Almost Mutually Best in Matching Markets: Rank-Fairness and Size of the Core

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Abstract

This paper studies the one-to-one two-sided marriage model (Gale and Shapley, 1962). If agents' preferences exhibit mutually best, there is a unique stable matching that is trivially rank-fair (i.e., in each matched pair the agents assign one another the same rank). We study in how far this result is robust for matching markets that are "close" to mutually best. Without a restriction on preference profiles, we find that natural "distances" to mutually best neither bound the size of the core nor the rank-unfairness of stable matchings. However, for matching markets that satisfy horizontal heterogeneity, "local" distances to mutually best provide bounds for the size of the core and the rank-unfairness of stable matchings.

Keywords: matching, mutually best, horizontal heterogeneity, stable matching, core, rank-fairness.

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1 Introduction

We study the one-to-one two-sided marriage model (Gale and Shapley, 1962) with the same number of men (or workers) and women (or firms). Each agent has strict and complete

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preferences over the agents on the other side of the market. We assume that being matched is preferred to being unmatched (and consuming some outside option). Hence, each agent's preferences can be represented by a ranking of the agents on the other side of the market.

If the agents' preferences satisfy *vertical heterogeneity*, i.e., there is a strict and objective ranking of men by women and of women by men, then there is a unique stable matching that is moreover rank-fair (within matched pairs). Stability is the central notion in matching theory and refers to the absence of a blocking pair, i.e., a man and a woman that would prefer being matched to one another rather than keeping their current mates. Rank-fairness (Jaramillo et al., 2019) refers to the property that if a man ranks his mate k^{th}, then she also ranks him k^{th}. This can be interpreted as a notion of fairness within a matched pair: no agent can argue that his mate is treated more favorably.

The results in Holzman and Samet (2014) imply a generalization of the previous observation: rank-unfairness and the size or diameter of the set of stable matchings (i.e., the core) are bounded by measures of the diversity of rankings (i.e., the distance to vertical heterogeneity).[1] The size of the core is measured by means of the rank gap that exists between each agent's least preferred and most preferred mate among those that are obtained at stable matchings. Rank-(un)fairness at a stable matching is measured by means of the differences in ranks that matched agents assign to one another. Loosely speaking, if the preference profile is "close" to vertical heterogeneity, then the core is "small" and stable matchings are "almost" rank-fair. As schematically depicted in Figure 1, these findings in Holzman and Samet (2014) hold on the entire domain of marriage problems.

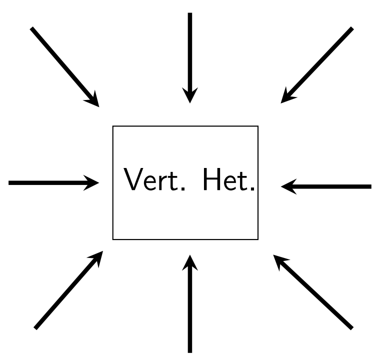


Figure 1: Holzman and Samet (2014) approaching vertical heterogeneity.

The starting point of this paper is a natural horizontal counterpart of vertical heterogeneity: mutually best. A marriage problem satisfies *mutually best* if top ranked agents are reciprocal, i.e., each man's most preferred woman also prefers him most. For instance, mutually best is satisfied if agents can be positioned along two concentric circles (one for each

[1]Jaramillo et al. (2019) prove similar results for a class of roommate problems that is not logically related to the class of marriage problems studied in Holzman and Samet (2014).

gender), and each agent prefers a possible mate to another if the former is closer by than the latter (Burdett and Wright, 1998; Kiyotaki and Wright, 1989; Smith, 2002). Obviously, if preferences are mutually best, there is a unique stable matching, which is moreover rank-fair. Similarly to Holzman and Samet (2014), our goal is to investigate whether preference profiles that are "almost" mutually best have a core that is "small" and stable matchings that are "almost" rank-fair.

Before we summarize our results it is convenient to note that the domain of problems that satisfy mutually best is the intersection of two domains that we explain next: horizontal heterogeneity and the domain in Eeckhout (2000, Theorem 1). *Horizontal heterogeneity* is the condition that all men most prefer different women and all women most prefer different men.[2] The condition in Eeckhout (2000, Theorem 1), which is sufficient but not necessary for the uniqueness of stable matchings, requires preferences to be mutually best on recursively defined subsets of agents.[3] Following Clark (2006), we will refer to this condition as Eeckhout's sequential preference condition (SPC). This condition is satisfied in marriage problems with vertical heterogeneity. For horizontal heterogeneity it is satisfied if and only if preferences satisfy mutually best, which is the case if and only if there is a unique stable matching. The Venn-diagram in Figure 2 summarizes the logical relations between the discussed domains.

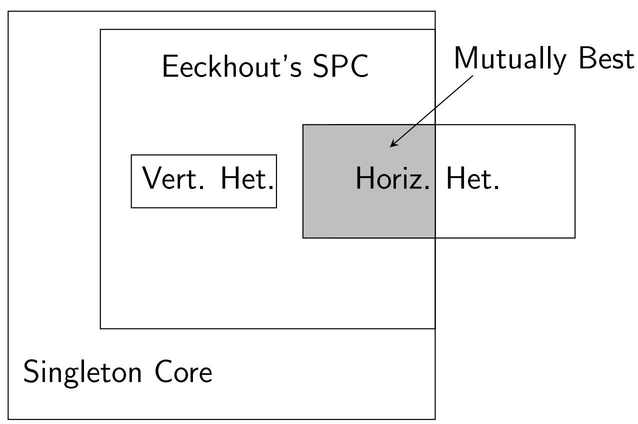


Figure 2: Venn-diagram of domains of preference profiles.

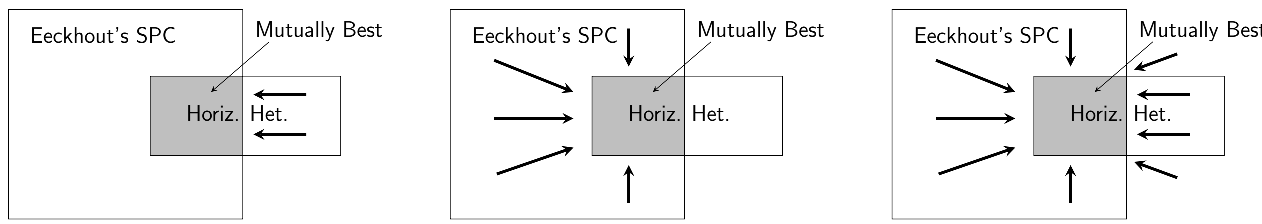
Given a marriage problem, to quantify the extent to which the preference profile violates mutually best (i.e., its "distance" from the domain of mutually best), we introduce two types of measures. The first type of measure is a "local" distance which first computes for each agent the rank with which his/her most preferred mate reciprocates and then aggregates by

[2]In Eeckhout (2000, Corollary 4) "horizontal heterogeneity" refers to our condition of mutually best. Our domain of horizontal heterogeneity contains the domain with the same name in Eeckhout (2000) but also contains profiles where all men prefer different women and all women prefer different men but where not all most preferred mates form mutually best pairs.

[3]The condition requires that agents can be relabeled $m_1, \ldots, m_n$ and $w_1, \ldots, w_n$ such that for each i, man m_i and woman w_i are mutually best after removing agents $m_1, \ldots, m_{i-1}$ and $w_1, \ldots, w_{i-1}$.

using generalized means. The second type of measure is a "global" distance which is based on the computation of a (well-defined) matching that is as close as possible to mutually best by minimizing some generalized mean. We show that these matchings can be obtained in polynomial time by applying the Hungarian algorithm (cf. Kuhn, 1955; Munkres, 1957) to an associated cost-minimizing job assignment problem.

Our goal is to determine whether rank-unfairness and the size of the core are bounded by the distance of the preference profile to the domain of mutually best. In other words, do small distances to mutually best lead to almost rank-fair stable matchings and a small core? In view of the Venn-diagram in Figure 2, it seems natural to study and "approach" mutually best from three different domains: horizontal heterogeneity, Eeckhout's SPC domain, and the general domain (i.e., without any restrictions) – see Figure 3.



(a) From the domain of horizontal heterogeneity.

(b) From the domain of Eeckhout's sequential preference condition.

(c) Without domain restrictions.

Figure 3: This paper: approaching mutually best from three different domains.

Our main findings are as follows.

(a) On the domain of horizontally heterogeneous problems: rank-unfairness of stable matchings and the size of the core are bounded in terms of local distances (Propositions 1 and 3) but not global distances (Propositions 2 and 4).

(b) On the domain of problems that satisfy Eeckhout's sequential preference condition: the size of the core is trivially bounded (as there is a unique stable matching) but rank-unfairness of stable matchings is not bounded in terms of local or global distances (Proposition 5).

(c) On the general domain: rank-unfairness of stable matchings and the size of the core are not bounded in terms of local or global distances (Proposition 6 and Corollaries 2 and 3 to results in (a) and (b)).

The remainder of the paper is organized as follows. In Section 2, we describe the model and define rank-fairness and the size of the core. In Section 3, we introduce restricted domains of problems and local/global distances to mutually best. In Section 4, we state and prove our findings. Section 5 concludes. The computation of global distances to mutually best based on the Hungarian algorithm is explained and illustrated in the Appendix.

2 The model

There are two finite disjoint sets of **men** M and **women** W of equal cardinality $n \in \mathbb{N}$ with $n \geq 2$. Each man $m \in M$ has a strict preference relation $\succ_m$ over W, and each woman $w \in W$ has a strict preference relation $\succ_w$ over M.[4] For each $m \in M$, man m's preferences can be represented by a **ranking**, i.e., a bijection $r_m : W \to \{1, \ldots, , n\}$ such that for $w, w' \in W$, $r_m(w) < r_m(w')$ if and only if $w \succ_m w'$. The integer $r_m(w)$ is the **rank** of w in man m's ranking. Hence, more preferred agents have a smaller rank. Similarly, for each $w \in W$, woman w's preferences can be represented by a ranking r_w. Let $r = (r_i)_{i \in M \cup W}$ be the profile of rankings. A **(marriage) problem** (Gale and Shapley, 1962) is given by (M, W, r), or r for short.

A **matching** is a function $\mu : M \cup W \to M \cup W$ such that for all $m \in M$ and all $w \in W$, $\mu(m) \in W$, $\mu(w) \in M$, and $\mu(m) = w \Leftrightarrow \mu(w) = m$. If $\mu(m) = w$, we say that m and w are matched to one another and that they are (each other's) mates at μ. Let $\mathcal{M}$ denote the set of matchings. For convenience we often denote a matching by its set of matched pairs.

A pair (m, w) blocks a matching μ if $r_m(w) < r_m(\mu(m))$ and $r_w(m) < r_w(\mu(w))$. A matching μ is **stable** if no pair blocks it. It is well-known that the core $\mathcal{C} \equiv \mathcal{C}(r)$ equals the set of stable matchings. Gale and Shapley (1962) showed that each marriage problem has a stable matching. In fact, they proved the existence of a **man-optimal** (and woman-pessimal) stable matching μ_M such that for all $m \in M$ and all $w \in W$, $r_m(\mu_M(m)) = \min_{\mu \in \mathcal{C}} r_m(\mu(m))$ and $r_w(\mu_M(w)) = \max_{\mu \in \mathcal{C}} r_w(\mu(w))$. Similarly, there exists a **woman-optimal** (and man-pessimal) stable matching μ_W.

Holzman and Samet (2014) focused on averages and maxima to define their main concepts. In order to carry out an analysis that goes beyond averages and maxima, we recall the definition of a generalized mean. Let $p > 0$. For any multiset[5] $\{x_1, \ldots, x_\ell\}$ that consists of $\ell \in \mathbb{N}$, $\ell \geq 1$ non-negative numbers $x_1, \ldots, x_\ell \in \mathbb{R}_+$, we define its **($p$-)generalized mean** by

$$\boldsymbol{M_p}(\{x_1, \ldots, x_\ell\}) \equiv \sqrt[p]{\frac{1}{\ell} \sum_{k=1}^{\ell} x_k^p}\ .$$

Note that M_1 is the arithmetic mean. It is also known (see, e.g., Bullen, 2003) that

$$\lim_{p \to \infty} M_p(\{x_1, \ldots, x_\ell\}) = \max\{x_1, \ldots, x_\ell\} \tag{1}$$

for any multiset $\{x_1, \ldots, x_\ell\}$ of non-negative numbers. For this reason it is convenient to define the **(∞-)generalized mean**:

$$\boldsymbol{M_\infty}(\{x_1, \ldots, x_\ell\}) \equiv \max\{x_1, \ldots, x_\ell\}.$$

[4]In particular, following Holzman and Samet (2014), it is assumed that all agents on the other side are "acceptable," i.e., each agent prefers to be matched to any agent from the other side rather than remaining unmatched (and consuming some outside option).

[5]Unlike a set, a multiset allows for multiple instances for each of its elements.

Rank-fairness. Following (Holzman and Samet, 2014), the rank gap of a pair $(m, w) \in M \times W$ is given by $|r_m(w) - r_w(m)|$. To aggregate the rank gaps of all n pairs at a matching we use generalized means. More specifically, for each $p \in (0, \infty]$, the **($\boldsymbol{p}$-aggregated) rank gap at matching $\boldsymbol{\mu}$** for problem r is defined as the p-generalized mean of all rank gaps, i.e.,

$$\boldsymbol{\Gamma_p(\mu)} \equiv M_p \left(\bigcup_{(m,w)\in\mu} \{|r_m(w) - r_w(m)|\} \right).$$

For $p = 1$ and $p = \infty$, the p-aggregated rank gap boils down to the average rank gap and the maximum rank gap, respectively. A matching μ is **rank-fair** (see also Jaramillo et al., 2019) if for some $p \in (0, \infty]$, $\Gamma_p(\mu) = 0$ or, equivalently, for all $p \in (0, \infty]$, $\Gamma_p(\mu) = 0$.

Size of the core. Following Holzman and Samet (2014), to measure the size of the core, we consider for each agent $i \in M \cup W$ the range of different ranks obtained at stable matchings which, by side-optimality of μ_M and μ_W, equals $|r_i(\mu_M(i)) - r_i(\mu_W(i))|$. To aggregate over all $2n$ agents we use again generalized means. Specifically, for $p \in (0, \infty]$, the **($\boldsymbol{p}$-)size/diameter of the core of problem $\boldsymbol{r}$** is defined by

$$\boldsymbol{D_p(\mathcal{C}(r))} \equiv M_p \left(\bigcup_{i\in M\cup W} \{|r_i(\mu_M(i)) - r_i(\mu_W(i))|\} \right).$$

The size of the core is nil if for some $p \in (0, \infty]$, $D_p(\mathcal{C}(r)) = 0$ or, equivalently, for all $p \in (0, \infty]$, $D_p(\mathcal{C}(r)) = 0$.

3 Domains and distances

3.1 Restricted domains of problems

A marriage problem (M, W, r) satisfies **horizontal heterogeneity** if all women most prefer different men and all men most prefer different women, i.e., for all $m, m' \in M$ and all $w, w' \in W$, $[r_w(m) = r_{w'}(m) = 1$ implies $w = w']$ and $[r_m(w) = r_{m'}(w) = 1$ implies $m = m']$. For given sets of men M and women W, we denote the class of problems r that satisfy horizontal heterogeneity by $\boldsymbol{H(M, W)}$. For marriage problems with horizontal heterogeneity, matching each woman to her most preferred man yields the woman-optimal stable matching, and similarly, matching each man to his most preferred woman yields the man-optimal stable matching.

A marriage problem (M, W, r) satisfies **Eeckhout's sequential preference condition** if there is an ordering of M, say w.l.o.g. $m_1, \ldots m_n$, and an ordering of W, say w.l.o.g. $w_1, \ldots, w_n$, such that for all $k, l = 1, \ldots, n$ with $l > k$, $r_{w_k}(m_k) < r_{w_k}(m_l)$ and $r_{m_k}(w_k) < r_{m_k}(w_l)$. For given sets of men M and women W, we denote the class of problems r that satisfy Eeckhout's sequential preference condition by $\boldsymbol{E(M, W)}$. Eeckhout (2000, Theorem 1) showed that this condition implies a singleton core, i.e., for each $r \in E(M, W)$, $|\mathcal{C}(r)| = 1$.

Finally, let $\boldsymbol{MB(M,W)}$ denote the class of problems with **mutually best** mates, i.e., for each $m \in M$ there is $w \in W$ with $r_m(w) = r_w(m) = 1$. If preferences are mutually best, the unique stable matching is rank-fair. Note $MB(M,W) = H(M,W) \cap E(M,W)$. It is easy to see that the domains $H(M,W)$ and $E(M,W)$ are logically unrelated. Finally, for $r \in H(M,W)$ we have $|\mathcal{C}(r)| = 1$ if and only if $r \in E(M,W)$. To see this, note that $|\mathcal{C}(r)| = 1$ if and only if the two side-optimal stable matchings coincide (and then constitute the unique stable matching), which for $r \in H(M,W)$ occurs if and only if the top preferences of the two sides are reciprocal, i.e., the problem satisfies mutually best.

3.2 Distances to mutually best

We consider two natural types of "distances." The first type is that of local distances, i.e., measures that consider the distance from mutually best separately for each agent.

Local distances. Let r be a marriage problem. For each $i \in M \cup W$, let $t(i)$ denote i's most preferred agent, i.e., $r_i(t(i)) = 1$. Hence, $r_{t(i)}(i)$ is the rank that i's favorite agent assigns to i. For any $p \in (0, \infty]$, the **local ($\boldsymbol{p}$-)distance from mutually best at $\boldsymbol{r}$** is

$$\boldsymbol{\Delta^l_p(r)} \equiv M_p\left(\bigcup_{i \in M \cup W} \{r_{t(i)}(i) - 1\}\right).$$

Note that for $p = 1$ and $p = \infty$, $\Delta^l_p(r)$ yields the average and the maximum of the numbers $\{r_{t(i)}(i) - 1\}_{i \in M \cup W}$, respectively. Moreover, for any $r \in MB(M,W)$ and any $p \in (0, \infty]$, $\Delta^l_p(r) = 0$.

The distances have a "local" flavor as they compute for each agent the rank with which his/her most preferred mate reciprocates – in particular, it does not require the function $i \mapsto t(i)$ to constitute a well-defined matching. Note also that the computation of local distances is easy as it only requires finding for each agent i the rank of i according to the ranking of agent $t(i)$.

Example 1. [Local distances.]
Consider problem r in Table 1. Here, each column represents the ranking of an agent. For instance, the 3^{rd} entry of the 2^{nd} column of the table on the left hand side shows that $r_{m_2}(w_4) = 3$. The bold-faced entries depict agents i in the ranking of $t(i)$. By looking at the bold-faced entries, we obtain $\Delta^l_1(r) = \frac{1}{8}(0+2+3+0+0+2+0+3) = \frac{5}{4}$ and $\Delta^l_\infty(r) = r_{m_1}(w_1) - 1 = 4 - 1 = 3$. For other values of $p \in (0, \infty]$, $\Delta^l_p(r)$ can be calculated similarly. $\diamond$

In order to measure the extent to which a problem fails to satisfy mutually best from a global point of view, we consider how close we can get to the most preferred mates for all agents simultaneously by a well-defined matching.

m_1	m_2	m_3	m_4
$\boldsymbol{w_4}$	$\boldsymbol{w_2}$	w_3	w_4
w_2	w_3	w_4	w_2
$\boldsymbol{w_3}$	w_4	w_1	w_3
$\boldsymbol{w_1}$	w_1	w_2	w_1

w_1	w_2	w_3	w_4
m_1	$\boldsymbol{m_2}$	m_1	$\boldsymbol{m_1}$
m_3	m_3	m_2	m_2
m_2	m_4	$\boldsymbol{m_3}$	m_3
m_4	m_1	m_4	$\boldsymbol{m_4}$

Table 1: Problem r in Example 1. Bold-faced entries indicate how most preferred agents reciprocate.

Global distances. Similarly to local distances, we consider global distances based on generalized means. Formally, the **global (p-)distance from mutually best at r**, is

$$\boldsymbol{\Delta^g_p(r)} \equiv \min_{\mu\in\mathcal{M}} M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu(i))-1\}\right).$$

Note that for $p=1$ and $p=\infty$, $\Delta^g_p(r)$ yields the minimum over all matchings of the average and the maximum of the numbers $\{r_i(\mu(i))-1\}_{i\in M\cup W}$, respectively. Moreover, for any $r\in MB(M,W)$ and any $p\in(0,\infty]$, $\Delta^g_p(r)=0$.

Finding the minimizer in $\Delta^g_p(r)$ for a given $p\in(0,\infty)$ is equivalent to solving the following (job) assignment problem. Let (M,W,r) be a marriage problem with $|M|=|W|=n$. Let $M=\{m_1,\ldots,m_n\}$ and $W=\{w_1,\ldots,w_n\}$. Consider the $n\times n$ matrix C where for all $k,l=1,\ldots,n$,

$$c_{kl}=(r_{m_k}(w_l)-1)^p+(r_{w_l}(m_k)-1)^p \tag{2}$$

is interpreted as the cost for "worker" m_k to carry out "job" w_l. The objective is to find a feasible assignment of jobs to workers that minimizes total costs:

$$\begin{aligned}
&\min\sum_{k=1}^{n}\sum_{l=1}^{n}c_{kl}x_{kl} \qquad (3)\\
&\text{s.t.}\sum_{l=1}^{n}x_{kl}=1\quad\text{for all } k=1,\ldots,n;\\
&\qquad\sum_{k=1}^{n}x_{kl}=1\quad\text{for all } l=1,\ldots,n;\\
&\qquad x_{kl}\geq 0\quad\text{for all } k,l=1,\ldots,n.
\end{aligned}$$

There is always an optimal solution to the program in (3) in integer values, which can be found in polynomial time, e.g., by using the Hungarian algorithm (cf. Kuhn, 1955; Munkres, 1957). An integer solution x to (3) corresponds to a minimizer μ^- in the computation of $\Delta^g_p(r)$ by taking $\mu^-(m_k)=w_l$ if and only if $x_{kl}=1$. As an example, for $p=1$ and problem r in Table 1, one of the minimizers is $\mu^-=\{m_1w_1,m_2w_2,m_3w_3,m_4w_4\}$, which yields $\Delta^g_1(r)=\frac{1}{8}\sum_{i\in M\cup W}(r_i(\mu^-(i))-1)=1$. The computation of $\Delta^g_\infty(r)$ is slightly more cumbersome but can still be done in polynomial time. More specifically, the computation consists of applying at most $n-1$ times the Hungarian algorithm to "perturbed" rankings to see whether all agents can be matched to a mate that is ranked at most $(n-1)^{\text{st}}$, at most $(n-2)^{\text{nd}}$, etc. In the Appendix, we provide a detailed discussion of computations of $\Delta_p^g(r)$ for $p \in (0,\infty]$ using the Hungarian algorithm.

Example 2. [Local and global distances are logically unrelated.]
The local and global p-distances are logically unrelated, i.e., for any $p \in (0,\infty]$, there is a problem for which Δ_p^g is strictly larger than Δ_p^l and there is also a problem for which Δ_p^g is weakly smaller than Δ_p^l.

To establish the first statement, consider problem r' exhibited in Table 2. Let $p \in (0,\infty)$. Looking at the bold-faced entries, one immediately verifies that $\Delta_p^l(r') = \sqrt[p]{\frac{1}{8}(1^p + 1^p)}$. Next, by simple but cumbersome checking of cases, one verifies that at any feasible matching (stable or not), at least one agent is matched to his/her second-ranked mate (or worse) and at least one other agent is matched to his/her third-ranked mate (or worse). Hence,

$$\Delta_p^g(r') = \min_{\mu \in \mathcal{M}} M_p\left(\bigcup_{i \in M \cup W} \{r_i'(\mu(i)) - 1\}\right) \geq \sqrt[p]{\frac{1}{8}((2-1)^p + (3-1)^p)}.$$

Therefore, $\Delta_p^g(r') > \Delta_p^l(r')$. Moreover, from the above and (1), $\Delta_\infty^g(r') \geq 2 > 1 = \Delta_\infty^l(r')$ as well.

m_1	m_2	m_3	m_4
$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$	w_3
$\boldsymbol{w_4}$	w_3	w_1	w_2
w_3	w_1	w_2	w_1
w_2	w_4	w_4	w_4

w_1	w_2	w_3	w_4
$\boldsymbol{m_1}$	$\boldsymbol{m_2}$	$\boldsymbol{m_3}$	m_1
m_2	m_3	$\boldsymbol{m_4}$	m_2
m_3	m_4	m_1	m_3
m_4	m_1	m_2	m_4

Table 2: Problem r' in Example 2.

The second statement follows from the next lemma. ◇

Lemma 1. *Let (M,W,r) be a marriage problem where all men most prefer different women or all women most prefer different men. Then, for any $p \in (0,\infty]$, $\Delta_p^g(r) \leq \Delta_p^l(r)$.*

Proof. Assume, without loss of generality, that all men most prefer different women. Then, assigning each man m to his most preferred woman $t(m)$ yields (the well-defined matching) μ_M. Let $p \in (0,\infty)$. We have

$$\Delta_p^g(r) = \min_{\mu \in \mathcal{M}} \sqrt[p]{\frac{1}{2n} \sum_{i \in M \cup W} (r_i(\mu(i)) - 1)^p}$$

$$
\begin{aligned}
&\leq \sqrt[p]{\frac{1}{2n}\sum_{i\in M\cup W}(r_i(\mu_M(i))-1)^p} \\
&= \sqrt[p]{\frac{1}{2n}[\sum_{m\in M}(1-1)^p+\sum_{m\in M}(r_{t(m)}(m)-1)^p]} \\
&\leq \sqrt[p]{\frac{1}{2n}\sum_{i\in M\cup W}(r_{t(i)}(i)-1)^p} \;=\; \Delta_p^l(r).
\end{aligned}
$$

Moreover, from the above and (1), $\Delta_\infty^g(r) \leq \Delta_\infty^l(r)$ as well. □

Corollary 1. [Local and global distances on domain of horizontal heterogeneity.] *Let $p \in (0,\infty]$. For any marriage problem (M,W,r) with $r \in H(M,W)$, $\Delta_p^g(r) \leq \Delta_p^l(r)$.*

In contrast, no clear-cut relation between Δ_p^g and Δ_p^l exists on the domain of Eeckhout's sequential preference condition. On the one hand, for $r' \in E(M,W)\backslash H(M,W)$ in Example 2 we have already seen that for any $p \in (0,\infty]$, $\Delta_p^g(r') > \Delta_p^l(r')$.

On the other hand, for profile $r \in E(M,W)\backslash H(M,W)$ in Table 3 we claim that for any $p \in (0,\infty]$, $\Delta_p^g(r) \leq \Delta_p^l(r)$.

m_1	m_2	m_3		w_1	w_2	w_3
$\boxed{\boldsymbol{w_1}}$	w_1	w_2		$\boxed{\boldsymbol{m_1}}$	m_1	m_2
w_3	$\boxed{w_2}$	w_1		m_3	$\boxed{m_2}$	m_1
$\boldsymbol{w_2}$	$\boldsymbol{w_3}$	$\boxed{w_3}$		$\boldsymbol{m_2}$	$\boldsymbol{m_3}$	$\boxed{m_3}$

Table 3: Problem r with $\Delta_p^g(r) \leq \Delta_p^l(r)$ for all $p \in (0,\infty]$.

To see this, let $p \in (0,\infty)$. Since $\mu^* = \{m_1w_1, m_2w_2, m_3w_3\}$ is a feasible matching,

$$
\begin{aligned}
\Delta_p^g(r) &= \min_{\mu\in\mathcal{M}} M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu(i))-1\}\right) \\
&\leq M_p\left(\bigcup_{i\in M\cup W}\{r_i(\mu^*(i))-1\}\right) \\
&= \sqrt[p]{\frac{1}{6}(0^p+1^p+2^p+0^p+1^p+2^p)} \\
&\leq \sqrt[p]{\frac{1}{6}(0^p+2^p+2^p+0^p+2^p+2^p)} \;=\; \Delta_p^l(r).
\end{aligned}
$$

Moreover, from the above and (1), $\Delta_\infty^g(r) \leq \Delta_\infty^l(r)$ as well.

4 Bounds on rank-unfairness and size of core

4.1 Approaching mutually best on the domain of horizontal heterogeneity

Consider a marriage problem (M, W, r) that satisfies horizontal heterogeneity, i.e., $r \in H(M, W)$. If the problem satisfies mutually best, then there is a unique and rank-fair stable matching. Below we explore what happens with the size of the core and the rank-fairness of the stable matchings if the problem does not satisfy mutually best. The first question we tackle is whether rank-unfairness can be bounded in terms of the distance to mutually best. It turns out that the answer depends on whether the local or global distance to mutually best is used.

Proposition 1. [Rank-unfairness bounded by local distances to mutually best.] *For any marriage problem (M, W, r) with $r \in H(M, W)$, for any $\mu \in \mathcal{C}(r)$, and for any $p \in (0, \infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\, \Delta^l_p(r)$ and $\Gamma_\infty(\mu) \leq \Delta^l_\infty(r)$.*

Proof. Let $\mu \in \mathcal{C}(r)$. Let $p \in (0, \infty)$. Then,

$$\begin{aligned}
\Gamma_p(\mu) &= \sqrt[p]{\frac{1}{n} \sum_{(m,w)\in\mu} |r_m(w) - r_w(m)|^p} \\
&\leq \sqrt[p]{\frac{1}{n} \sum_{(m,w)\in\mu} \max\{(r_m(\mu_W(m)) - 1)^p, (r_w(\mu_M(w)) - 1)^p\}} \\
&\leq \sqrt[p]{\frac{1}{n} \sum_{(m,w)\in\mu} (r_m(\mu_W(m)) - 1)^p + \frac{1}{n} \sum_{(m,w)\in\mu} (r_w(\mu_M(w)) - 1)^p} \\
&= \sqrt[p]{\frac{1}{n} \sum_{i\in M\cup W} (r_{t(i)}(i) - 1)^p} \\
&= \sqrt[p]{2}\, \Delta^l_p(r),
\end{aligned}$$

where the second equality follows from $r \in H(M, W)$. Hence, for any $p \in (0, \infty)$, $\Gamma_p(\mu) \leq \sqrt[p]{2}\, \Delta^l_p(r)$. Then, from (1) it follows that $\Gamma_\infty(\mu) \leq \Delta^l_\infty(r)$. □

Proposition 1 says that on the domain of problems that satisfy horizontal heterogeneity, if for a given problem either of the local distances to mutually best is small, then no stable matching can be very rank-unfair.

However, as the following result shows, rank-unfairness cannot be bounded in terms of the *global* distances to mutually best.

Proposition 2. [Rank-unfairness *cannot* be bounded by global distances to mutually best.]
For any $\alpha > 0$, there is a marriage problem (M, W, r) with $r \in H(M, W)$ and $\mu \in \mathcal{C}(r)$ such that for any $p \in (0, \infty]$, $\Gamma_p(\mu) > \alpha \Delta_p^g(r)$.

Proof. Let $n \geq 3$ be an integer such that $n - 1 > \alpha$. Let $p \in (0, \infty)$. Consider the marriage problem (M, W, r) exhibited in Table 4.

m_1	m_2	m_3	$\dots$	m_n
w_1	w_2	w_3	$\dots$	w_n
$\boxed{w_2}$	$\boxed{w_3}$	$\boxed{w_4}$	$\dots$	$\boxed{w_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_n	w_1	w_2	$\dots$	w_{n-1}

w_1	w_2	w_3	$\dots$	w_n
m_2	m_3	m_4	$\dots$	m_1
$\boxed{m_n}$	$\boxed{m_1}$	$\boxed{m_2}$	$\dots$	$\boxed{m_{n-1}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m_1	m_2	m_3	$\dots$	m_n

Table 4: Problem r exhibits "mutually second-best" (boxed matching).

Note that $r \in H(M, W)$. At any of the two side-optimal stable matchings μ, the rank gap is $n - 1$, i.e., $\Gamma_p(\mu) = n - 1$.

Let μ^* be the matching where each agent is matched to his/her second most preferred mate. We have

$$\begin{aligned}
\Delta_p^g(r) &= \min_{\mu \in \mathcal{M}} M_p \left(\bigcup_{i \in M \cup W} \{r_i(\mu(i)) - 1\} \right) \\
&\leq M_p \left(\bigcup_{i \in M \cup W} \{r_i(\mu^*(i)) - 1\} \right) \\
&= \sqrt[p]{\frac{1}{2n}(1^p + \dots + 1^p)} \\
&= 1.
\end{aligned}$$

So, $\Delta_p^g(r) \leq 1$. Hence, $\Gamma_p(\mu) > \alpha \Delta_p^g(r)$.

Moreover, from the above and (1), $\Gamma_\infty(\mu) = n - 1 > \alpha \geq \alpha \Delta_\infty^g(r)$ as well. □

Next, we turn to the size of the core.

Proposition 3. [Size of core bounded by local distance to mutually best.]
For any marriage problem (M, W, r) with $r \in H(M, W)$ and any $p \in (0, \infty]$, $D_p(\mathcal{C}(r)) = \Delta_p^l(r)$.

Proof. Note that for any $m \in M$,

$$|r_m(\mu_M(m)) - r_m(\mu_W(m))| = r_m(\mu_W(m)) - 1 = r_{t(\mu_W(m))}(\mu_W(m)) - 1.$$

Therefore,

$$\{|r_m(\mu_M(m)) - r_m(\mu_W(m))|\}_{m\in M} \text{ and } \{r_{t(w)}(w) - 1\}_{w\in W}$$

are identical multisets. Similarly,

$$\{|r_w(\mu_W(w)) - r_w(\mu_M(w))|\}_{w\in W} \text{ and } \{r_{t(m)}(m) - 1\}_{m\in M}$$

are identical multisets. Hence,

$$D_p(\mathcal{C}(r)) = M_p\left(\bigcup_{i\in M\cup W} \{|r_i(\mu_M(i)) - r_i(\mu_W(i))|\}\right) = M_p\left(\bigcup_{i\in M\cup W} \left\{r_{t(i)}(i) - 1\right\}\right) = \Delta^l_p(r)$$

follows. □

As the following result shows, it is not possible to replace local distance by global distance in Proposition 3: the size of the core cannot be bounded in terms of the *global* distance to mutually best.

Proposition 4. [Size of the core *cannot* be bounded by global distances to mutually best.]
For any $\alpha > 0$, there is a marriage problem (M, W, r) with $r \in H(M, W)$ such that for any $p \in (0, \infty]$, $D_p(\mathcal{C}(r)) > \alpha\Delta^g_p(r)$.

Proof. Let $n \geq 3$ be an integer such that $n - 1 > \alpha$. Let $p \in (0, \infty]$. For profile $r \in H(M, W)$ in Table 4 we already know from the proof of Proposition 2 that $\Delta^g_p(r) \leq 1$. Since $D_p(\mathcal{C}(r)) = n - 1 > \alpha$, the statement follows. □

4.2 Approaching mutually best on the domain of Eeckhout's sequential preference condition

For any problem that satisfies Eeckhout's sequential preference condition, there is a unique stable matching so that the diameter of the core is nil and hence trivially bounded.

The following result shows that on the domain of Eeckhout's sequential preference condition rank-unfairness cannot be bounded in terms of distances to mutually best.

Proposition 5. [Rank-unfairness *cannot* be bounded by distances to mutually best.]
For any $\alpha > 0$, there is a marriage problem (M, W, r) with $r \in E(M, W)$ such that for any $\mu \in \mathcal{C}(r)$ and any $p \in (0, \infty]$, $\Gamma_p(\mu) > \alpha\Delta^l_p(r), \alpha\Delta^g_p(r)$.

Proof. Consider problem (M, W, r) depicted in Table 5 where $n > \max\{\alpha + 2, 3\}$. It is easy to see that $r \in E(M, W)$. For the unique stable matching $\mu \in \mathcal{C}(r)$, which is depicted in boxes, we have $\Gamma_p(\mu) = \sqrt[p]{\frac{1}{n}(n-2)^p}$ for all $p \in (0, \infty)$.

m_1	m_2	m_3	m_4	m_5	$\dots$	m_n
$\boxed{\boldsymbol{w_1}}$	w_1	$\boxed{\boldsymbol{w_3}}$	$\boxed{\boldsymbol{w_4}}$	$\boxed{\boldsymbol{w_5}}$	$\dots$	$\boxed{\boldsymbol{w_n}}$
$\boldsymbol{w_2}$	$\boxed{w_2}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$					

w_1	w_2	w_3	w_4	w_5	$\dots$	w_n
$\boxed{\boldsymbol{m_1}}$	m_1	$\boxed{\boldsymbol{m_3}}$	$\boxed{\boldsymbol{m_4}}$	$\boxed{\boldsymbol{m_5}}$	$\dots$	$\boxed{\boldsymbol{m_n}}$
$\boldsymbol{m_2}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\boxed{m_2}$					

Table 5: Problem $r \in E(M, W)$ in Proposition 5.

From the bold-faced entries it follows that for any $p \in (0, \infty)$, $\Delta_p^l(r) = \sqrt[p]{\frac{1}{2n}(1^p + 1^p)} = \frac{1}{n^{1/p}}$. Similarly, from the encircled matching it follows that for any $p \in (0, \infty)$, $\Delta_p^g(r) \leq \frac{1}{n^{1/p}}$. Hence,

$$\Gamma_p(\mu) = \frac{n-2}{n^{1/p}} > \alpha \frac{1}{n^{1/p}} \geq \alpha \Delta_p^l(r), \alpha \Delta_p^g(r) \quad \text{for all} \quad p \in (0, \infty). \tag{4}$$

From (1) and (4) one obtains

$$\Gamma_\infty(\mu) = n - 2 > \alpha \geq \alpha \Delta_\infty^l(r), \alpha \Delta_\infty^g(r).$$

□

4.3 Approaching mutually best without domain restrictions

As indicated by Proposition 5, rank-unfairness cannot be bounded by distances to mutually best for marriage problems on the domain of Eeckhout's sequential preference condition. This directly implies the following first negative result on the general domain.

Corollary 2. [Rank-unfairness *cannot* be bounded by distances to mutually best.]
For any $\alpha > 0$, there is a marriage problem (M, W, r) such that for any $\mu \in \mathcal{C}(r)$ and any $p \in (0, \infty]$, $\Gamma_p(\mu) > \alpha \Delta_p^l(r), \alpha \Delta_p^g(r)$.

Similarly, as Proposition 4 demonstrates that the size of the core is not bounded by global distances to mutually best on the domain of horizontal heterogeneity, we get the second negative result on the general domain.

Corollary 3. [Size of the core *cannot* be bounded by global distances to mutually best.]
For any $\alpha > 0$, there is a marriage problem (M, W, r) such that for any $p \in (0, \infty]$, $D_p(\mathcal{C}(r)) > \alpha \Delta_p^g(r)$.

Given Corollaries 2 and 3, the only remaining question for approaching mutually best from the general domain is: can the size of the core be bounded in terms of *local* distances to mutually best? Our last result, below, answers also this question in the negative.

Proposition 6. [Size of the core *cannot* be bounded by local distances to mutually best.]
For any $\alpha > 0$ and any $p \in (0, \infty]$, there is a marriage problem (M, W, r) such that $D_p(\mathcal{C}(r)) > \alpha \Delta^l_p(r)$.

Proof. Let $p \in (0, \infty)$. Let $n > 4$ be an integer such that

$$\sqrt[p]{[6 + 2^p + (n-1)^p]} > \alpha \sqrt[p]{[5 + 3 \times 2^p]}.$$

Consider problem (M, W, r) depicted in Table 6. The man-optimal stable matching μ_M is depicted in boxes, while the woman-optimal stable matching μ_W is depicted in circles. From

m_1	m_2	m_3	m_4	m_5	$\ldots$	m_n
$\boxed{w_2}$	w_2	$\boxed{w_3}$	$\boxed{w_4}$	$\boxed{\boldsymbol{w_5}}$	$\ldots$	$\boxed{\boldsymbol{w_n}}$
$\boldsymbol{w_1}$	$\boxed{w_1}$	w_4	$\boldsymbol{w_3}$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\boldsymbol{w_4}$	$\boldsymbol{w_2}$	$\vdots$			
	$\vdots$	$\vdots$				

w_1	w_2	w_3	w_4	w_5	$\ldots$	w_n
m_1	m_3	m_4	m_2	$\boxed{\boldsymbol{m_5}}$	$\ldots$	$\boxed{\boldsymbol{m_n}}$
m_3	$\boxed{\boldsymbol{m_1}}$	$\boxed{\boldsymbol{m_3}}$	$\boxed{\boldsymbol{m_4}}$	$\vdots$	$\vdots$	$\vdots$
m_4	$\boldsymbol{m_2}$	$\vdots$	$\vdots$			
$\vdots$	$\vdots$					
$\boxed{m_2}$						

Table 6: Problem r in Proposition 6.

these two stable matchings and the bold-faced entries it follows that

$$D_p(\mathcal{C}(r)) = \sqrt[p]{\frac{1}{2n}[6 + 2^p + (n-1)^p]} \text{ and } \Delta^l_p(r) = \sqrt[p]{\frac{1}{2n}[5 + 3 \times 2^p]}\ , \tag{5}$$

respectively. Hence, $D_p(\mathcal{C}(r)) > \alpha \Delta^l_p(r)$.

Finally, using (1) and (5) or directly the definitions of $D_\infty(\mathcal{C}(r))$ and $\Delta^l_\infty(r)$, we find that for (M, W, r) depicted in Table 6 with $n > \max\{4, 2\alpha + 1\}$, $D_\infty(\mathcal{C}(r)) > \alpha \Delta^l_\infty(r)$. □

5 Conclusion

For the analysis of matching problems, it is arguably not only crucial to consider the existence of stable matchings but also the corresponding conflict within matched pairs (as measured by the rank-(un)fairness of a stable matching) and across market sides (as measured by the diameter of the core).

While Holzman and Samet (2014) demonstrate that preference profiles that are close to vertical heterogeneity (i.e., agents of each market side almost agree on the ranking of the other market side) yield a small core and stable matchings that are almost rank-fair, our study indicates that a generalization of these findings beyond the case of vertical heterogeneity is not straightforward.

Rather than considering vertical heterogeneity, we start out with a horizontal counterpart: profiles that exhibit mutually best (i.e., for each man, his most preferred woman also considers him her best mate). As in the case of vertical heterogeneity, mutually best implies that there is a unique and rank-fair stable matching. But in contrast to vertical heterogeneity, the fact that a profile almost satisfies mutually best does not necessarily mean that it has a small core nor that its stable matchings are almost rank-fair. Only if the profile satisfies horizontal heterogeneity, results similar to those of Holzman and Samet (2014) can be established for local distances (Propositions 1 and 3 being the (horizontal) counterparts of Theorems 1, 2, 3, and 4 in Holzman and Samet, 2014). However, as is shown in Propositions 2 and 4, these results cannot be extended to global distances. Finally, we notice that, qualitatively, none of our (positive and negative) results depend on the particular generalized mean that is used to measure distances, rank-unfairness, and the size of the core.

Appendix: Computation of global distances

In the Appendix, we discuss the computation of the global distances $\Delta_p^g(\cdot)$ through an application of the Hungarian algorithm (cf. Kuhn, 1955; Munkres, 1957). We provide a detailed description for $p = 1$ and $p = \infty$. For any other value $p \in (0, \infty)$ the computation is similar to that for $p = 1$, the only difference being the definition of the matrix C.

Computation of global 1-distance from mutually best

Let (M, W, r) be a marriage problem with $|M| = |W| = n$. Let $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$. We first construct the $n \times n$ matrix C where for all $k, l = 1, \ldots, n$, $c_{kl} = r_{m_k}(w_l) + r_{w_l}(m_k)$. (In general, for any $p \in (0, \infty)$, the matrix C is defined by (2). For the sake of convenience we simplified the matrix C for $p = 1$ by not subtracting 1 from each rank.)

Example 3. [Matrix *C* for problem *r* in Table 1.]
Using problem r in Table 1, straightforward calculations yield

$$C = \begin{pmatrix} 5 & 6 & 4 & 2 \\ 7 & 2 & 4 & 5 \\ 5 & 6 & 4 & 5 \\ 8 & 5 & 7 & 5 \end{pmatrix}.$$

$\diamond$

Next, we apply the Hungarian algorithm to matrix C. The algorithm consists of five steps. Steps 1 and 2 are executed only once, while Steps 3 and 4 are repeated until we move to Step 5 which outputs a matching that is a minimizer for $\Delta_1^g(r)$.

HUNGARIAN ALGORITHM:

INPUT: Non-negative matrix C.

STEP 1: For each row in the matrix C, find the smallest element and subtract it from each element in that row. Let C^1 denote the resulting matrix.

STEP 2: For each column, find the smallest element in C^1 and subtract it from each element in that column. Let C^2 denote the resulting matrix. Set $q \equiv 0$.

STEP 3: Cover all zeros in C^{2+q} using a minimum number of horizontal and vertical lines. If n lines are required, go to Step 5. Otherwise, go to Step 4.

STEP 4: Find a smallest element in C^{2+q} , say c_{kl}^{2+q}, that is not covered by the minimal cover of Step 3. Subtract c_{kl}^{2+q} from all uncovered elements and add c_{kl}^{2+q} to all elements that are covered twice. Let C^{2+q+1} denote the resulting matrix. Set $q \equiv q+1$ and go to Step 3.

STEP 5 (OUTPUT): A well-defined matching μ^- such that for all $k, l = 1, \ldots, n$, $\mu^-(m_k) = w_l$ implies $c_{kl}^{2+q} = 0$.[6]

Matching μ^- induces an integer solution x to (3) by taking $x_{kl} = 1$ if and only if $\mu^-(m_k) = w_l$. Hence, μ^- is a minimizer for $\Delta_1^g(r)$.

Example 4. [Applying the Hungarian algorithm to obtain $\boldsymbol{\Delta_1^g(r)}$ for problem $\boldsymbol{r}$ in Table 1.]

We apply the Hungarian algorithm to matrix C in Example 3. In Step 1, we subtract the row minima from C:

$$C = \begin{pmatrix} 5 & 6 & 4 & 2 \\ 7 & 2 & 4 & 5 \\ 5 & 6 & 4 & 5 \\ 8 & 5 & 7 & 5 \end{pmatrix} \implies \text{row minima} \begin{pmatrix} 2 \\ 2 \\ 4 \\ 5 \end{pmatrix} \implies C^1 = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 5 & 0 & 2 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 \end{pmatrix}.$$

In Step 2, we subtract the column minima from C^1:

$$C^1 = \begin{pmatrix} 3 & 4 & 2 & 0 \\ 5 & 0 & 2 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 0 & 2 & 0 \end{pmatrix} \implies \text{column minima} \begin{pmatrix} 1\,0\,0\,0 \end{pmatrix} \implies C^2 = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$

[6] As, in Step 5, C^{2+q} requires n lines to cover its zeros, existence of at least one such matching μ^- is guaranteed.

In Step 3, we cover all zeros in C^2 using a minimum number of horizontal and vertical lines:

$$C^2 = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix} \quad \Longrightarrow \quad \text{only } 3 < n \text{ lines are required.}$$

In Step 4, we find a smallest element in C^2 that is not covered by the minimal cover of Step 3, subtract it from all uncovered elements, and add it to all elements that are covered twice:

$$C^2 = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{pmatrix} \quad \Longrightarrow \quad \text{smallest uncovered element is } 2 \quad \Longrightarrow \quad C^3 = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We return to Step 3 and cover all zeros in C^3 using a minimum number of horizontal and vertical lines:

$$C^3 = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad 4 = n \text{ lines are required.}$$

Since $n = 4$ lines are required to cover all zeros in C^3, we go to Step 5. Looking at the five patterns of gray cells in C^3 below

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

one easily verifies that there are (exactly) five matchings μ^- such that for all $k, l = 1, \ldots, n$, $\mu^-(m_k) = w_l$ implies $c^3_{kl} = 0$. These matchings are $\{m_1w_1, m_2w_2, m_3w_3, m_4w_4\}$, $\{m_1w_3, m_2w_2, m_3w_1, m_4w_4\}$, $\{m_1w_4, m_2w_2, m_3w_1, m_4w_3\}$, $\{m_1w_4, m_2w_3, m_3w_1, m_4w_2\}$, and $\{m_1w_4, m_2w_2, m_3w_3, m_4w_1\}$. The five matchings are the (only) minimizers in the computation of $\Delta^g_1(r)$. Using for instance the first of these matchings,

$$\Delta^g_1(r) = \frac{1}{2n} \sum_{k=1}^{n} (r_{m_k}(w_k) - 1 + r_{w_k}(m_k) - 1) = \frac{1}{8}(3 + 0 + 0 + 0 + 0 + 2 + 0 + 3) = 1.$$

◇

Computation of global ∞-distance from mutually best

The integer $\Delta^g_\infty(r)$ can be computed by applying at most $n-1$ times the Hungarian algorithm. To see this, observe first that the rank agent i's most preferred agent, $t(i)$, assigns to i in any matching is at most n. The following algorithm now checks whether there is a matching where for each i, $t(i)$ assigns at most a rank of $(n-1)$, $(n-2)$, etc.

Computation of global ∞-distance:
Input: A marriage problem (M, W, r) where $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$. Let f be an integer such that $f > n^3$.[7]
Step $k = 1, ..., n$: Let $\bar{r} = n - k$. If $\bar{r} = 0$, then $\Delta^g_\infty(r) = 1$ and stop. Otherwise, construct the following "perturbed" rankings and their associated cost matrix. For each $m \in M$, let $\tilde{r}_m : W \to \mathbb{N}$ be the function defined by

$$\tilde{r}_m(w) = \begin{cases} r_m(w) & \text{if } r_m(w) \le \bar{r}; \\ f & \text{if } r_m(w) > \bar{r}. \end{cases}$$

Similarly, for each $w \in W$, let $\tilde{r}_w : M \to \mathbb{N}$ be the function defined by

$$\tilde{r}_w(m) = \begin{cases} r_w(m) & \text{if } r_w(m) \le \bar{r}; \\ f & \text{if } r_w(m) > \bar{r}. \end{cases}$$

Construct the $n \times n$ matrix $\tilde{C}$ where for all $k, l = 1, \ldots, n$, $\tilde{c}_{kl} = \tilde{r}_{m_k}(w_l) + \tilde{r}_{w_l}(m_k)$. Apply the Hungarian algorithm to $\tilde{C}$ and let μ^- be the resulting output. If $\sum\limits_{i \in M \cup W} \tilde{r}_i(\mu^-(i)) \ge f$, then $\Delta^g_\infty(r) = \bar{r} + 1\ (= n - k + 1)$ and stop. Otherwise, set $k \equiv k + 1$ and go to Step k.

To see that the algorithm indeed computes $\Delta^g_\infty(r)$, note first that for each $k = 1, \ldots, n-1$ and for each matching μ such that for each agent i, $r_i(\mu(i)) \in \{1, \ldots, n-k\}$, we have that $\sum\limits_{i \in M \cup W} r_i(\mu(i)) \le \sum\limits_{i \in M \cup W} (n-k) \le n^2(n-1) < f$. Then, since the Hungarian algorithm finds μ^- that minimizes $\sum\limits_{i \in M \cup W} \tilde{r}_i(\mu^-(i))$ and r and $\tilde{r}$ coincide for each agent's $(n-k)$ first ranked mates, the inequality $\sum\limits_{i \in M \cup W} \tilde{r}_i(\mu^-(i)) \ge f$ holds at the end of Step k if and only if there is no matching μ such that for each agent i, $r_i(\mu(i)) \in \{1, \ldots, n-k\}$.

Example 5. [Computation of $\boldsymbol{\Delta^g_\infty(r)}$ for problem $\boldsymbol{r}$ in Table 1.]
Note that for problems r with a "small" number of agents, $\Delta^g_\infty(r)$ can often be quickly calculated without recurring to the Hungarian algorithm. However, below we illustrate the algorithm described above by applying it to problem r in Table 1.

Let $f = 100 > n^3$. In Step 1, the perturbed cost matrix is

$$\tilde{C} = \begin{pmatrix} 101 & 102 & 4 & 2 \\ 103 & 2 & 4 & 5 \\ 5 & 102 & 4 & 5 \\ 200 & 5 & 103 & 101 \end{pmatrix}.$$

[7] The bound n^3 is clearly not the smallest bound for which the algorithm works. However, our goal is only to show that for sufficiently large values of f the algorithm works.

Applying the Hungarian algorithm to $\tilde{C}$ yields the unique matching $\mu^- = \{m_1w_4, m_2w_3, m_3w_1, m_4w_2\}$. Since $\sum_{i\in M\cup W} \tilde{r}_i(\mu^-(i)) = 16 < 100 = f$, we go to Step 2.

In Step 2, the perturbed matrix is

$$\tilde{C} = \begin{pmatrix} 101 & 102 & 101 & 2 \\ 200 & 2 & 4 & 102 \\ 102 & 102 & 101 & 102 \\ 200 & 102 & 200 & 101 \end{pmatrix}.$$

Applying the Hungarian algorithm to $\tilde{C}$ yields again the same minimizer $\mu^- = \{m_1w_4, m_2w_3, m_3w_1, m_4w_2\}$. Since this time $\sum_{i\in M\cup W} \tilde{r}_i(\mu^-(i)) = 210 \geq 100 = f$, we conclude that $\Delta^g_\infty(r) = n - 2 + 1 = 3$. $\diamond$

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