Comparative Statics in the Multiple-Partners Assignment Game

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COMPARATIVE STATICS
IN THE MULTIPLE-PARTNERS ASSIGNMENT GAME[1]

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ABSTRACT

The multiple partners game (Sotomayor, 1992) extends the assignment game to a matching model where the agents can have several partners, up to their quota, and the utilities are additively separable. The present work fills a gap in the literature of that game by studying the effects on agents' payoffs caused by the entrance of new agents in the market under both the cooperative and the competitive approaches. The results obtained have no parallel in the one-to-one assignment game.

Keywords: matching, stability, competitive equilibrium, comparative statics.

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1. INTRODUCTION

Two-sided matching markets where a set of possibly heterogeneous agents from one side meet with another set of possibly heterogeneous agents from the other side are very common in practice. Examples are the markets for firms and workers, buyers and sellers, and venture capital funds and start-ups. These environments can be approached cooperatively and competitively.

Distinct matching models have been proposed by several authors to represent such markets, aiming to give them some mathematical treatment that leads to a better understanding of their organization as real markets. These models differ in the structure of the agents' preferences as well as in the rules of the market. The simplest one was introduced by Shapley and Shubik (1972) in terms of buyers and sellers, and it is referred to as the assignment game. The main assumption is that each agent can form one partnership at most and utility is identified with money. That paper shows that the core of this market is non-empty and is a complete lattice. It contains a special allocation that gives, among all core allocations, the highest payoff to each buyer and the lowest payoff to each seller, and another allocation with symmetric properties. Moreover, the core of the game coincides with the set of stable allocations and with the set of competitive equilibrium allocations.[4]

A generalization of the assignment game was obtained by Demange and Gale (1985), by allowing the utility functions, although continuous in the money variable, to not necessarily be linear. Several papers propose other extensions of this game by assuming that the agents from one side or from both sides of the market can form several partnerships and can negotiate their payoffs, either as a block or individually (we can cite Kelso and Crawford, 1982, Roth, 1984, Kaneko, 1982, Thompson, 1980, Sotomayor, 1992, 2002a, and 2002b, among others). These are the so-called many-to-one and many-to-many matching models, respectively.

The simplest many-to-many matching model was introduced in Sotomayor (1992) and it is obtained by introducing quotas in the assignment game. The quota of an agent is the maximum number of partnerships the agent can enter. The main

[4] The competitive one-to-one market was proposed in Gale (1960), who proved the existence of equilibrium prices.

characteristic of this game is that players negotiate their individual payoffs: if agents i and j belong to opposite sides and become partners, they undertake an activity together that produces a gain a_{ij}, which is divided between them the way both agree: u_{ij} for i and $v_{ij} = a_{ij} - u_{ij}$ for j. Therefore, an outcome of this game is a matching, that is, a set of partnerships that does not violate the quotas of the players, along with individual payoffs u_{ij}'s and v_{ij}'s.

This model is called multiple-partners assignment game (*multiple partners game*, for short). Sotomayor (1999a) shows that the set of stable outcomes of the multiple partners game is a non-empty complete lattice, although it does not coincide with the core of the game, which is a larger set and it is not always a lattice (Sotomayor, 1992).

The competitive approach of the multiple partners game is considered in Sotomayor (2007b), through an economic structure in terms of buyers and sellers supported on a general concept of competitive equilibrium. It can be interpreted that each seller has a number of identical objects to sell and each buyer wants to acquire a number of distinct objects. Roughly speaking, facing a vector of prices, one price for each object, the buyers behave like price-takers by demanding the most preferred bundles of distinct objects, of a size up to their quotas. The preferences of the buyers over the bundles are defined by their total payoffs. Sotomayor (2007b) shows that the set of competitive equilibrium allocations in this environment also forms a complete lattice, which is a sublattice (and may be distinct) of that of the stable allocations.[5]

The present work fills a gap in the literature of the multiple partners game by studying the effects on the agents' payoffs caused by the entrance of new agents in the market. This is a problem of economic interest as these effects capture fundamental differences and similarities between the roles played by agents on opposite sides of the market.

In the literature of matching games, meaningful comparative static results of adding agents to the market have been approached in models where the core is endowed with a complete lattice structure. It is then assumed that the agents are allocated according to one of the extreme points of that lattice, because these allocations always

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[5] Pérez-Castrillo and Sotomayor (2017) study the manipulability of competitive equilibrium allocation rules in the multiple partners game.

exist and can be obtained via some well-known algorithms. For the assignment game, the comparative static result is given by Proposition 8.17 of Roth and Sotomayor (1990) and it is the restriction, to this model, of the proposition proved in Demange and Gale (1985). It asserts that when agents' payoffs correspond to one of the extreme points of the lattice of the core, it is the case that if some firms (or buyers) enter the market, no current firm is made better off and no current worker (or seller) is made worse off. Equivalently, if some workers enter the market, no current worker will be made better off and no current firm will be made worse off.

The core, the set of stable payoffs, and the set of competitive equilibrium payoffs coincide in the one-to-one matching model. Therefore, the previous comparative static effects hold for the three sets.[6]

Given that the core of the multiple partners game is not always a complete lattice, and that the existence of the optimal core allocations is an open problem (see Sotomayor, 1992), we will follow an different approach from the previous authors. We will compare core allocations that are not optimal for either of the two sides, but are

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[6] For the assignment game, two important references besides Demange and Gale (1985) are Shapley (1962) and Mo (1988). Shapley (1962) shows that the optimal core payoff for an agent weakly decreases when another agent is added to the same side and weakly increases when another agent is added to the other side. Mo (1988) proves that if the incoming firm is allocated to a worker in some core outcome for the new market, there is a set of agents such that every worker is better off and every firm is worse off in the new market than in the previous one. The symmetric result is valid when the incoming agent is a worker.

Comparative statics results for the marriage markets are obtained in Gale and Sotomayor (1985) and Roth and Sotomayor (1990). For the discrete many-to-one matching market, Kelso and Crawford (1982) show that, within the context of firms and workers, the addition of firms to the market weakly improves the workers' payoffs, and the addition of workers weakly improves the firms' payoffs, under the firm-optimal core allocation (the case of the worker-optimal core allocation was not studied). Crawford (1991) obtains some comparative statics result for a discrete many-to-many matching model, for the firm-optimal and the worker-optimal pairwise-stable outcomes. However, pairwise-stable outcomes may be out of the core in that model (Blair, 1988), so they may be unstable (Sotomayor, 1999b). Finally, Sotomayor (2007a) extends some comparative static results to a hybrid model which includes the marriage market and the assignment game.

optimal stable or optimal competitive, for one side. These allocations always exist and can be reached by using an algorithm.[7]

Intuitively, one feels that there is no connection between the comparative static effects on the agents' payoffs and the assumption of the model concerning their quotas. We show that this intuition turns out to be right:[8] *In the extreme points of either the lattice of the stable payoffs or the lattice of the competitive equilibrium payoffs of the multiple partners game, the entrance of new firms cannot make the current firms better off or the current workers worse off. And the reverse happens after the entrance of new workers.*

This result is obtained here as a corollary of a non-intuitive and stronger theorem that has no parallel in the one-to-one case. The theorem states the comparative statics effects on the agents' individual trades. Specifically, we first prove the following:

When agents are allocated according to one of the extreme points of the lattice of the stable payoffs, it is always the case that if some firms enter the market, no current firm will be able to make a trade with a new partner obtaining a higher individual payoff than some of its current individual payoffs, nor will she be able to make a more profitable trade with one of its current partners. The opposite conclusion holds for the workers. Moreover, a symmetric result holds if it is assumed that some workers enter the market.

We also prove a second theorem that states *the same result for the extreme points of the lattice of the competitive equilibrium payoffs* instead of the lattice of the stable payoffs.

The proof of these results involves the comparison for each agent of his/her/its set of individual payoffs at some allocation in the original market, with his/her/its set of individual payoffs at some allocation after the entrance of the new agents. Therefore, we must define, for each agent, some ordering of the individual payoffs in both sets. The technical difficulty is that the allocations in comparison belong to distinct markets which do not share the same set of optimal matchings. Then, the existent vectorial

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[7] Sotomayor (2009) presents an algorithm that produces the buyer-optimal stable allocation. It is proved in Sotomayor (2007) that this allocation always coincides with the buyer-optimal competitive equilibrium allocation.

[8] However, the proofs are not generalizations of existing proofs.

representation of the stable allocations defined in Sotomayor (2007b) cannot be used, since some coordinates of both vectors may be indexed by different optimal matchings and so the vectors cannot be compared. We solve this problem by providing a new ordering of the set of individual payoffs of all agents in both allocations. Moreover, we show that, under such a vectorial representation, if the highest payoff of some agent is obtained in, say, the first vector, then this vector is greater than or equal to the second vector in each component.

The paper is organized as follows. Section 2 introduces the framework for the labor market of heterogeneous firms and workers operating cooperatively. Section 3 presents and proves the comparative statics results for the model described in section 2. Section 4 analyzes the comparative statics effects on the optimal competitive equilibrium allocations for each side of the market, in terms of buyers and sellers. The final remarks are given in section 5.

2. FRAMEWORK FOR THE LABOR MARKET OF HETEROGENEOUS FIRMS AND WORKERS

We will think of the multiple partners game as a labor market of firms and workers. There are two finite and disjoint sets of agents, $F = \{i_1, i_2,\ldots, i_m\}$, the set of firms, and $W = \{j_1, j_2,\ldots, j_n\}$, the set workers. We will use the letters i and j to represent, generically, any element of F and W, respectively. Each agent has a quota representing the maximum number of partnerships he/she/it can enter. Thus, the quota $s(j)$ of worker j is the maximum number of jobs he/she can take. The quota $r(i)$ of firm i is the maximum number of workers it can hire.

Without loss of generality, we assume that every agent has a reservation utility of 0. For each pair (i, j), there is a non-negative number $a_{ij} \geq 0$, representing the productivity of worker j in firm i. We also assume that agents' preferences are separable across pairs, in the sense that the payoff from a partnership does not depend on the other partnerships formed. If firm i hires worker j at salary v_{ij}, then its individual payoff in this transaction is $u_{ij} = a_{ij} - v_{ij}$ whereas worker j receives v_{ij}. For notational simplicity, both sets F and W include one dummy firm and one dummy worker, both players are denoted by 0. The quotas of the dummy agents are enough to guarantee that the non-dummy agents in the market can fill their quotas. The productivity of any worker at the dummy firm, as well as the productivity of the dummy worker at any firm is zero. That is, $a_{0j} = a_{i0} = 0$ for all $j \in W$ and all $i \in F$.

The market described above will often be denoted by M, or $M = (F, W, a)$, without reference to the quotas of the players when this simplification does not lead to confusion.

Definition 2.1. *A* ***feasible matching*** *is a* $m \times n$ *matrix* $x = (x_{ij})_{(i,j) \in F \times W}$ *of non-negative integer numbers such that* $x_{ij} \in \{0, 1\}$ *if* $i \in F - \{0\}$ *and* $j \in W - \{0\}$*. Furthermore,* $x_{00} = 0$, $\sum_F x_{ij} = s(j)$ *for all* $j \in W - \{0\}$ *and* $\sum_W x_{ij} = r(i)$ *for all* $i \in F - \{0\}$*.*[9]

If $x_{ij} > 0$ (respectively, $x_{ij} = 0$), then we say that firm i and worker j are (respectively, are not) matched at x. An *allowable set of partners* for firm $i \in F - \{0\}$ is an array with $r(i)$ distinct workers, except some of them may be repetitions of the dummy worker. Similarly, an *allowable set of partners* for worker $j \in W - \{0\}$ is an array with $s(j)$ distinct firms, except some of them may be repetitions of the dummy firm. Therefore, a feasible matching specifies an allowable set of partners for each agent. Given a feasible matching x, we denote by $C_i(x)$ the allowable set of partners assigned to firm i at x and by $C_j(x)$ the allowable set of partners assigned to worker j at x. The set of pairs $(i, j) \in F \times W$ that are assigned to each other at x is denoted by $C(x)$.

A matching x is *optimal* if it attains the maximum value among all feasible matchings. Formally,

Definition 2.2. *A* ***matching*** x *is* ***optimal*** *if* (a) *it is feasible and* (b) $\sum_{F \times W} a_{ij}\, x_{ij} \geq \sum_{F \times W} a_{ij}\, x'_{ij}$ *for all feasible matchings* x'*.*

Definition 2.3. *Given two feasible matchings,* x *and* x'*, we say that* ***firm*** i ***is a non-essential partner of worker*** j ***at*** x ***with respect to*** x' *if* $i \in C_j(x) - C_j(x')$*;* ***worker*** j ***is***

[9] We will use the notation $\sum_A$ to denote the sum over all elements of A.

a non-essential partner of firm i at x with respect to x' *if $j \in C_i(x) - C_i(x')$. In both cases, we say that (i, j) is a non-essential partnership at x with respect to x'.*

Definition 2.4. *A **feasible allocation** for M, denoted by (u, v, x), is a feasible matching x and a pair of payoffs (u, v), where the individual payoffs of each $i \in F$ and $j \in W$ are given by the arrays of numbers $u_{ij} \geq 0$ and $v_{ij} \geq 0$, respectively, only defined if $x_{ij} > 0$, and such that $u_{ij} + v_{ij} = a_{ij}$. Consequently, $u_{i0} = u_{0j} = v_{i0} = v_{0j} = 0$ in case these payoffs are defined.*

If $(u, v; x)$ is a feasible allocation we say that (u, v) is compatible with x and vice-versa.

The total payoff of firm i and worker j is denoted by $U_i \equiv \Sigma_{j\in C_i(x)}\, u_{ij}$ and $V_j \equiv \Sigma_{i\in C_j(x)}\, v_{ij}$. Also, $\underline{u}_i \equiv \min\{u_{ij}; j \in C_i(x)\}$ denotes the smallest individual payoff of firm i and $\underline{v}_j \equiv \min\{v_{ij}; i \in C_j(x)\}$ denotes the smallest individual payoff of worker j.

The key concept is that of a stable allocation.[10] In Sotomayor (1992) it is proved that this notion is equivalent to the following:

Definition 2.5. *The allocation $(u, v; x)$ is **stable** for M if it is feasible and, for all (i, j) such that $x_{ij} = 0$,*

$$\underline{u}_i + \underline{v}_j \geq a_{ij}. \qquad (1)$$

If $(u, v; x)$ is stable then we say that x is a *stable matching* and (u, v) is a *stable payoff compatible with x.*

The interpretation of Definition 2.5 is standard. If condition (1) is not satisfied then there would be some pair of agents (i, j) who are not partners but who could both get a higher payoff by forming a partnership while at the same time dissolving one of

[10] In the present model, the idea of stability is captured by the concept of setwise-stability. An allocation is setwise-stable if there is no coalition of players who, by forming new partnerships only among themselves − possibly dissolving some partnerships to remain within their quotas and possibly keeping other partnerships − can all obtain a higher payoff. Setwise-stability is equivalent to pairwise-stability (Sotomayor, 1992).

their current partnerships (so as to stay within their quotas). In this case, we say that (i, j) *destabilizes* the allocation, or that (i, j) is a destabilizing pair.

The existence of stable allocations was originally proved in Sotomayor (1992).[11]

NOTATION: If $\sigma = (u, v, x)$ is a feasible allocation then $\{u_{ij}\}_{j\in C_i(x)}$ and $\{v_{ij}\}_{i\in C_j(x)}$ stands for the arrays of individual payoffs of firm i and worker j, respectively. Sotomayor (1999a) shows that if σ is stable then we can order the elements of $\{u_{ij}\}_{j\in C_i(x)}$ and $\{v_{ij}\}_{i\in C_j(x)}$, for all $(i, j) \in F\times W$, in some convenient way, so that the resulting vectors are independent of x. We will denote such vectors by $\sigma_i \in R^{r(i)}$ and $\sigma_j \in R^{s(j)}$, respectively. We will keep the same notation $\sigma = (u, v, x)$ for $u = (\sigma_i)_{i\in \mathrm{F}}$ and $v = (\sigma_j)_{j\in \mathrm{W}}$, when this does not cause any confusion.

It is proved in Sotomayor (1992) that *every stable matching is optimal*. The vectorial representation of the set of individual payoffs of the agents allows *the set of stable allocations to be regarded as the Cartesian product of the set of stable payoffs by the set of optimal matchings*. Therefore, we can characterize *the optimal matchings as the stable matchings*. That is,

Proposition 2.1 (Sotomayor, 1999a, 1992). (a) *Let $(u, v; x)$ be a stable allocation and x' an optimal matching. Then $(u, v; x')$ is also a stable allocation.*

(b) *If $(u, v; x)$ is a stable allocation then x is an optimal matching.*

Definition 2.6. *A stable payoff is called a* ***firm-optimal stable payoff*** *if every firm weakly prefers it to any other stable payoff. We define the* ***worker-optimal stable payoff*** *similarly.*

That is, a firm-optimal stable payoff gives to each firm the maximum total

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[11] There are two existence proofs in Sotomayor (1992). One of them uses linear programming duality theory. The other one is based on a replication of the agents, the number of times of their quotas, together with a convenient income matrix that reduces the model to a related one-to-one assignment game. There is also another proof in Sotomayor (2009) given by a mechanism that mimics an auction procedure.

payoff among all stable payoffs whereas a worker-optimal stable payoff gives to each worker the maximum total payoff among all stable payoffs.

By using the vectorial representation of the agents' individual payoffs, mentioned above, Sotomayor (1999a) defines two partial order relations, $\geq_F$ and $\geq_W$, in the set of stable payoffs, such that if σ and τ are stable payoffs then $\sigma \geq_F \tau$ if $\sigma_i \geq \tau_i$ for all $i \in F$ and $\sigma \geq_W \tau$ if $\sigma_j \geq \tau_j$ for all $j \in W$. She shows that there is a polarization of interests between the two sides of the market along the whole set of stable payoffs, that is, $\sigma \geq_F \tau$ if and only if $\tau \geq_W \sigma$, for all stable payoffs σ and τ. Sotomayor (1999a) also shows that the set of stable payoffs is endowed with a complete lattice structure under each partial order, where one is the dual of the other. As a consequence, there exists one and only one maximal element and one and only one minimal element in each lattice. Due to the polarization of interests between firms and workers, the maximal element of the lattice under $\geq_F$ is the minimal element of the lattice under $\geq_W$, and vice versa. Formally,

Proposition 2.2 (Sotomayor, 1999a). *Let* σ^+ *be the maximal element of the lattice of stable payoffs under the partial order* $\geq_F$ *and* σ_+ *be the maximal element of the lattice of stable payoffs under the partial order* $\geq_W$*. Then,* $\sigma^+_i \geq \sigma_i$, $\sigma_j \geq \sigma^+_j$, $\sigma_i \geq \sigma_{+I}$ *and* $\sigma_{+j} \geq \sigma_j$ *for every stable payoff* σ *and all* $(i, j) \in F{\times}W$.

Corollary 2.1. *Let* σ^+ *be the maximal element of the lattice of stable payoffs under the partial order* $\geq_F$ *and* σ_+ *be the maximal element of the lattice of stable payoffs under the partial order* $\geq_W$*. Then,* σ^+ *is the firm-optimal stable payoff and* σ_+ *is the worker-optimal stable payoff.*

In an allocation, the payoff that a firm obtains with the workers it hires may depend on the identity of the workers. Also, a worker who is hired by several firms may obtain a different individual payoff from each of them. When the payoff of a worker is the same in all the firms he/she works for, we say that the allocation is *F-non-discriminatory*; and similarly for *W-non-discriminatory* allocations. That is,

Definition 2.7. (a) *The feasible allocation* $(u, v; x)$ *is* ***F-non-discriminatory*** *if*

$v_{ij} = v_j$ *for all* $j \in W$ *and* $i \in C_j(x)$.

(b) *The feasible allocation* $(u, w; x)$ *is* ***W-non-discriminatory*** *if*

$u_{ij} = u_i$ *for all* $i \in F$ *and* $j \in C_i(x)$.

(c) *The feasible allocation* $(u, w; x)$ *is* **non-discriminatory** *if it is F-non-discriminatory and W-non-discriminatory.*

Proposition 2.3 (Sotomayor, 2007b). (a) *Let* $(u, v; x)$ *be a firm-optimal stable payoff. Then,* $(u, v; x)$ *is an F-non-discriminatory stable allocation.*

(b) *Let* $(u, v; x)$ *be a worker-optimal stable payoff. Then,* $(u, v; x)$ *is a W-non-discriminatory stable allocation.*

3. COMPARATIVE STATIC EFFECTS ON THE FIRM-OPTIMAL AND WORKER-OPTIMAL STABLE ALLOCATIONS

In this section, we will consider the cooperative game structure of the labor market of firms and workers introduced in section 2 and we will analyze the comparative static effects when some agents from one side are added to the market. More specifically, we are going to provide comparative static effects on the payoffs of firms and workers in the firm-optimal and worker-optimal stable allocations when there is entry of a group of firms or a group of workers.

To obtain our main results, we first study the comparison between the payoffs of firms and workers in stable allocations in two markets with the same sets of agents but that are different because, in the second market, some of the agents on one side become non-productive. We develop this analysis in subsection 3.1 and state our main results in subsection 3.2.

3.1. PRELIMINARY RESULTS

In this subsection, we study two markets that involve the set of firms $F = F^1 \cup F^2$ and the set of workers W. We denote the markets $M \equiv (F, W, a)$ and $M' \equiv (F, W, a')$. The only difference between M and M' is that the firms in the subset F^2 have a productivity of 0 with any worker in M' but not necessarily in M. That is, $a'_{ij} = a_{ij}$ for all $(i, j) \in F^1 \times W$ and $a'_{ij} = 0$ for all $i \in F^2$ and $j \neq 0$. Also, we consider any stable allocations $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ of M and M', respectively. Without loss of generality, we assume that $x'_{ij} = 0$ for all $i \in F^2$ and $j \neq 0$.

To simplify, when there is no confusion, given feasible allocations $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$, for M and M', respectively, we sometimes write "non-essential partners at x," meaning "non-essential partners at x with respect to x'." Also, we denote C_h and C'_h the sets $C_h(x)$ and $C_h(x')$, for $h \in F \cup W$.

Given the stable allocations τ and τ', we denote $F(\tau)$ the set of firms in F^1 that have non-essential partners at x and have a higher minimum individual payoff under τ than under τ'. Similarly, we denote $W(\tau')$ the set of workers who have non-essential partners at x and a higher minimum individual payoff under τ' than under τ. Formally,

$F(\tau) \equiv \{i \in F^1;\ u_i > u'_i \text{ and } C_i \neq C'_i\}$,

$W(\tau') \equiv \{j \in W;\ v'_j > v_j \text{ and } C_j \neq C'_j\}$.

Lemma 3.1 shows that all the non-essential partners at x of the firms in $F(\tau)$ have a higher minimum individual payoff under τ' than under τ. It also states that all the non-essential partners at x' with respect to x of the workers in $W(\tau')$ have a higher minimum individual payoff under τ than under τ'.

Lemma 3.1. *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be stable allocations for M and M', respectively. Then,*

(a) *if $i \in F(\tau)$ then $i \neq 0$ and $C_i - C'_i \subseteq W(\tau')$;*

(b) *if $j \in W(\tau')$ then $j \neq 0$ and $C'_j - C_j \subseteq F(\tau)$.*

Proof. (a) Let $i \in F(\tau)$. Then, $C_i \neq C'_i$, so $C_i - C'_i \neq \emptyset$. Let $j \in C_i - C'_i$. Then, $u_{ij} \geq u_i > u'_i \geq 0$, which implies $i \neq 0$ and $j \neq 0$. Suppose by contradiction that $j \notin W(\tau')$. Given that $j \in C_i - C'_i$, it is also true that $i \in C_j - C'_j$; hence $C_j \neq C'_j$. Therefore, $j \notin W(\tau')$ requires $v_j \geq v'_j$. It would then be the case that $a_{ij} = u_{ij} + v_{ij} \geq u_i + v_j > u'_i + v'_j \geq a'_{ij}$, by stability of $(u', v'; x')$ and the fact that $j \notin C'_i$. Now use that $a'_{ij} = a_{ij}$ for $i \in F^1$ to get a contradiction. Hence, $j \in W(\tau')$ and then $C_i - C'_i \subseteq W(\tau')$.

(b) Let $j \in W(\tau')$. Then, $C'_j \neq C_j$, so $C'_j - C_j \neq \emptyset$. Let $i \in C'_j - C_j$. Then, $v'_{ij} \geq v'_j > v_j \geq 0$, so $j \neq 0$ and $i \neq 0$. This implies that $i \in F^1$ (because x' associates every $i \in F^2$ to 0). We claim that $i \in F(\tau)$. Otherwise (given that $C'_i \neq C_i$ holds) it would be the case that $u'_{ij} \geq u'_i \geq u_i$. Then, $a'_{ij} = u'_{ij} + v'_{ij} > u_i + v_j \geq a_{ij}$, by stability of $(u, v; x)$ and the fact that $j \notin C_i$. Now use that $a'_{ij} = a_{ij}$ for $i \in F^1$ to get a contradiction. Therefore, $i \in F(\tau)$ and then $C'_j - C_j \subseteq F(\tau)$. ■

Using the previous framework, let α denote the set of all non-essential partnerships (i, j) at x that involve firms in $F(\tau)$. Similarly, let β denote the set of all non-essential partnerships (i, j) at x' for which $j \in W(\tau')$. That is,

$$\alpha \equiv \{(i, j);\ i \in F(\tau) \text{ and } j \in C_i - C'_i\} \text{ and } \beta \equiv \{(i, j);\ j \in W(\tau') \text{ and } i \in C'_j - C_j\}.$$

Proposition 3.1 asserts that the set α coincides with the set of all non-essential partnerships at x that involve workers in $W(\tau')$ and that the set β coincides with the set of all non-essential partnerships at x' that involve firms in $F(\tau)$. Therefore, if (i, j) is a non-essential partnership at x or at x', then $i \in F(\tau)$ if and only if $j \in W(\tau')$.

In the proof of Proposition 3.1 we use Lemma 3.2, which states that at both matchings x and x' the total number of non-essential partners for the firms in $F(\tau)$ is equal to the total number of non-essential partners for the workers in $W(\tau')$.

Lemma 3.2. *Consider the stable allocations τ and τ' for the markets M and M'. Then:*

$\sum_{i \in F(\tau)} |C_i - C'_i| = \sum_{j \in W(\tau')} |C_j - C'_j|$ *and*

$\sum_{j \in W(\tau')} |C'_j - C_j| = \sum_{i \in F(\tau)} |C'_i - C_i|$.

Proof. Since $C_i - C'_i \subseteq W(\tau')$ for all $i \in F(\tau)$ by Lemma 3.1 (a), we have that

$$\sum_{i \in F(\tau)} |C_i - C'_i| \leq \sum_{j \in W(\tau')} |C_j - C'_j|. \qquad (2)$$

Similarly, because $C'_j - C_j \subseteq F(\tau)$ for all $j \in W(\tau')$ by Lemma 3.1 (b), we have:

$$\sum_{j \in W(\tau')} |C'_j - C_j| \leq \sum_{i \in F(\tau)} |C'_i - C_i|. \qquad (3)$$

On the other hand, $|C_i - C'_i| = |C'_i - C_i|$ for all $i \in F$ and $|C'_j - C_j| = |C_j - C'_j|$ for all $j \in W$. Then, by (2) and (3), we obtain

$$\sum_{i \in F(\tau)} |C_i - C'_i| \leq \sum_{j \in W(\tau')} |C_j - C'_j| = \sum_{j \in W(\tau')} |C'_j - C_j| \leq \sum_{i \in F(\tau)} |C'_i - C_i| = \sum_{i \in F(\tau)} |C_i - C'_i|. \quad (4)$$

Therefore, all inequalities in (4) are equalities. In particular, $\sum_{i \in F(\tau)} |C_i - C'_i| = \sum_{j \in W(\tau')} |C_j - C'_j|$ and $\sum_{j \in W(\tau')} |C'_j - C_j| = \sum_{i \in F(\tau)} |C'_i - C_i|$. ■

We can now state and prove Proposition 3.1.

Proposition 3.1. *Let* $\tau = (u, v; x)$ *and* $\tau' = (u', v'; x')$ *be stable allocations for* M *and* M'*, respectively. Then,*

$\alpha = \{(i, j);\ j \in W(\tau')$ *and* $i \in C_j - C'_j\}$ *and*

$\beta = \{(i, j);\ i \in F(\tau)$ *and* $j \in C'_i - C_i\}$.

Proof. First, notice that according to Lemma 3.1, $C_i - C'_i \subseteq W(\tau')$ for all $i \in F(\tau)$. Also, $j \in C_i - C'_i$ implies that $i \in C_j - C'_j$, Therefore, $\alpha \equiv \{(i, j);\ i \in F(\tau)$ and $j \in C_i - C'_i\} \subseteq \{(i, j);\ j \in W(\tau')$ and $i \in C_j - C'_j\}$. Then,

$$\sum_{i \in F(\tau)} |C_i - C'_i| = |\alpha| \leq |\{(i, j); j \in W(\tau') \text{ and } i \in C_j - C'_j\}| = \sum_{j \in W(\tau')} |C_j - C'_i|. \qquad (5)$$

Under Lemma 3.2, the inequality in (5) must be equality, so $\alpha = \{(i, j);\ j \in W(\tau')$ and $i \in C_j - C'_j\}$.

For the second equality of the proposition, Lemma 3.1 also implies that $C'_j - C_j \subseteq F(\tau)$ for all $j \in W(\tau')$, so $\beta \subseteq \{(i, j);\ i \in F(\tau)$ and $j \in C'_i - C_i\}$. Using that $\sum_{j \in W(\tau')} |C'_j - C_j| = \sum_{i \in F(\tau)} |C'_i - C_i|$ by Lemma 3.2, we get that $\beta = \{(i, j);\ i \in F(\tau)$ and $j \in C'_j - C_i\}$. ■

Proposition 3.1 implies that if (i, j) is a non-essential partnership at x or at x' then firm $i \in F^1$ and $u_i > u'_i$ if and only if $v'_j > v_j$. Proposition 3.2, which is our next result, adds that if (i, j) is a non-essential partnership at x (x', respectively), and $i \in F(\tau)$ or $j \in W(\tau')$, then i and j obtain their minimum individual payoff in τ (τ', respectively) in their partnership.

Proposition 3.2. *Let* $\tau = (u, v; x)$ *and* $\tau' = (u', v'; x')$ *be stable allocations for* M *and* M'*, respectively. Then,*

$u_{ij} = u_i$ *and* $v_{ij} = v_j$ *for all* $(i, j) \in \alpha$ *and*

$u'_{ij} = u'_i$ *and* $v'_{ij} = v'_j$ *for all* $(i, j) \in \beta$.

Proof. We can write:

$$\sum_\alpha a_{ij} = \sum_\alpha (u_{ij} + v_{ij}) = \sum_{i \in F(\tau)} \sum_{j \in C_i - C'_i} u_{ij} + \sum_{j \in W(\tau')} \sum_{i \in C_j - C'_j} v_{ij} \geq$$

$$\sum_{i \in F(\tau)} \sum_{j \in C_i - C'_i} u_i + \sum_{j \in W(\tau')} \sum_{i \in C_j - C'_j} v_j = \sum_{i \in F(\tau)} |C_i - C'_i|\, u_i + \sum_{j \in W(\tau')} |C_j - C'_j|\, v_j =$$

$$\sum_{i \in F(\tau)} |C'_i - C_i|\, u_i + \sum_{j \in W(\tau')} |C'_j - C_j|\, v_j =$$

$$\sum_{i \in F(\tau)} \sum_{j \in C'_i - C_i} u_i + \sum_{j \in W(\tau')} \sum_{i \in C'_j - C_j} v_j = \sum_\beta (u_i + v_j) \geq \sum_\beta a_{ij}, \qquad (6)$$

where the first equality in (6) follows from the feasibility of $(u, v; x)$ and the definition of the set α, the second equality follows from Proposition 3.1, and the last inequality follows from the stability of $(u, v; x)$ and the fact that $x_{ij} = 0$ for every $(i, j) \in \beta$. Then,

$$\textstyle\sum_{\alpha} a_{ij} \geq \sum_{\beta} a_{ij}. \qquad (7)$$

Also,

$$\textstyle\sum_{\beta} a_{ij} = \sum_{\beta} (u'_{ij} + v'_{ij}) = \sum_{i \in F(\tau)} \sum_{j \in C'_i - C_i} u'_{ij} + \sum_{j \in W(\tau')} \sum_{i \in C'_j - C_j} v'_{ij} \geq$$

$$\textstyle\sum_{i \in F(\tau)} \sum_{j \in C'_i - C_i} u'_i + \sum_{j \in W(\tau')} \sum_{i \in C'_j - C_j} v'_j = \sum_{i \in F(\tau)} |C'_i - C_i|\, u'_i + \sum_{j \in W(\tau')} |C'_j - C_j|\, v'_j =$$

$$\textstyle\sum_{i \in F(\tau)} |C_i - C'_i|\, u'_i + \sum_{j \in W(\tau')} |C_j - C'_j|\, v'_j =$$

$$\textstyle\sum_{i \in F(\tau)} \sum_{j \in C_i - C'_i} u'_i + \sum_{j \in W(\tau')} \sum_{i \in C_j - C'_j} v'_j = \sum_{\alpha} (u'_i + v'_j) \geq \sum_{\alpha} a_{ij}, \qquad (8)$$

where the first equality in (8) follows from the feasibility of $(u', v'; x')$ and the definition of the set β, the second equality follows from Proposition 3.1, and the last inequality follows from the stability of $(u', v'; x')$ and the fact that $x'_{ij} = 0$ in β. Then,

$$\textstyle\sum_{\beta} a_{ij} \geq \sum_{\alpha} a_{ij}. \qquad (9)$$

Therefore, $\sum_{\alpha} a_{ij} = \sum_{\beta} a_{ij}$ by (7) and (9), which implies that all inequalities are equalities in (6) and (8). Hence, $u_{ij} = u_i$ and $v_{ij} = v_j$ for all $(i, j) \in \alpha$. Similarly, $u'_{ij} = u'_i$ and $v'_{ij} = v'_j$ for all $(i, j) \in \beta$, as we wanted to prove. ■

Corollary 3.1 uses the result of Proposition 3.2 to provide additional information concerning the payoffs of firms and workers in the stable allocations of markets M and M'.

Corollary 3.1. *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be stable allocations for M and M', respectively. Then,*

(a) *If $i \in F(\tau)$, $j \in C_i$ and $k \in C'_i - C_i$, then $u_{ij} > u'_{ik}$. If $j \in W(\tau')$, $i \in C'_j$ and $t \in C_j - C'_j$, then $v'_{ij} > v_{tj}$.*

(b) *If $i \in F^1 - F(\tau)$, $j \in C_i - C'_i$ and $k \in C'_i$, then $u'_{ik} \geq u_{ij}$. If $j \in W - W(\tau')$, $i \in C_j$ and $t \in C'_j - C_j$, then $v_{ij} \geq v'_{tj}$.*

(c) *If $i \in F^1$ and $j \in C_i \cap C'_i$, then $u'_{ij} > u_{ij}$ if and only if $v_{ij} > v'_{ij}$; $u_{ij} > u'_{ij}$ if and only if $v'_{ij} > v_{ij}$; and $u'_{ij} = u_{ij}$ if and only if $v_{ij} = v'_{ij}$*

(d) *If $i \in F^1$ and $W(\tau') = \varnothing$ then $u'_{ik} \geq u_{ij}$ for all $k \in C'_i$ and $j \in C_i - C'_i$.*

Proof. (a) Let $i \in F(\tau)$, $j \in C_i$ and $k \in C'_i - C_i$. Then, $u_{ij} \geq u_i > u'_i = u'_{ik}$, where the equality follows from Proposition 3.2 because $(i, k) \in \beta$ under Proposition 3.1. For the other assertion, let $j \in W(\tau')$, $i \in C'_j$ and $t \in C_j - C'_i$. Then, $v'_{ij} \geq v'_j > v_j = v_{tj}$, where the equality follows from Proposition 3.2.

(b) Let $i \in F^1 - F(\tau)$, $j \in C_i - C_i(x')$, and $k \in C'_i$. According to Proposition 3.1, $i \notin F(\tau)$ implies $j \notin W(\tau')$. Hence, $v_j \geq v'_j$. Now suppose, by way of contradiction, that $u_{ij} > u'_{ik}$. Then, we have an absurd because $a_{ij} = u_{ij} + v_{ij} > u'_{ik} + v_j \geq u'_i + v'_j \geq a'_{ij} = a_{ij}$, where the last inequality is implied by the stability of $(u', v'; x')$ and the fact that $j \notin C'_i$. Hence, $u'_{ik} \geq u_{ij}$.

For the other assertion, take $j \in W - W(\tau')$, $i \in C_j$ and $t \in C'_j - C_j$. Proposition 3.1 implies that $t \notin F(\tau)$. Then, $u'_t \geq u_t$ if $t \in F^1$ and $u_t \geq u'_t = 0$ if $t \in F^2$. In the last case, $j = 0$, so $v'_{tj} = 0$ and then $0 = v_{ij} \geq v'_{tj}$. In the former case, suppose by way of contradiction that $v'_{tj} > v_{ij}$. Then, we get an absurd because $a_{tj} = u'_{tj} + v'_{tj} > u'_t + v_{ij} \geq u_t + v_j \geq a_{tj}$, where the last inequality is implied by the stability of $(u, v; x)$ and the fact that $t \notin C_j$. Hence, $v_{ij} \geq v'_{tj}$.

(c) The proof follows from the fact that if $j \in C_i \cap C'_i$ and $i \in F^1$ then $u_{ij} + v_{ij} = a_{ij} = a'_{ij} = u'_{ij} + v'_{ij}$.

(d) The proof is immediate from (b) and (c) since $W(\tau') = \varnothing$ implies, first, that $F(\tau) = \varnothing$ by Proposition 3.1 and, second, that $v_j \geq v'_j$ for all $j \in W$. ■

Our next results concern properties of allocations that are constructed as follows. We start with two allocations: $\tau = (u, v; x)$ for M and $\tau' = (u', v'; x')$ for M'. Also, we assume that τ and τ' are either F-non-discriminatory or W-non-discriminatory, that is, the agents of a given side of the market do not discriminate among the agents of the other side. Then, we construct two new allocations that are based on either the minimum payoffs for the workers or the minimum payoffs for the firms between their payoffs in τ and τ'. For this purpose, we first define the matching y so that it agrees with x on $F(\tau) \times W(\tau')$ and with x' otherwise. Also, we define the matching z so that it agrees with x' on $F(\tau) \times W(\tau')$ and with x otherwise. Under Proposition 3.1, matchings y and z are feasible for M' and M, respectively. We construct the new allocations in two cases:

Case 1. *τ and τ' are F-non-discriminatory stable allocations for M and M',* respectively. Then, define $\tau^{\#} = (u^{\#}, v_{\#}; y)$, where $v_{\#ij} = v_{\#j} = \min\{v_j, v'_j\}$ for all $j \in W$ and $i \in C_j(y)$, and $u^{\#}$ is feasibly defined.

Case 2. *τ and τ' are W-non-discriminatory stable allocations for M and M',* respectively. Then, define $\tau_{\#} = (u_{\#}, v^{\#}; z)$, where $u_{\#ij} = u_{\#i} = \min\{u_i, u'_i\}$ for all $i \in F^1$ and $u_{\#ij} = u_{\#i} = u_i$ for all $i \in F^2$ and all $j \in C_i(z)$, and $v^{\#}$ is feasibly defined.

Remark 3.1. (a) Notice that if $i \in F^2$ then $C_i(y) = C_i(x')$. Hence, $j = 0$ if $y_{ij} > 0$, so $u^{\#}_{ij} = 0$ and $v_{\#j} = 0$. In this case, $u^{\#}_i = u'_i = 0$. Also, we claim that if $i \in F^2$ and $x_{ij} > 0$ then $j \notin W(\tau')$. In fact, if $j \in W(\tau')$ then $j \neq 0$, so $i \in C_j(x) - C_j(x')$. Then, under Proposition 3.1, $i \in F(\tau)$, which implies that $i \notin F^2$, in contradiction with $i \in F^2$. Therefore, we have that $v_j \geq v'_j$, so $v_{\#j} = v'_j$.

(b) Notice that it follows from Proposition 3.1 that if $i \in F^1 - F(\tau)$ then all non-essential partners of i at x and x' are in $W - W(\tau')$. Also, if $j \in W - W(\tau')$ then all non-essential partners of j at x and x' are in $F^1 - F(\tau)$.

The allocation $\tau^{\#}$ (the allocation $\tau_{\#}$, respectively) is defined by taking the minimum of the payoffs for the workers (the firms, respectively) in the stable allocations τ and τ' when τ and τ' are F-non-discriminatory (W-non-discriminatory, respectively). Proposition 3.3 provides information about the payoffs for the firms (workers, respectively) in $\tau^{\#}$ ($\tau_{\#}$, respectively).

Proposition 3.3 (a) *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be F-non-discriminatory stable allocations for M and M', respectively. Then, $u^{\#}_{ij} = \max\{u_{ij}, u'_{ij}\}$ for all $i \in F^1$ and $j \in C_i \cap C'_i$. Furthermore, if $i \in F^1$ we must have that $u^{\#}_i \geq \max\{u_i, u'_i\}$.*

(b) *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be W-non-discriminatory stable allocations for M and M', respectively. Then, $v^{\#}_{ij} = \max\{v_{ij}, v'_{ij}\}$ for all $j \in W$ and $i \in C_j \cap C'_j$. Furthermore, if $j \in W$ we must have that $v^{\#}_j \geq \max\{v_j, v'_j\}$.*

Proof. We will prove (a); the proof of part (b) follows dually.

Let $i \in F^1$ and $j \in C_i \cap C'_i$. By definition of $v_{\#}$, if $v'_j \geq v_j$ then $v_{\#ij} = v_{\#j} = v_j$, and if $v_j \geq v'_j$ then $v_{\#ij} = v_{\#j} = v'_j$. In the first case, Corollary 3.1 (c) implies that

$u_{ij} \geq u'_{ij}$. Then, also using that $(u, v; x)$ is F-non-discriminatory, we get that $u^{\#}_{ij} = a_{ij} - v_{\#ij} = a_{ij} - v_j = u_{ij} \geq u'_{ij}$. In the second case, Corollary 3.1 (c) implies that $u'_{ij} \geq u_{ij}$. Then, using similar arguments to those above, we get that $u^{\#}_{ij} = u'_{ij} \geq u_{ij}$. Hence, $u^{\#}_{ij} = \max\{u_{ij}, u'_{ij}\}$.

For the second assertion, if $i \in F(\tau)$ then $u_i > u'_i$ and $C_i - C'_i \neq \varnothing$ by definition of $F(\tau)$. Take $j \in C_i - C'_i$. Then, $(i, j) \in \alpha$. According to Proposition 3.1, $j \in W(\tau')$, so $v_{\#j} = v_j$ and $y_{ij} = x_{ij} > 0$; hence, $u^{\#}_{ij} = u_{ij}$ by definition of $u^{\#}$. On the other hand, Proposition 3.2 implies that $u_{ij} = u_i$. Thus, $u^{\#}_{ij} = u_i > u'_i$ for all $j \in C_i - C'_i$. Furthermore, if $k \in C_i \cap C'_i$, it follows from the previous assertion that $u^{\#}_{ik} = \max\{u_{ik}, u'_{ik}\}$, so $u^{\#}_{ik} \geq u_{ik} \geq u_i = u^{\#}_{ij}$. Therefore, $u^{\#}_i = u_i = \max\{u_i, u'_i\}$.

Now suppose $i \in F^1 - F(\tau)$. Then $u'_i \geq u_i$. We therefore have to show that $u^{\#}_{ij} \geq u'_i$ for all $j \in C_i(y) = C'_i$. If $j \in C'_i - C_i$ then, as follows from Remark 3.1, we have that $j \in W - W(\tau')$, so $v_{\#j} = v'_j$ and $u^{\#}_{ij} = u'_{ij}$ by definition of $u^{\#}$. Moreover, $u'_{ij} \geq u'_i$, so $u^{\#}_{ij} \geq u'_i$. If $j \in C_i \cap C'_i$, it follows from the previous assertion that $u^{\#}_{ij} \geq u'_{ij} \geq u'_i$. Then, $u^{\#}_{ij} \geq u'_i$ for all $j \in C_i(y)$. In particular, $u^{\#}_i \geq u'_i \geq \max\{u_i, u'_i\}$, and the proof is complete. ■

Our last result in this subsection states that the allocation $\tau^{\#}$ that we have defined for the situations where τ and τ' are F-non-discriminatory stable allocations for M and M' is also F-non-discriminatory stable for M'. It states a symmetric result for the allocation $\tau_{\#}$.

Lemma 3.3. (a) *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be F-non-discriminatory stable allocations for M and M', respectively. Then, $\tau^{\#} = (u^{\#}, v_{\#}; y)$ is a F-non-discriminatory stable allocation for M'.*

(b) *Let $\tau = (u, v; x)$ and $\tau' = (u', v'; x')$ be W-non-discriminatory stable allocations for M and M', respectively. Then, $\tau_{\#} = (u_{\#}, v^{\#}; z)$ is a W-non-discriminatory stable allocation for M.*

Proof. We will prove (a); the proof of part (b) follows dually. We have to prove that (I) $\tau^{\#}$ is feasible for M', (II) $v_{\#ij} = v_{\#j}$ for all $i \in C_j(y)$, and (III) $u^{\#}_i + v_{\#j} \geq a'_{ij}$ for all $(i, j) \in F \times W$ such that $y_{ij} = 0$.

Under Proposition 3.1, y is a feasible matching for M'. Furthermore, the individual payoffs are defined so that the allocation $\tau^{\#}$ is compatible with y in the market M'. Then, condition (I) is satisfied. Also, condition (II) is implied by the definition of $\tau^{\#}$.

To prove that (III) holds, take $(i, j) \in F \times W$ such that $y_{ij} = 0$. If $i \in F^2$, then $a'_{ij} = 0$ for all $j \in W$. The non-negativity of $u^{\#}_i$ and $v_{\#j}$ implies that $u^{\#}_i + v_{\#j} \geq 0 = a'_{ij}$, so condition (III) is satisfied. If $i \in F^1$, Proposition 3.3 asserts that $u^{\#}_i \geq \max\{u_i, u'_i\}$. Then, condition (III) is clearly satisfied if $(i, j) \in F(\tau) \times W(\tau')$ or $(i, j) \in F^1 \times (W - W(\tau'))$. In the first case, $u^{\#}_i + v_{\#j} = u^{\#}_i + v_j \geq u_i + v_j \geq a_{ij} = a'_{ij}$, where the last inequality follows from the stability of $(u, v; x)$ and the fact that $x_{ij} = y_{ij} = 0$. In the second case, $u^{\#}_i + v_{\#j} = u^{\#}_i + v'_j \geq u'_i + v'_j \geq a'_{ij}$, where the last inequality follows from the stability of $(u', v'; x')$ and the fact that $x'_{ij} = y_{ij} = 0$. For the remaining case, where $i \in F^1 - F(\tau)$ and $j \in W(\tau')$, we have that $y_{ij} = x'_{ij} = 0$. Also, $x_{ij} = 0$ (because if $x_{ij} > 0$ then it would be case that $j \in C_i - C'_i$ and Proposition 3.1 would imply that $i \in F(\tau)$, which is a contradiction). Then, $u^{\#}_i + v_{\#j} = u^{\#}_i + v_j \geq u_i + v_j \geq a_{ij} = a'_{ij}$, where the last inequality follows from the stability of $(u, v; x)$ and the fact that $x_{ij} = 0$. Hence, $\tau^{\#}$ is F-non-discriminatory stable allocation for M' and the proof is complete. ■

3.2. MAIN RESULTS FOR THE COMPARATIVE STATIC EFFECTS ON STABLE ALLOCATIONS

In this subsection, we use the previous properties to obtain our main results, which involve the comparative static effects on the firm-optimal and worker-optimal payoffs when some firms or some workers are added to the market.

When we study the effect of the entrance of a set of firms F^2 we denote, as in the previous subsection, $M \equiv (F, W, a)$ and $M' \equiv (F, W, a')$, where $F = F^1 \cup F^2$, $a'_{ij} = a_{ij}$ for all $(i, j) \in F^1 \times W$ and $a'_{ij} = 0$ for all $i \in F^2$ and $j \neq 0$. Also, the original market, before the entrance of new firms, is denoted by $M^1 = (F^1, W, a^1)$, where $a^1_{ij} = a_{ij}|_{(F^1 \times W)}$. If x^1 is a feasible matching for M^1, we denote $C^1_h \equiv C_h(x^1)$ for all $h \in F^1 \cup W$.

By exchanging the roles between firms and workers, we can define the markets M and M' accordingly: $W = W^1 \cup W^2$, $M \equiv (F, W, a)$ and $M' \equiv (F, W, a')$, where $a'_{ij} = a_{ij}$ for all $(i, j) \in F \times W^1$ and $a'_{ij} = 0$ for all $i \in F$ and $j \in W^2$. By the symmetry of the model, all the results of subsection 3.1 hold for these markets. Thus, the original market, before the entrance of new workers, is denoted by $M^1 = (F, W^1, a^1)$, where $a^1{}_{ij} = a_{ij}|_{(F\times W^1)}$.

We prove that the comparative static result obtained in Demange and Gale (1985) for the one-to-one assignment game generalizes to the multi partners assignment game: when agents are allocated according to the firm-optimal or the worker-optimal stable payoffs, it is always the case that if some firms enter the market, no current firm will be made better off and no current worker will be made worse off. Equivalently, as follows from the symmetry of the model, if some workers enter the market, no current worker will be made better off and no current firm will be made worse off. Indeed, Theorem 3.1 provides a stronger result which has that generalization as an immediate corollary. Theorem 3.1 concerns not only the change in the agents' total payoffs in the extreme points of the lattice of the stable payoffs, but also the change in the individual payoffs of each agent in these allocations. As an illustration, Theorem 3.1 establishes that if the agents are allocated according to the firm-optimal or the worker-optimal stable allocations, and some firms are added to the market, then for any of the firms present in the original market, any of its current individual payoffs are at least as high as any of its ex-post individual payoffs obtained with non-essential partners. Also, with any of its essential partners, such a firm obtains an individual payoff that is at least as high in the original market as in the new market.

In order to state Theorem 3.1, it is convenient to redefine the vectorial representation of a firm's individual payoffs as follows. Consider any firm $i \in F^1$ and any allocations τ^1 and τ of M^1 and M, respectively. If, say, $|C^1{}_i \cap C_i| = p$ then we can reindex the elements of W so that $\{j_1, j_2,\ldots, j_p\} = C^1{}_i \cap C_i$. We can then represent the p first coordinates of allocation $\tau^1{}_i$ of M^1 and allocation τ_i of M as $u^1{}_{i_1}, u^1{}_{i_2},\ldots, u^1{}_{i_p}$ and $u_{i_1}, u_{i_2},\ldots, u_{i_p}$, respectively. The remaining $r(i)$-p coordinates in $\tau^1{}_i$ and τ_i are arbitrarily ordered. Analogously, we redefine the vectorial representation of $\tau^1{}_j$ and τ_j for all $j \in W$. Such a new ordering of $\tau^1{}_h$ and τ_h, for all $h \in F^1 \cup W$, will be said to *preserve h's essential partnerships*.

Theorem 3.1. (a) *Let $\tau = (u, v; x)$ be a firm-optimal (worker-optimal, respectively) stable allocation for market $M = (F^1 \cup F^2, W, a)$. Let $\tau^1 = (u^1, v^1; x^1)$ be a firm-optimal* *(worker-optimal, respectively) stable allocation for market* $M^1 = (F^1, W, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F^1\times W)}$. *Suppose* τ *and* τ^1 *preserve the agents' essential partnerships. Then, whatever the order of the agents' non-essential partners in* τ *and* τ^1, *we have that* $\tau^1_i \geq \tau_i$ *and* $\tau_j \geq \tau^1_j$ *for all* $i \in F^1$ *and* $j \in W$.

(b) *Let* $\sigma = (u, v; x)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M = (F, W^1 \cup W^2, a)$. *Let* $\sigma^1 = (u^1, v^1; x^1)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M^1 = (F, W^1, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F\times W^1)}$. *Suppose* σ *and* σ^1 *preserve the agents' essential partnerships. Then, whatever the order of the agents' non-essential partners in* σ *and* σ^1, $\sigma_i \geq \sigma^1_i$ *and* $\sigma^1_j \geq \sigma_j$ *for all* $i \in F$ *and* $j \in W^1$.

Proof. (a) Suppose that $\tau = (u, v; x)$ and $\tau^1 = (u^1, v^1; x^1)$ are firm-optimal stable allocations for M and M^1, respectively. The proof of the case in which τ and τ^1 are worker-optimal stable allocations for M and M^1, respectively, follows dually. Let $\tau' = (u', v'; x')$ be some firm-optimal stable allocation for market $M' = (F, W, a')$. Clearly, $x'|_{(F^1\times W)}$ is an optimal matching for M^1. Then, without loss of generality we can set $x^1 \equiv x'|_{(F^1\times W)}$.

Notice that, by construction of the markets, any stable allocation for M^1 can be extended to a stable allocation of M' by assigning the firms in F^2 to the dummy worker, with payoffs of 0. This way, the allocation τ^1 can be extended to a stable allocation for M'. By the firm optimality of τ' in M' and Proposition 2.2 we get that

$$u'_{ij} \geq u^1_{ij} \text{ for all } i \in F^1 \text{ and } j \in C'_i \text{ and}$$
$$v'_{ij} \leq v^1_{ij} \text{ for all } j \in W \text{ and } i \in C'_j. \tag{10}$$

Also, the restriction of any stable allocation of M' to M^1 is a stable allocation for M^1. Then, such a restriction of τ' is a stable allocation for M^1. By the firm optimality of τ^1 in M^1 and Proposition 2.2 we get that

$$u^1_{ij} \geq u'_{ij} \text{ for all } i \in F^1 \text{ and } j \in C'_i \text{ and}$$
$$v^1_{ij} \leq v'_{ij} \text{ for all } j \in W \text{ and } i \in C'_j. \tag{11}$$

Under (10) and (11) we have

$$u^1_{ij} = u'_{ij} \text{ for all } i \in F^1 \text{ and } j \in C'_i \text{ and}$$
$$v^1_{ij} = v'_{ij} \text{ for all } j \in W \text{ and } i \in C'_j. \tag{12}$$

According to Proposition 2.3 (a), the firm-optimal stable allocations are F-non-discriminatory. Then, Lemma 3.3 (a) implies that $\tau^{\#} = (u^{\#}, v_{\#}; y)$ (defined using τ and τ') is stable and F-non-discriminatory for M'. Therefore, $(u^{\#}, v_{\#})$ can be indexed by x' (see Proposition 2.1). Thus, using (12), the firm optimality of τ' in M', and Proposition 2.2, it follows that

$$u^1_{ij} \geq u^{\#}_{ij} \text{ and } v^1_{ij} \leq v_{\#ij} \text{ for all } i \in F^1 \text{ and } j \in C'_i. \qquad (13)$$

By definition, $v_{\#j} \leq v_j$, so $v^1_j \leq v_j$ for all $j \in W$ (here, it is used that both allocations $(u^{\#}, v_{\#}; x')$ and $(u^1, v^1; x')$ are F-non-discriminatory). Since $(u, v; x)$ is F-non-discriminatory, we must have that $\tau_j \geq \tau^1_j$ for all $j \in W$ whatever the order of the non-essential partners in τ_j and τ^1_j.

For the other assertion, notice that $v'_j = v^1_j \leq v_j$ for all $j \in W$ implies that $W(\tau')$ $= \varnothing$. If $i \in F^1$, $j \in C'_i - C_i$, and $k \in C_i - C'_i$, Corollary 3.1 (d) implies that $u^1_{ij} = u'_{ij} \geq u_{ik}$. If $i \in F^1$ and $j \in C_i \cap C'_i$ then, since $v'_j \leq v_j$, Corollary 3.1 (c) implies that $u^1_{ij} = u'_{ij} \geq u_{ij}$. Hence, whatever the order of the non-essential partners in τ^1_i and τ_i, it follows that $\tau^1_i \geq \tau_i$ for all $i \in F^1$.

(b) The proof of part (b) is obtained by reversing the roles between firms and workers in the proof of part (a) and by taking $\tau_{\#} = (u_{\#}, v^{\#}; z)$ instead of $\tau^{\#} = (u^{\#}, v_{\#}; y)$. Hence the proof of Theorem 3.1 is complete. ■

Remark 3.2. Notice that while Theorem 3.1 (a) proves that $\tau^1_i \geq \tau_i$ for all $i \in F^1$, Corollary 3.1 (d) allows us to state a stronger result: $\min_{j \in C_i(x^1)}\{u^1_{ij}\} \geq \max_{j \in C_i(x) - C_i(x^1)}\{u_{ij}\}$ for all $i \in F^1$. Then if, say, firm i gets the set of individual payoffs $\{4, 5, 3\}$ in the original market, it cannot expect to obtain an individual payoff larger than 3 with any new partner in the new market.

The generalization to many-to-many matching models of the comparative static result obtained in Demange and Gale (1985) for the one-to-one model is stated in Corollary 3.2. Additionally, Corollary 3.3 states the comparative static effects on the minimum and maximum individual payoffs of firms and workers.

Corollary 3.2. (a) *Let* $\tau = (u, v; x)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M = (F^1 \cup F^2, W, a)$. *Also, let* $\tau^1 = (u^1, v^1; x^1)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M^1 = (F^1, W, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F^1 \times W)}$. *Then,* $U^1_i \geq U_i$ *for all* $i \in F^1$ *and* $V^1_j \leq V_j$ *for all* $j \in W$.

(b) *Let* $\sigma = (u, v; x)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M = (F, W^1 \cup W^2, a)$. *Also, let* $\sigma^1 = (u^1, v^1; x^1)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M^1 = (F, W^1, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F \times W^1)}$. *Then,* $U^1_i \leq U_i$ *for all* $i \in F^1$ *and* $V^1_j \geq V_j$ *for all* $j \in W$.

Proof. (a) For all $h \in F^1 \cup W$, take any reordering of τ_h and τ^1_h that preserves h's essential partnerships. Then, part (a) follows immediately from the property established in Theorem 1 (a) that $\tau^1_i \geq \tau_i$ for all $i \in F^1$ and $\tau^1_j \leq \tau_j$ for all $j \in W$.

(b) For all $h \in F \cup W^1$ take any reordering of σ_h and σ^1_h that preserves h's essential partnerships. Then, part (b) follows from the property established in Theorem 1 (b) that $\sigma^1_i \leq \sigma_i$ for all $i \in F$ and $\sigma^1_j \geq \sigma_j$ for all $j \in W^1$. ■

Corollary 3.3. (a) *Let* $\tau = (u, v; x)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M = (F^1 \cup F^2, W, a)$. *Also, let* $\tau^1 = (u^1, v^1; x^1)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M^1 = (F^1, W, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F^1 \times W)}$.. *Then:*

$$max_{j \in C^1_i}\{u^1_{ij}\} \geq max_{j \in C_i}\{u_{ij}\} \text{ and } min_{j \in C^1_i}\{u^1_{ij}\} \geq min_{j \in C_i}\{u_{ij}\} \text{ for all } i \in F^1;$$
$$max_{i \in C^1_j}\{v^1_{ij}\} \leq max_{i \in C_j}\{v_{ij}\} \text{ and } min_{i \in C^1_j}\{v^1_{ij}\} \leq min_{i \in C_j}\{v_{ij}\} \text{ for all } j \in W.$$

(b) *Let* $\sigma = (u, v; x)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M = (F, W^1 \cup W^2, a)$. *Also, let* $\sigma^1 = (u^1, v^1; x^1)$ *be a firm-optimal (worker-optimal, respectively) stable allocation for market* $M^1 = (F, W^1, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F \times W^1)}$. *Then:*

$$max_{j \in C^1_i}\{u^1_{ij}\} \leq max_{j \in C_i}\{u_{ij}\} \text{ and } min_{j \in C^1_i}\{u^1_{ij}\} \leq min_{j \in C_i}\{u_{ij}\} \text{ for all } i \in F;$$
$$max_{i \in C^1_j}\{v^1_{ij}\} \geq max_{i \in C_j}\{v_{ij}\} \text{ and } min_{i \in C^1_j}\{v^1_{ij}\} \geq min_{i \in C_j}\{v_{ij}\} \text{ for all } j \in W^1.$$

Proof. Immediate from Theorem 3.1. (Given two vectors A and B, if $A \geq B$ then the max and the min of the components of A are larger than or equal to the max and the min, respectively, of the components of B). ■

4. COMPARATIVE STATIC EFFECTS ON THE BUYER-OPTIMAL AND SELLER-OPTIMAL COMPETITIVE EQUILIBRIUM ALLOCATIONS

In this section, we will look at players as being buyers and sellers and will analyze the comparative static effects on the competitive (instead of the cooperative) game structure of the buyer-seller market, when agents from the same side are added to the market. Then, we will interpret F as a set of buyers and W as a set of sellers. The rest of the model described in section 2 has the natural interpretation. For completeness, we repeat here some of the elements of the model. In particular, each seller $j \in W$ has $s(j)$ identical objects to sell, which will also be denoted by j. Each buyer $i \in F$ wants to acquire $r(i)$ objects, at most. As before, both sets F and W include one dummy buyer and one dummy seller, both denoted by 0. For this interpretation, we add several null objects, also denoted by 0, owned by the dummy seller, in order to fulfil the demand of the buyers. No buyer can acquire more than one object of the same seller, except from the dummy seller.

The number a_{ij} is the maximum amount of money buyer i will consider paying for an object of seller j. If buyer i purchases object j at price p_j then her individual payoff in this transaction is $u_{ij} = a_{ij} - p_j$ whereas seller j receives $v_{ij} = p_j$. For notational simplification, the market is denoted by $M = (F, W, a)$, without reference to the set of objects.

A *bundle (of objects) for buyer* $i \in F$ is a set with $r(i)$ objects that contains at most one object of the same non-dummy seller (although it may include several null objects). An *assignment* of the buyers to the objects assigns each non-dummy buyer to a bundle of objects for her[12] and each non-null object to one buyer (who might be the dummy buyer). Of course, the dummy buyer may be assigned to any number of objects and the null object may be allocated to any number of buyers. If an object is assigned to a buyer then the seller who owns this object is matched to that buyer. If an object is assigned to the dummy buyer, we say that it is left unsold. Therefore, there is no loss in identifying an assignment between the set of buyers and the set of objects with a matching between F and W. Thus, as before, we will represent a matching

[12] We will refer to a buyer as "she" and to a seller as "he."

(assignment) by a matrix $x = (x_{ij})_{(i,j)\in F\times W}$. If $x_{ij} > 0$ we say, indistinctly, that buyer i is matched to seller j or that one of the objects of seller j is allocated to buyer i at x. For a matter of simplification, if no confusion is caused, $C_i(x)$ will also denote the bundle of objects for buyer i that is allocated to her at matching x. Matching x is feasible if it satisfies Definition 2.1.

In the competitive approach of the market, a *feasible price vector* is a non-negative vector which associates a price to each object and the price 0 to the null objects. The *demand set of buyer i at prices p* is defined as follows:

$D_i(p) = \{S \subseteq W;$ S is a bundle of objects for i and

$\Sigma_{j\in S}(a_{ij} - p_j) \geq \Sigma_{k\in S'}(a_{ik} - p_k)\}$ for any bundle S' of objects for $i\}$.

That is, $S \in D_i(p)$ if i weakly prefers S to any other bundle.

The key concept in this section is that of a *competitive equilibrium allocation*.

Definition 4.1. *A* ***competitive equilibrium*** *is a pair (p, x), where p is a feasible price vector and x is a feasible matching such that $C_i(x) \in D_i(p)$ for all $i \in F$ and $p_j = 0$ if object j is left unsold. A* ***competitive equilibrium allocation*** *is a triple $(u, p; x)$, where (p, x) is a competitive equilibrium and $u_{ij} = a_{ij} - p_j$ for all $i \in F$ and $j \in C_i(x)$.*

Therefore, matching x assigns every buyer to a bundle in her demand set at prices p. If the allocation (p, x) is a competitive equilibrium, we say that p is an *equilibrium price vector*, x is a *competitive matching*, and x is *compatible* with p and vice versa. If $(u, p; x)$ *is a* competitive equilibrium allocation then (u, p) is called a *competitive equilibrium payoff* compatible with x.

Every seller sells all his objects at the same price at a competitive equilibrium (Sotomayor, 2007b). This implies that if $(u, p; x)$ is a competitive equilibrium allocation then it is F-non-discriminatory. The condition that every unsold object has price 0 implies that $(u, p; x)$ is also feasible. Finally, the condition that $C_i(x) \in D_i(p)$ for all $i \in F$ implies that $(u, p; x)$ does not have any destabilizing pair. Therefore, $(u, p; x)$ is an F-non-discriminatory stable allocation. Sotomayor (2007b) proves that the converse of this assertion is also true. That is,

Proposition 4.1 *The feasible allocation $(u, v; x)$ is a competitive equilibrium allocation if and only if it is an F-non-discriminatory stable allocation.*

It is not hard to construct examples of a multiple partners game in which non-discriminatory stable allocations do not exist (see Sotomayor, 2017). Nevertheless, the set of F-non-discriminatory stable allocations (as well as the set of W-non-discriminatory stable allocations) is always non-empty and has the algebraic structure of a complete sublattice of the lattice of the stable allocations (Sotomayor, 2007b). Therefore, there exists the *optimal F-non-discriminatory stable payoff* (and the *optimal W-non-discriminatory stable payoff*) for each side of the market. The F-non-discriminatory stable allocations play a relevant role in the competitive game structure of the buyer-seller market because they are precisely the competitive equilibrium allocations. Thus, the buyer-optimal F-non-discriminatory stable payoff is the *buyer-optimal competitive equilibrium payoff*. Similarly, the seller-optimal F-non-discriminatory stable payoff is the *seller-optimal competitive equilibrium payoff*. Formally,

Definition 4.2. *A competitive equilibrium payoff is called a* ***buyer-optimal competitive equilibrium payoff*** *if every buyer weakly prefers it to any other competitive equilibrium payoff. We similarly define the* ***seller-optimal competitive equilibrium payoff***.

That is, the buyer-optimal (seller-optimal, respectively) competitive equilibrium payoff gives to each buyer (seller, respectively) the maximum total payoff among all competitive equilibrium payoffs.

The central result of this section is Theorem 4.1. It shows that the comparative static effects caused at the extreme points of the lattice of the competitive equilibrium payoffs by the entrance of new agents in the market are similar to those caused at the corresponding extreme points of the lattice of the stable payoffs. However, the proof of Theorem 4.1 is distinct from that of Theorem 3.1. It uses the following propositions from Sotomayor (2007b).

Proposition 4.2. *Let* $(u, v; x)$ *be a stable allocation. Set* $v'_{ij} = v_j$ *and* $u'_{ij} = a_{ij} - v'_{ij}$ *if* $x_{ij} > 0$. *Then,* $(u', v'; x)$ *is a competitive equilibrium allocation. Furthermore, if* $(u, v; x)$ *is an optimal stable allocation for one side of the market then* $(u', v'; x)$ *is the corresponding optimal competitive equilibrium allocation for that side.*

Proposition 4.3. *Let* (u, v) *be the buyer-optimal stable payoff. Then,* (u, v) *is the buyer-optimal competitive equilibrium payoff.*

Sotomayor (2007b) shows that the seller-optimal stable payoff and the seller-optimal competitive equilibrium payoff may be distinct.

Theorem 4.1. (a) *Let* $\tau = (u, v; x)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M = (F^1 \cup F^2, W, a)$*. Also, let* $\tau^1 = (u^1, v^1; x^1)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M^1 = (F^1, W, a^1)$*, where* $a^1{}_{ij} = a_{ij}|_{(F^1 \times Q)}$*. Suppose* τ *and* τ^1 *preserve the agents' essential partnerships. Then, whatever the order of the agents' non-essential partners in* τ *and* τ^1*, we have that* $\tau^1{}_i \geq \tau_i$ *and* $\tau_j \geq \tau^1{}_j$ *for all* $i \in F^1$ *and* $j \in W$*.*

(b) *Let* $\sigma = (u, v; x)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M = (F, W^1 \cup W^2, a)$*. Also, let* $\sigma^1 = (u^1, v^1; x^1)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M^1 = (F, W^1, a^1)$*, where* $a^1{}_{ij} = a_{ij}|_{(F \times Q^1)}$*. Suppose* σ *and* σ^1 *preserve the agents' essential partnerships. Then, whatever the order of the agents' non-essential partners in* σ *and* σ^1*,* $\sigma^1{}_i \leq \sigma_i$ *and* $\sigma_j \leq \sigma^1{}_j$ *for all* $i \in F$ *and* $j \in W^1$*.*

Proof. Under Proposition 4.3, the buyer-optimal stable payoff and the buyer-optimal competitive equilibrium payoff coincide. Then, according to Theorem 3.1, the result holds when the allocations in comparison are buyer-optimal competitive equilibrium allocations.

For the case where the allocations in comparison are seller-optimal competitive equilibrium payoffs, Theorem 3.1 does not apply because these allocations are not necessarily seller-optimal stable payoffs. We will prove part (a) for this case. Part (b) is obtained by reversing the roles between buyers and sellers in the proof of part (a), with the due adaptations. Thus, we assume that the allocations τ and τ^1 are seller-optimal competitive equilibrium allocations for the markets M and M^1, respectively. For this proof, we also use the auxiliary market $M' = (F, W, a')$, where $a'_{ij} = a_{ij}$ for all $(i, j) \in F^1 \times W$ and $a'_{ij} = 0$ for all $i \in F^2$ and $j \in W$. Let $\tau' = (u', v'; x')$ be some seller-optimal competitive equilibrium allocation for market M'. We have that $x'|_{(F^1 \times W)}$

is an optimal matching for M^1. Then, without loss of generality we can set $x^1 \equiv x'|_{(F^1 \times W)}$.

Let $\tau^* = (u^*, v^*; x)$, $\tau^{*1} = (u^{*1}, v^{*1}; x^1)$ and $\tau^{*'} = (u^{*'}, v^{*'}; x')$ be seller-optimal stable allocations for M, M^1, and M', respectively. It follows from Proposition 4.2 that, first, $v_j = v_{ij} = v^*_j$ for all $j \in W$ and $i \in C_j(x)$ and, second, $v^1_j = v^1_{tj} = v^{*1}_j$ and $v'_j = v_{tj}' = v^{*'}_j$ for all $j \in W$ and $t \in C_j(x')$. Corollary 3.3 (a) then implies that $v^*_j \geq v^{*1}_j$, so $v_j \geq v^1_j$ for all $j \in W$. Consequently, $\tau_j \geq \tau^1_j$ for all $j \in W$.

For the other assertion, it is easy to show, by using the seller-optimality of τ^1 in M^1 and of τ' in M', we can identify τ^1 with τ', by assigning the buyers in F^2 to the dummy seller, with payoffs of 0. It then follows that $v^1_j = v'_j$, and so $v_j \geq v'_j$ for all $j \in W$, from which follows that $W(\tau') = \emptyset$. The result then follows from Corollary 3.1 (d), by using that τ and τ' are stable allocations in M and M', respectively. Hence, the proof is complete. ■

The proofs of the following corollaries follow the arguments used in the proofs of Corollary 3.2 and Corollary 3.3, respectively.

Corollary 4.1 (a) *Let $(u, v; x)$ be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M = (F^1 \cup F^2, W, a)$. Also, let $(u^1, v^1; x^1)$ be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M^1 = (F^1, W, a^1)$, where $a^1_{ij} = a_{ij}|_{(B^1 \times Q)}$. Then, $U^1_i \geq U_i$ for all $i \in F^1$ and $V^1_j \leq V_j$ for all $j \in W$.*

(b) *Let $(u, v; x)$ be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M = (F, W^1 \cup W^2, a)$. Also, let $(u^1, v^1; x^1)$ be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M^1 = (F, W^1, a^1)$, where $a^1_{ij} = a_{ij}|_{(B \times Q^1)}$. Then, $U^1_i \leq U_i$ for all $i \in F^1$ and $V^1_j \geq V_j$ for all $j \in W$.*

Corollary 4.2. (a) *Let $(u, v; x)$ be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M = (F^1 \cup F^2, W, a)$. Also, let $(u^1, v^1; x^1)$*

be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market $M^1 = (F^1, W, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(B^1 \times Q)}$. *Then,*

$max_{j \in C_i(x^1)}\{u^1_{ij}\} \geq max_{j \in C_i(x)}\{u_{ij}\}$ *and* $min_{j \in C_i(x^1)}\{u^1_{ij}\} \geq min_{j \in C_i(x)}\{u_{ij}\}$ *for all* $i \in F^1$;
$max_{i \in C_j(x^1)}\{v^1_{ij}\} \leq max_{i \in C_j(x)}\{v_{ij}\}$ *and* $min_{i \in C_j(x^1)}\{v^1_{ij}\} \leq min_{i \in C_j(x)}\{v_{ij}\}$ *for all* $j \in W$.

(b) *Let* $\sigma = (u, v; x)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M = (F, W^1 \cup W^2, a)$. *Also, let* $\sigma^1 = (u^1, v^1; x^1)$ *be a buyer-optimal (seller-optimal, respectively) competitive equilibrium allocation for market* $M^1 = (F, W^1, a^1)$, *where* $a^1_{ij} = a_{ij}|_{(F \times W^1)}$. *Then*

$max_{j \in C_i(x^1)}\{u^1_{ij}\} \leq max_{j \in C_i(x)}\{u_{ij}\}$ *and* $min_{j \in C_i(x^1)}\{u^1_{ij}\} \leq min_{j \in C_i(x)}\{u_{ij}\}$ *for all* $i \in F$;
$max_{i \in C_j(x^1)}\{v^1_{ij}\} \geq max_{i \in C_j(x)}\{v_{ij}\}$ *and* $min_{i \in C_j(x^1)}\{v^1_{ij}\} \geq min_{i \in C_j(x)}\{v_{ij}\}$ *for all* $j \in W^1$.

5. FINAL REMARKS

The theory developed in this paper has allowed us to provide comparative statics results for four specific core allocations: the maximal and the minimal elements of the lattice of the stable payoffs, and the maximal and the minimal points of the lattice of the competitive equilibrium payoffs. Nevertheless, such comparative statics effects are not restricted to these four allocations of the core. Indeed, there are infinitely many pairs of points in the core that reflect the same comparative static effects. For example, any convex combination of the two extreme points of the lattice of the stable payoffs yields the same comparative statics effects as those produced at each of the two extreme allocations. Moreover, since the lattice of the stable payoffs in any market is a convex set in some Euclidean space then the convex combinations are stable allocations in the corresponding markets (Sotomayor, 2007b).

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