z)$ in the same region, or a zero or pole of $M(z)$ in $r\mathbb{D}^c \cap \rho\mathbb{D}$, contradicting the definition of $r$. Thus, $M_f((r/\rho)z)M_0(r/\rho)$ has no zeros or poles in $\rho\mathbb{D}^c$. □

*Proof of Theorem 3.1.* (i) Existence of ILWHF follows from the existence of the WHFs mentioned in Proposition 3.1. In each case, the matrix function to be factorized is a non-singular rational matrix function with no zeros or poles on the contour with respect to which it is to be factorized. The result then follows from Theorem 1.6 of Gohberg et al. (2003).

(ii) By Proposition 3.1 (i), both ILWHFs can be considered WHFs relative to a contracted contour. The result then follows from the corresponding result for WHF (e.g. Theorem I.1.1 of Clancey & Gohberg (1981)). □

*Proof of Theorem 3.2. The "if" part.* Since $M_f(z)$ is a polynomial in $z^{-1}$, its elements can have poles only at zero. Thus, $M_f(z)$ has no poles in $\rho\mathbb{D}^c$. On the other hand, $\det(M_f(z^{-1})) \neq 0$ for all $z \in \rho^{-1}\overline{\mathbb{D}}$ if and only if $\det(M_f(z)) \neq 0$ for all $z \in \rho\mathbb{D}^c$ so $M_f(z)$ can have no zeros in $\rho\mathbb{D}^c$. A similar argument proves that $M_b(z)$ can have no zeros or pole in $\rho\mathbb{D}$.

*The "only if" part.* This follows from the proof of Theorem A.1 of the online supplement.

The highest power of $z$ achievable in the factorization, $\deg(M_b(z)) + \kappa_1$, must be bounded above by $p$ because $M(z) \in \mathbb{R}^{n\times n}_{pq}(z)$. Thus $\kappa_1 \leq p$ and $\deg(M_b(z)) \leq p$ whenever $\kappa_1 = 0$. By a similar argument, $-\deg(M_f(z^{-1})) + \kappa_n \geq -q$, which implies that $\kappa_n \geq -q$ and $\deg(M_f(z^{-1})) \leq q$ whenever $\kappa_n = 0$. It follows that $\deg(M_b(z)) - q \leq \deg(M_b(z)) + \kappa_n \leq \deg(M_b(z)) + \kappa_1 \leq p$, which implies that $\deg(M_b(z)) \leq p + q$. The bound on $\deg(M_f(z^{-1}))$ is proven similarly. □

*Proofs of Lemmas 4.1 and 4.2.* The proofs involve routine applications of power series methods and the usual "limsupery" as Williams (1991) puts it. Therefore they are relegated to Section B of the online supplement. □

*Proof of Theorem 4.1.* First note that by Definition 3.3 (i), $M_f^{-1}(z)$ has no poles in $\mathbb{D}^c$. It follows from Lemma 4.1 (iii) that $\{E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) : t \in \mathbb{Z}\} \in \mathcal{S}^n$.

(i) By Lemma 4.2, the observation above and (2) imply that $X \in \mathcal{S}^n$. Since the right hand side of (2) is $\mathscr{I}_t$-measurable and $M_b(0)$ is invertible, $X$ is adapted to $\mathscr{I}$. Thus Definition 4.2 (ii) is satisfied. To see that Definition 4.2 (iv) is satisfied, apply the operator $E(M_f(L)(\cdot)|\mathscr{I}_t)$ to both sides of (2) and use Lemma 4.1 (ii). Finally, let $\hat{X}$ be another solution so that

$$E(M(L)(X_t - \hat{X}_t)|\mathscr{I}_t) = E(M(L)X_t|\mathscr{I}_t) - E(M(L)\hat{X}_t|\mathscr{I}_t) = 0 \quad \text{a.s.} \quad t \geq 0. \tag{6}$$

Since $X - \hat{X} \in \mathcal{S}^n$, $M(L)(X - \hat{X}) \in \mathcal{S}^n$. Applying the operator $E(M_f^{-1}(L)(\cdot)|\mathscr{I}_t)$ to both sides of (6) and using Lemma 4.1 (ii), we have that $E(M_b(L)(X_t - \hat{X}_t)|\mathscr{I}_t) = 0$ a.s. for all $t \geq 0$. But since $X$ and $\hat{X}$ are adapted to $\mathscr{I}$, $M_b(L)(X_t - \hat{X}_t) = 0$ a.s. for all $t \geq 0$. Finally, Lemma 4.2 (ii) implies that $\hat{X}$ is indistinguishable from $X$.

(ii) Take the expected value of (3) with respect to $\mathscr{I}_t$, the result then follows from exactly the same argument as used in (i).

(iii) Suppose a solution $(X, \mathscr{I})$ exists. Then applying the operator $E(M_f(L)(\cdot)|\mathscr{I}_t)$ to both sides of Definition 4.2 (iv) we obtain

$$E(M_0(L)M_b(L)X_t|\mathscr{I}_t) = E(M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t) \quad \text{a.s.} \quad t \geq 0. \tag{7}$$

If any partial index is positive then $\kappa_1 > 0$ and the first equation of (7) can be written as $e_1' M_b(L) X_{t-\kappa_1} = E(e_1' M_f^{-1}(L)\varepsilon_t|\mathscr{I}_t)$ a.s. for all $t \geq 0$, where $e_1 = (1, 0, \ldots, 0)' \in \mathbb{R}^n$. For $t = 0$ in particular, $e_1' M_b(L) X_{-\kappa_1} = E(e_1' M_f^{-1}(L)\varepsilon_0|\mathscr{I}_0)$ a.s. Since $M_b(0)$ and $M_f(\infty)$ are invertible, it is always possible to choose $\varepsilon$ and/or initial conditions of $X$ that violate this last equation and therefore equation (7) as well. □

*Proof of Corollary 4.1.* Follows from (2). □

*Proof of Theorem 4.2.* The result is proven by repeated application of the Argument Principle (Ahlfors, 1979, Section 5.2). By Theorem 3.2 (i), $\det(M_f(z))$ can have zeros only in $\mathbb{D}$ and since it is also a polynomial in $z^{-1}$ with $\det(M_f(\infty)) \neq 0$, it has an equal number of zeros and poles inside $r\mathbb{T}$. Thus, the winding number of $\det(M_f(z))$ about the origin is zero. By Theorem 3.2 (iii), $\det(M_b(z))$ is analytic and non-zero inside and on $r\mathbb{T}$, thus it winds zero times around the origin. Finally, the winding number of $\det(M_0(z)) = z^{\sum_{i=1}^n \kappa_i}$ around the origin is $\sum_{i=1}^n \kappa_i$. Adding it all up, the total number of times that $\det(M(z))$ winds around the origin is $\sum_{i=1}^n \kappa_i$. By Theorem A.3 of the online supplement, the set of non-singular $M(z) \in \mathbb{R}_{pq}^{n\times n}(z)$ with $\kappa_1 \leq \kappa_n + 1$ is generic. Thus, for a generic LREM, $\sum_{i=1}^n \kappa_i$ is zero, negative, or positive, according to whether the partial indices are all zero, all non-positive with some negative, or all non-negative with some positive respectively. □

*Proof of Corollary 4.2.* By the Argument Principle again, if $r$ is as in Proposition 3.1, the number of times that $\det(M(z))$ winds around the origin as $r\mathbb{T}$ is traversed counter clockwise is the number of zeros of $\det(M(z))$ in $r\mathbb{D}$ minus the number of poles of $\det(M(z))$ in $r\mathbb{D}$. $\square$

*Proof of Proposition 5.1.* The solution is unstable if and only if $M_b(z)$ has a zero on $\mathbb{T}$. However, for $|z_0| \geq 1$, $M_f(z_0)$ is invertible and, therefore, such a $z_0$ can be a zero of $M_b(z)$ if and only if it is also a zero of $M(z)$. $\square$

*Sketch of Proof of Proposition 5.2.* Using basic state space techniques, it is easy to show that $F_{t+1|t}, \Psi P_t \in l_n^\infty$ a.s. (see Section B of the online supplement). Since $M(z)$ has no zeros or poles on $\mathbb{T}$, its ILWHF relative to $\mathbb{T}$ is also a WHF and since, additionally, its partial indices are zeros, this implies that $\Theta$ is an invertible operator on $l_n^\infty$ (Gohberg & Fel'dman, 1974, Theorem VIII.4.2). Thus $F_{t+1|t}$ is a.s. the unique solution in $l_n^\infty$ of $\Theta F_{t+1|t} + \Psi P_t = 0$. $\square$

*Proof of Proposition 5.3.* By the definition of the ILWHF relative to $\mathbb{T}$, $M_f(z)$ is invertible for $z \in \mathbb{D}^c$ and so $M_f(1)$ is invertible. Thus, $M_b(1) = M_f^{-1}(1)M(1)$ it follows that $M_f'(1)\alpha_\perp$ and $\beta_\perp$ span the left and right null spaces of $M_b(1)$ respectively. Theorem 4.2 of Johansen (1995) then implies that the result is true if and only if $\det\left(\alpha_\perp' M_f(1)M_b^*(1)\beta_\perp\right) \neq 0$, where $M_b^*(z) = (1-z)^{-1}(M_b(z) - M_b(1)z)$. Finally, $\alpha_\perp' M_f(1)M_b^*(1)\beta_\perp = \alpha_\perp' M_f(1)\left(-\frac{dM_b(1)}{dz} + M_b(1)\right)\beta_\perp = -\alpha_\perp' M_f(1)\frac{dM_b(1)}{dz}\beta_\perp = -\alpha_\perp'\left(M_f(1)\frac{dM_b(1)}{dz} + \frac{dM_f(1)}{dz}M_b(1)\right)\beta_\perp = -\alpha_\perp'\frac{dM(1)}{dz}\beta_\perp = \alpha_\perp'(M^*(1) - M(1))\beta_\perp = \alpha_\perp' M^*(1)\beta_\perp$. $\square$

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