Restricted Environments and Incentive Compatibility in Interdependent Values Models

Salvador Barberà
Dolors Berga
Bernardo Moreno

This version: March 2020
(March 2018)

Barcelona GSE Working Paper Series

Working Paper nº 1024

Restricted environments and incentive compatibility in interdependent values models[1]

by

Salvador Barberà[†], Dolors Berga[‡], and Bernardo Moreno[§]

March 3rd, 2020

[1] This is a new version of a paper that was previously circulated under the title "The role of domain restrictions in mechanism design: ex post incentive compatibility and Pareto efficiency".

[†] MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. Mailing address: Departament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain. E-mail: salvador.barbera@uab.cat

[‡] Departament d'Economia, C/ Universitat de Girona, 10; Universitat de Girona, 17071 Girona, Spain. E-mail: dolors.berga@udg.edu

[§] *Corresponding author.* Departamento de Teoría e Historia Económica, Facultad de Ciencias Económicas y Empresariales, Campus de El Ejido, 29071 Málaga, Spain. E-mail: bernardo@uma.es

Abstract: We study general restrictions allowing to design satisfactory ex post incentive compatible single valued direct mechanisms in interdependent values environments, characterized by the set of agents' type profiles and by their induced preference profiles. For environments that we call partially knit and strict, ex post incentive compatibility extends to groups, and strategy-proofness implies strong group strategy-proofness in the special case of private values environments. For those called knit and strict, only constant mechanisms can be ex post incentive compatible. The results extend to mechanisms operating on non-strict domains under an additional requirement of respectfulness. We discuss voting, assignment and auctions environments where our theorems apply.

Journal of Economic Literature Classification Numbers: D70, D71, D80, D82, D44, C78.
Keywords: Mechanisms, Interdependent Types, Ex Post Incentive Compatibility, Strategy-Proofness, Pareto Efficiency, Preference Functions, Jury Decisions, Allocation Problems, Auctions.

1 Introduction

A major concern when designing economic mechanisms is to provide agents with incentives to reveal their true characteristics. Setting aside some obviously unsatisfactory solutions, it is well understood that attaining this objective is not always possible. Moreover, when it is, a conflict often arises between the mechanism's efficiency and its incentive compatibility. These generic statements hold for different formulations of the mechanism design problem, and for various concepts of equilibrium.

Hence, a mechanism can only meet attractive lists of desiderata if the class of problems to be dealt with is somewhat constrained. In social choice theory, where mechanisms are defined as functions whose domains are subsets of preference profiles, these constraints on the relevant situations to be considered are called domain restrictions, in contrast to the notion of universal domain that was the basis of fundamental theorems like Arrow's or Gibbard and Satterthwaite's. Of course, some restricted domains may admit satisfactory mechanisms, and others not. But the analysis of domain restrictions provides a systematic tool to explore the frontiers between possibility and impossibility results.

The notion of domain restrictions is not always explicitly used in the larger literature on mechanism design. There, assumptions on what economic situations lie within the scope of each model are usually predicated directly on the structure of the set of alternatives, or on the types of agents.

Our purpose in the present paper is to study when it is possible to design satisfactory mechanisms even in the case where agent's values are interdependent. But before we introduce our present endeavor, it will be useful to refer to our previous and parallel work regarding mechanism design in private values environments (Barberà, Berga, and Moreno, 2016). There, we started from the observation that, under specific circumstances, it may be possible to define mechanisms that are not only individually strategy-proof but also (weakly) group strategy-proof. Then, we studied the common characteristics of those models for which it was known that non-trivial mechanisms satisfying group strategy-proofness could be defined, in problems of matching, rationing, housing, public good provision and auctions. We also identified numerous models and domains under which impossibilities arose.

In the same unifying spirit as in that earlier work, we consider here the case where agents' types may be interdependent and we look for the possibility of defining non-trivial ex post group incentive compatible mechanisms. The shift to the notion of ex post group incentive compatibility is a natural counterpart, in the case of interdependent values, to our view that group strategy-proofness guarantees a degree of efficiency.

Ex post incentive compatibility, when focused on individual incentives has been the object of study of many papers. It is an important solution concept because it guarantees belief-free (often called "robust) implementation, an for social choice functions is equivalent to it (Corollary 1 in Bergemann and Morris, 2005). From the 1980s, the literature on interdependent value environments has obtained positive and negative results regarding the possibility of designing ex post incentive compatible mechanisms. Most papers are mainly motivated by or applied to auctions. A number of authors have shown that ex post incentive compatibility and efficiency are compatible when signals are one-dimensional and a single-crossing property is satisfied: see Crémer and McLean (1985), Maskin (1992), Ausubel (1997), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Bergemann and Välimäki (2002), Perry and Reny (2002), for example. When it comes to situations where types are multi-dimensional, the general wisdom is that negative results prevail. Under additional assumptions, no mechanism or only constant ones are implementable: see for example Maskin (1992), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006). There are also a small number of works that have found impossibility results for other settings. Austen-Smith and Feddersen (2006) for the voting in the deliverative jury problem and Che, Kim, and Kojima (2015) for a house allocation problem without transfers, for instance. Dizdar and Moldovanu (2016) studied a two-sided matching model with a finite number of agents, two-sided incomplete information, interdependent values, and multi-dimensional attributes and show that premuneration values corresponding to uniform, fixed-proportion sharing are essentially the only efficient ones in their setting. A recent work by Pourpouneh, Ramezanianz, and Sen (2018) show the existence of ex post incentive compatible and ex-post stable rules in the marriage problem with specific interdependent preferences.

Part of our work also refers to the individual notion of ex post incentive compatibility, but we also introduce and study the more demanding condition of ex post group incentive compatibility. In interdependent and non-private values settings, a different concept to control for group manipulations has been used in Che, Kim, and Kojima (2015) who consider group manipulations using first-order stochastic dominance. As we have already observed, one of our concerns is to unify different strands of literature where good incentives and efficiency are found to be compatible. Much in the same way as group strategy-proofness was interpreted as a form of efficiency in private values environments, we also consider that meeting the strong property of ex post group incentive compatibility is a form of efficiency when types are interdependent.

Our search across the literature did not produce as many positive results for genuinely interdependent values contexts as it did for the case of private values (to which our first general result also apply) but the examples we found come from different fields and in a variety of models, so that they are hardly comparable.

Our starting point toward a general model starts from the observation that, in contexts where values are interdependent, the incentives provided by a mechanism not only depend on the type profiles in its domain, but also on the properties of the preference function associating a profile of agent's preferences to each one of types.

We define an environment as a pair formed by the set of admissible type profiles and an associated preference function, and argue that what matters to determine whether an environment admits satisfactory mechanisms depends on how it is restricted. Restrictions on preference domains are a particular case of our general framework.

The two classes of environments we define, those that we call partially knit and knit, must meet requirements regarding the possibility to connect admissible pairs of type profiles through sequences of changes in individual types, which are defined in reference to certain alternatives and through the use of the preference function.[1] The set of pairs of type profiles and reference alternatives for which the requirements must be met for an environment to

[1]Our purpose here is to present the reader with a general roadmap. Formal definitions are in Section 2 and illustrations in Section 4. An intuition behind the conditions is presented in Section 5.

be knit is larger than for it to be partially knit. Thus, the latter is a weaker condition. Their joint consideration allows us to illustrate the fact that, in order to draw the line between possibility and impossibility in mechanism design, what matters is not the size of environments, but rather the connections that can be established between the different potential situations that may arise as type profiles do change.

Let's mention that partial knitness is satisfied, for example, by models developed by Pourpouneh, Ramezanianz, and Sen (2018), Dasgupta and Maskin (2000) and by three examples in voting, house allocation, and auctions, respectively, that we provide in our Subsection 4.2 below (see Examples 3, 5, and 7, inspired by Austen-Smith and Feddersen, 2006, Che, Kim, and Kojima, 2015, and Dasgupta and Maskin, 2000, respectively). Moreover, in private values environments several classical domains, like the universal domain, the set of single-peaked profiles, and the set of preferences profiles in the housing problem (see Subsection 4.1 below) satisfy partial knitness.

Concerning our second and stronger condition on environments, knitness, we present in Examples 2, 4, and 6 in Subsection 4.2 below three examples of knit environments in different settings.

Let's now describe two related demands we may impose on mechanisms. One first attractive and well-studied requirement is that of ex post incentive compatibility, guaranteeing truthful revelation of types to be a Nash equilibrium in all the games that result from any specification of possible type profiles.[2] We also introduce a second concept, that of ex post group incentive compatibility, under which truthful revelation is required to be a strong Nash equilibrium. These are our main target properties, and we can obtain possibility and impossibility results regarding them, for those environments that we call strict, where agents are never indifferent between alternatives. In the general case where some agents may be indifferent among several alternatives, we need to use an additional condition that we call respectfulness. This condition, when applied to private values is a relative of non-bossiness (Satterthwaite and Sonnenschein, 1981), but less demanding than this or other similar conditions analyzed in Thomson (2016). It essentially rules out manipulations by one agent that could affect others while not gaining anything in exchange, thus opening the way to bribes.

Our Theorem 1 opens the door to the possibility of defining ex post group incentive compatible mechanisms, and to ensure efficiency in addition to good incentive properties. Theorem 1 refers to partially knit environments and applies both to the case of interdependent and that of private values. It states that under partially knit environments all respectful and ex post incentive compatible mechanisms will also be ex post group incentive compatible. Corollary 1 reaches the same conclusion for strict environments (those with strict preferences) without need to invoke respectfulness. Moreover, recall that, in the case of private values, ex post incentive compatibility is equivalent to strategy-proofness. Likewise, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness.

[2]The study of incentive compatibility in Bayesian terms was started by d'Aspremont and Gérard-Varet (1979), and Arrow (1979), and its appropriate formulation and results depend on the information that will be available to the agents at the time where the analysis is carried out. The case of interdependent values was first studied by D'Aspremont, Crémer, and Gérard-Varet (1990). The notion of ex post incentive compatibility corresponds to the time where agents have received all possible information, and can be defined without attributing cardinal utility to agents, as it does not require Bayesian updating. See Jackson (2003).

Hence, a corollary for the case of private values is that, under the conditions of our first theorem, individual and strong group strategy-proofness become equivalent (see Corollary 2). This parallels the results in Barberà, Berga, and Moreno (2010, 2016) we referred to above, connecting individual and weak group strategy-proofness.[3]

Our Theorem 2 and Corollary 3 are impossibility results. Theorem 2 states that only constant mechanisms can be ex post incentive compatible and respectful in knit environments. Corollary 3 reaches the same conclusion for strict environments without need to invoke respectfulness, which trivially holds in that case. In fact, the results only apply to the case of interdependent values because, as we prove later on (see Subsection 4.1), no private values environment can be knit. The conclusion of Theorem 2 is the same that the one obtained by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006), but the analogy stops here as we mention in Section 3. Again, the same conclusion that mechanisms must be constant is reached by Che, Kim, and Kojima (2015) in the specific context of house allocation; by Austen-Smith and Feddersen (2006) in a voting context, and by Dasgupta and Maskin (2000) for the case of auctions. Examples 2, 4 and 6 below also contain negative results for special cases within the settings proposed by Austen-Smith and Feddersen (2006), Che, Kim, and Kojima (2015), and Dasgupta and Maskin (2000), respectively, analyzed in Subsection 4.2.

The paper proceeds as follows. In the next Section 2 we present the general framework and define the restrictions on environments that we propose, and the kind of mechanisms we shall concentrate on. Section 3 contains the main results in the paper. Section 4 provides examples of applications and ties them in with our general framework and main results. Some discussion is presented in Section 5. Finally, Appendix A contains the proofs of the main results in the paper, Appendix B presents the proofs of the results related to the applications, while in Appendix C we illustrate some basic definitions and an observation presented in the further comments section.

2 The model

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$ and A be a set of *alternatives*.

Let $\widetilde{\mathcal{R}}$ be the set of all complete, reflexive, and transitive binary relations on A and $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}$ be the set of those preferences that are allowed for individual i. While $R_i \in \widetilde{\mathcal{R}}$ denotes agent i's preferences, let P_i and I_i be the strict and the indifference part of R_i, respectively. For any $x \in A$ and $R_i \in \widetilde{\mathcal{R}}$, $U(R_i, x) = \{y \in A : yR_ix\}$ is the *(weak) upper contour set of* R_i *at* x and $\overline{U}(R_i, x) = \{y \in A : yP_ix\}$ is the *strict upper contour set of* R_i *at* x. Consider the next definition on preferences.

Definition 1 *We say that* $R'_i \in \widetilde{\mathcal{R}}$ *is an* x-***monotonic transform of*** $R_i \in \widetilde{\mathcal{R}}$ *if* $U(R'_i, x) \subseteq U(R_i, x)$ *and* $\overline{U}(R'_i, x) \subseteq \overline{U}(R_i, x)$.[4]

[3] A pioneering paper by Shenker (1993) investigated the connections between individual and group strategy-proof non-bossy social choice rules in economic environments. For a recent reference on efficiency in general environments, see Copic (2017).

[4] In words, R'_i is an x-monotonic transform of R_i if there exists a subset of x's indifference class in R_i, containing x, such that the relative position of its elements has weakly improved when going from R_i to R'_i (in our previous paper Barberà, Berga, and Moreno, 2012b, we present a similar condition but with an

A special class of monotonic transforms that are easy to identify are those where two preference relations have exactly the same weak and also the same strict upper contour sets for a given alternative x. Then we say that they are *reshufflings* of each other, and each of the two preferences are, in particular, monotonic transforms of the other.

Elements $R = (R_1, ..., R_n)$ in $\times_{i \in N} \mathcal{R}_i$ are called *preference profiles.* The next property on preferences will be used in defining our main conditions.

Each agent $i \in N$ is endowed with a *type* θ_i belonging to a set Θ_i. Each θ_i includes all the information in the hands of i. We denote by $\Theta = \times_{i \in N} \Theta_i$ the set of type profiles. A *type profile* is an n-tuple $\theta = (\theta_1, ..., \theta_n) \in \Theta$ that we will write as $\theta = (\theta_C, \theta_{N \setminus C})$ when we want to stress the role of coalition C in N.

Once type profiles are fully determined, so are agents' preferences. We formalize this dependence through the notion of a *preference function.*

Definition 2 *Let Θ be a set of type profiles. A* ***preference function*** *$\mathfrak{R}$ on Θ, $\mathfrak{R} : \Theta \to \times_{i \in N} \mathcal{R}_i$, assigns a preferences profile $\mathfrak{R}(\theta)$ to each type profile $\theta \in \Theta$.*

We call $\mathfrak{R}(\theta) = (R_1(\theta), ..., R_n(\theta))$ the preference profile induced by the type profile θ while $R_i(\theta) \in \mathcal{R}_i$ stands for the induced preferences of agent i at θ. As usual $P_i(\theta)$ and $I_i(\theta)$ denote the strict and the indifference part of $R_i(\theta)$, respectively. Notice that $\mathcal{R}_i$ may be different for each agent.[5] Moreover, the domain of the preference function $\mathfrak{R}$ is a Cartesian product including all possible type profiles, but its range may be a non-Cartesian strict subset of $\times_{i \in N} \mathcal{R}_i$.

An *environment* is a pair $(\Theta, \mathfrak{R})$ formed by a set of type profiles and a preference function. Following standard use, *private values environments* are those where each agent's component of the preference function only depends on her type. That is, $R_i(\theta) = R_i(\theta_i, \theta'_{N \setminus \{i\}})$ for each agent $i \in N$, $\theta \in \Theta$, and $\theta'_{N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \Theta_j$. Otherwise, we are in *interdependent values environments.* In private values environments, abusing notation, we will write $R_i(\theta_i)$ instead of $R_i(\theta)$.

Elements in the range of a preference function may be restricted to satisfy further conditions. In particular, an *environment* $(\Theta, \mathfrak{R})$ is *strict* if for any $\theta \in \Theta$ and any agent $i \in N$, $R_i(\theta) \in \mathcal{R}_i$ is a strict preference.

Our results refer to direct mechanisms. In fact, the properties we discuss are best analyzed with reference to the direct mechanism associated to any general one that might be described in terms of different message spaces and outcome functions.

A *direct mechanism* (*on* Θ) is a function $f : \Theta \to A$. From now on, we drop the term "direct" and the reference to the set of type profiles and simply talk about mechanisms, without danger of ambiguity.

Notice that, by letting Θ be the domain of f, we implicitly assume that all type profiles within this set are considered to be feasible by the designer.

additional requirement).

[5]This is the case, for example, in economies with private goods when individuals are selfish.

2.1 Properties of environments

We shall now identify conditions on environments, that we call *partially knit* and *knit* (see Definitions 5 and 6), that may or may not be satisfied. Both conditions are similar in that they analyze sequences of type profiles and their induced sequences of preference profiles. But they are different in that the type sequences on which they impose restrictions are not the same. In both cases, the sequences of type profiles must be related to each other so that each of its elements differs from its predecessor and its follower in the type of only one individual. The sequence of preference profiles will be required to be *satisfactory* in a sense to be defined below.

We now formalize the kind of sequences of both type and preference profiles that we consider and the relationship between them.

Let $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$ be a sequence of individual types of length t_S, such that for each $h \in \{1, ..., t_S\}$, $\theta^S_{i(S,h)} \in \Theta_{i(S,h)}$. Agents may appear in that sequence several times or not at all. $I(S) = \{i(S,1), ..., i(S,t_S)\}$ is the sequence of agents whose types appear in S and $i(S,h)$ is the agent in position h in S.

Given $\theta \in \Theta$ and $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$, we consider the sequence of type profiles $m^h(\theta, S)$ that results from changing one at a time the types of agents according to S, starting from θ. Formally, $m^h(\theta, S) \in \Theta$ is defined recursively so that $m^0(\theta, S) = \theta$ and for each $h \in \{1, ..., t_S\}$, $m^h(\theta, S) = \left(\left(m^{h-1}(\theta, S)\right)_{N \setminus i(S,h)}, \theta^S_{i(S,h)}\right)$.

Let $\theta \in \Theta$, and $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$. We call the sequence of type profiles $\left\{m^h(\theta, S)\right\}_{h=0}^{t_S}$ **the passage from θ to θ' through** S if $m^{t_S}(\theta, S) = \theta'$ for $\theta' \in \Theta$. More informally, we say that θ leads to θ' through S.

Notice that a given passage from θ to θ' through S induces a corresponding sequence of preference profiles, $R^h(\theta, S) = (R^h_1(\theta, S), ..., R^h_n(\theta, S)) \in \times_{i \in N} \mathcal{R}_i$ for $h \in \{0, 1, ..., t_S\}$ where for each agent $i \in N$, we define $R^h_i(\theta, S) \equiv R_i\left(m^h(\theta, S)\right) \in \mathcal{R}_i$, that is, as the ith component of the preference function at the type profile $m^h(\theta, S)$.

We can now establish a condition (Definition 3) on the connection between sequences of changes in type profiles and the changes in preference profiles that they induce by means of the preference function.

Definition 3 *Let $x \in A$, $\theta, \theta' \in \Theta$. We will say that the passage from θ to θ' through S is x-**satisfactory** if for each $h \in \{1, ..., t_S\}$, $R^h_{i(S,h)}(\theta, S)$ is an x-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.*

We say that x is the reference alternative when going from θ to θ'. Notice that in the case of private values the order of individuals in S could be changed and the new sequence would still serve the same purpose. This is because the changes in the type of each agent only induce changes in the preferences of this agent. By contrast, the precise order of agents $I(S)$ may be crucial in the case of interdependent values.

Armed with our previous definitions we now define when two pairs, each of them formed by an alternative and a type profile, are *pairwise knit*. Whether or not two pairs are pairwise knit will depend on how the preference function determines what sequences are satisfactory.

Definition 4 *Two pairs formed by an alternative and a type profile each,* (x, θ) *and* $(z, \widetilde{\theta})$*, are* ***pairwise knit*** *in the environment* $(\Theta, \mathfrak{R})$ *if* (x, θ)*,* $(z, \widetilde{\theta}) \in A \times \Theta$*,* $\theta \neq \widetilde{\theta}$*, and there exist* $\theta' \in \Theta$ *and sequences of types* S *and* $\widetilde{S}$*, such that the passage from* θ *to* θ' *through* S *is* x*-satisfactory and the passage from* $\widetilde{\theta}$ *to* θ' *through* $\widetilde{S}$ *is* z*-satisfactory.*

We shall now define knit environments. This condition requires that any two pairs formed by an alternative and a type profile must be pairwise knit.

Definition 5 *We say that an environment* $(\Theta, \mathfrak{R})$ *is* ***knit*** *if any two pairs* (x, θ)*,* $(z, \widetilde{\theta}) \in A \times \Theta$*,* $\theta \neq \widetilde{\theta}$*,* $x \neq z$ *are pairwise knit in* $(\Theta, \mathfrak{R})$*.*

A weaker condition is that of partial knitness, which does not require pairwise knitness for all pairs, but for some of them. The relevant pairs will be determined by the following two sets of agents: for any $\theta \in \Theta$ and $x, z \in A$, let $C(\theta, z, x) = \{i \in N : zR_i(\theta)x\}$ and $\overline{C}(\theta, z, x) = \{j \in N : zP_j(\theta)x\}$.

Definition 6 *We say that an environment* $(\Theta, \mathfrak{R})$ *is* ***partially knit*** *if any two pairs formed by an alternative and a type profile each,* (x, θ)*,* $(z, \widetilde{\theta}) \in A \times \Theta$*,* $\theta \neq \widetilde{\theta}$*, such that* $\overline{C}(\theta, z, x) \neq \emptyset$*,* $\#C(\theta, z, x) \geq 2$*, and* $\widetilde{\theta}_j = \theta_j$ *for any* $j \in N \backslash C(\theta, z, x)$ *are pairwise knit in* $(\Theta, \mathfrak{R})$*.*

By definition, if an environment is knit it is also partially knit. It is worth noting that whether or not an environment is knit or partially knit will depend on the way how the preference function determines what sequences are considered to be satisfactory. In Section 5 we present an intuition behind our two conditions on environments.

To end this Section, we use Example 1 to illustrate the concept of satisfactory and non-satisfactory passages and knitness in an interdependent values environment (see Remarks 4 and 5 in Appendix C).[6]

Example 1 *Let* $N = \{1, 2\}$ *and* $A = \{a, b, c\}$*. Each agent* i *has two possible types:* $\Theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}$*. The preference function* $\mathfrak{R}$ *is defined in Table 1. We write, in each cell, the preferences of both agents for a given type profile represented by an ordered list from better to worse, with parenthesis in case of indifferences. Observe that agent 2's preferences over* b *and* c *depend on agent 1's type:* $bP_2(\underline{\theta}_1, \underline{\theta}_2)c$ *while* $cP_2(\overline{\theta}_1, \underline{\theta}_2)b$*, that is, we are in an interdependent values environment.*

$\mathfrak{R}$	$\underline{\theta}_2$		$\overline{\theta}_2$	
$\underline{\theta}_1$	$R_1(\underline{\theta}_1, \underline{\theta}_2)$ acb	$R_2(\underline{\theta}_1, \underline{\theta}_2)$ $b(ac)$	$R_1(\underline{\theta}_1, \overline{\theta}_2)$ bca	$R_2(\underline{\theta}_1, \overline{\theta}_2)$ $a(bc)$
$\overline{\theta}_1$	$R_1(\overline{\theta}_1, \underline{\theta}_2)$ $c(ab)$	$R_2(\overline{\theta}_1, \underline{\theta}_2)$ $c(ab)$	$R_1(\overline{\theta}_1, \overline{\theta}_2)$ $c(ab)$	$R_2(\overline{\theta}_1, \overline{\theta}_2)$ $c(ab)$

Table 1. Preference function for Example 1.

[6] This example adapts, in ordinal terms, the one proposed by Bergemann and Morris (2005) as their Example 1.

Notice that the range of $\mathfrak{R}$ *is not a Cartesian product, since* $\mathcal{R}_1 = \{acb, bca, c(ab)\}$ *and* $\mathcal{R}_2 = \{b(ac)), a(bc), c(ab)\}$ *but the preferences profile* $(acb, a(bc))$ *is not in the range of the preference function* $\mathfrak{R}$*.*
Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \underline{\theta}_2)$, *and* $S = \{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\}$ *be a sequence of individual types. Note that,* $I(S) = \{2, 1, 2\}$ *and* $t_S = 3$*. The passage from* θ *to* θ' *through* S *is a-satisfactory.*
Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, *and* $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ *be a sequence of individual types. Note that,* $I(S) = \{1, 2\}$ *and* $t_S = 2$*. The passage from* θ *to* θ' *through* S *is not a-satisfactory.*

2.2 Properties of mechanisms

Until now, we have concentrated on the properties of potential environments. We now turn attention to some properties of the mechanisms themselves, first looking at incentives.

Definition 7 *Let* $(\Theta, \mathfrak{R})$ *be an environment. We say that* ***an agent*** $i \in N$ ***can ex post profitably deviate under mechanism*** f ***at*** $\theta \in \Theta$ *if there exists* $\theta'_i \in \Theta_i$ *such that* $f(\theta'_i, \theta_{N\setminus\{i\}})P_i(\theta)f(\theta)$*. A mechanism* f *is* ***ex post incentive compatible*** *in* $(\Theta, \mathfrak{R})$ *if no agent can ex post profitably deviate at any type profile.*[7]

Therefore, the play where all agents reveal their true type must be a Nash equilibrium of the revelation game induced by the environment $(\Theta, \mathfrak{R})$.

In addition to individuals, coalitions of agents may also jointly deviate if they find it profitable. This leads us to propose the following definition.

Definition 8 *Let* $(\Theta, \mathfrak{R})$ *be an environment. We say that* ***a coalition*** $C \subseteq N$ ***can ex post profitably deviate under mechanism*** f ***at*** $\theta \in \Theta$ *if there exists* $\theta'_C \in \times_{i \in C}\Theta_i$ *such that for all agent* $i \in C$, $f(\theta'_C, \theta_{N\setminus C})R_i(\theta)f(\theta)$ *and for some* $j \in C$, $f(\theta'_C, \theta_{N\setminus C})P_j(\theta)f(\theta)$*. A mechanism* f *is* ***ex post group incentive compatible*** *in* $(\Theta, \mathfrak{R})$ *if no coalition of agents can ex post profitably deviate at any type profile.*[8]

Finally, we may require our mechanisms to satisfy a condition that we call respectfulness. This is a condition similar to those imposed in the literature when dealing with environments where agents' preferences allow for non-degenerate indifference classes (See Thomson, 2016). Relative to other technical conditions of the same sort, ours is among the weakest, because it only applies to some limited changes in type profiles, and has no bite in some important cases (for example, in public good economies where agents' preferences are strict). The condition essentially demands that for those specific changes in type profiles, no agent should affect the outcome (for her and for others) unless she changes her level of satisfaction.

[7]This property is called uniform incentive compatibility by Holmstrom and Myerson (1983). See also Bergemann and Morris (2005).

[8]Notice that we allow for some agents to participate in the profitable deviation without strictly gaining from it. Moreover, we also allow for some agents not to change their types. That facilitates the deviation by groups. Che, Kim, and Kojima (2015) consider group manipulations using the concept of first order stochastical dominance.

Definition 9 *Let* $(\Theta, \mathfrak{R})$ *be an environment. A mechanism* f *is* ***(outcome) respectful*** *in* $(\Theta, \mathfrak{R})$ *if*

$$f(\theta) I_i(\theta) f(\theta_i', \theta_{N \setminus \{i\}}) \text{ implies } f(\theta) = f(\theta_i', \theta_{N \setminus \{i\}}),$$

for each $i \in N$, $\theta \in \Theta$, *and* $\theta_i' \in \Theta_i$ *such that* $R_i(\theta_i', \theta_{N \setminus \{i\}})$ *is a* $f(\theta)$*-monotonic transform of* $R_i(\theta)$.[9]

For short, we call this condition respectfulness.

The following two Paretian notions of efficiency will be used in our discussion of results.

Definition 10 *Let* $(\Theta, \mathfrak{R})$ *be an environment. A mechanism* f *is* ***Pareto efficient on the range*** *in* $(\Theta, \mathfrak{R})$ *if for all* $\theta \in \Theta$, *there is no alternative* x *in the range of* f *such that* $xR_i(\theta)f(\theta)$ *for all* $i \in N$ *and* $xP_j(\theta)f(\theta)$ *for some* $j \in N$. *If, in addition, the mechanism is onto* A *we say that it is* ***fully efficient*** *in* $(\Theta, \mathfrak{R})$.

Notice that ex post group incentive compatibility implies Pareto efficiency on the range, since otherwise the grand coalition could profitably deviate. From now on we can omit reference to the environments on properties of f when no confusion arises.

Remark that the definition of ex post incentive compatibility is purely ordinal. Since we concentrate on this property, our whole framework is expressed in ordinal terms.[10]

3 The results

We now present our main results and discuss about their consequences. The necessity of partial knitness and knitness in our two theorems is commented in Section 5 while all the proofs are placed in Appendix A.

Our first result shows the equivalence between ex post individual and group incentive compatibility in partially knit environments and has bite for both private and interdependent values environments.

Theorem 1 *Let* $(\Theta, \mathfrak{R})$ *be any partially knit environment and* f *be any respectful mechanism in* $(\Theta, \mathfrak{R})$. *Then,* f *is ex post incentive compatible in* $(\Theta, \mathfrak{R})$ *if and only if* f *is ex post group incentive compatible in* $(\Theta, \mathfrak{R})$.

The theorem restricts attention to mechanisms that are respectful, but note that the latter requirement does not always have bite: it is irrelevant when the environment is strict, that is, when the preferences of all agents under all type profiles are strict (see Corollary 1).

[9]Respectfulness is an analogous condition to the one we use in Barberà, Berga, and Moreno (2016) but requiring here invariance in outcomes instead of indifferences in outcomes. Examples of mechanisms satisfying respectfulness are provided in Section 4. An example of a mechanism violating it is the Gale-Shapley mechanism (see Barberà, Berga, and Moreno, 2016).

[10]We do not resort to the use of utility functions like other authors do, mostly because they also analyze other notions of incentive compatibility that require it. The use of utility functions that represent the preferences of expected utility maximizers is especially useful to analyze incentive compatibility notions that involve uncertainty regarding the types.

Corollary 1 *Let* $(\Theta, \mathfrak{R})$ *be any partially knit strict environment and* f *be any mechanism. Then,* f *is ex post incentive compatible in* $(\Theta, \mathfrak{R})$ *if and only if* f *is ex post group incentive compatible in* $(\Theta, \mathfrak{R})$.

Let us discuss the content and implications of the Theorem.

A first consequence of ex post group incentive compatibility is Pareto efficiency on the mechanism's range. Hence, the implications that having a good performance regarding incentives may be compatible with efficiency is an invitation to investigate those cases where this may be a promising possibility. This is not always the case, even with ordinal preferences as observed by Yamashita and Zhu (2018). We have identified different papers where non-trivial mechanisms that are ex post incentive compatible and efficient do exist in the case of interdependent values environments. This includes environments considered in Pourpouneh, Ramezanianz, and Sen (2018), in Proposition 4 in Dasgupta and Maskin (1979),[11] and also Examples 3, 5, and 7 that we discuss in Subsection 4.2 of this paper. The list of environments where Theorem 1 opens the door to the existence of non-trivial, ex post incentive compatible, and respectful mechanisms also include cases with private values, since the latter are just a special case of interdependent ones. In Subsection 4.1 we show that our Theorem 1 applies to several well-known and important private values environments because they are partially knit (see Propositions 2, 3, 4, and 5).

Second, observe that in private values situations where environments are partially knit, the result in Theorem 1 admits a second reading. This is because ex post incentive compatibility then becomes equivalent to strategy-proofness,[12] since each agent i's preferences depend on θ only through θ_i. For the same reason, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness. These remarks lead us to the following corollary.

Corollary 2 *Let* $(\Theta, \mathfrak{R})$ *be any partially knit environment in private values and let* f *be any respectful mechanism in* $(\Theta, \mathfrak{R})$. *Then,* f *is strategy-proof in* $(\Theta, \mathfrak{R})$ *if and only if* f *is strongly group strategy-proof in* $(\Theta, \mathfrak{R})$.

The equivalence between individual and group ex post incentive compatibility may hold in rather vacuous ways, because there are cases where the only ex post incentive compatible rules lack any interest. But there are other cases where there is a real possibility of making these desiderata compatible in non-trivial ways.

Here are some relevant examples of mechanisms for which the result holds non-trivially in private values environments. One of them is the family of generalized median voters rules defined on the set of all strict single-peaked preferences (see Moulin, 1980 and our Proposition 3). Another case is provided by the top trading cycle mechanism for house

[11] Proofs showing that the environments in these two problems are partially knit are available upon request.

[12] We say that a mechanism f is weakly group manipulable at $\theta \in \Theta$ if there exist a coalition $C \subseteq N$ and $\theta'_C \in \times_{i \in C} \Theta_i$ ($\theta'_i \neq \theta_i$ for any $i \in C$) such that $f(\theta'_C, \theta_{-C}) R_i(\theta_i) f(\theta)$ for all $i \in C$ and $f(\theta'_C, \theta_{-C}) P_j(\theta_j) f(\theta)$ for some $j \in C$. A mechanism f is strongly group strategy-proof in an environment (Θ, R) if f is not weakly group manipulable at any $\theta \in \Theta$. When the condition is imposed only on singleton coalitions $C = \{i\}$, we say that f is strategy-proof (also called dominant strategy incentive compatible). In words, strategy-proofnes requires that all agents prefer truthtelling at a given type profile θ, whatever all the other agents report.

allocation (see Shapley and Scarf, 1974 and our Proposition 4). A third example is given by veto rules or serial dictators in cases where only two alternatives are at stake and agent's preferences are strict (see Barberà, Berga, and Moreno, 2012a, Manjunath, 2012, Larsson and Svensson, 2006; and Proposition 2 below for two alternatives). A fourth example is the class of peak rules defined by Saporiti (2009) for single-crossing preferences with three alternatives at stake (see Grandmont, 1978 and our Proposition 5).

In all four cases we are dealing with partially knit private values environments (see Subsection 4.1) where a type for an agent can be identified with her preference relation, the mechanisms are individual and strongly group strategy-proof, and by no means trivial. Also remark that for the case where the mechanism has more than two alternatives on the range, only dictatorship is strategy-proof on the universal set of preferences, by the Gibbard-Satterthwaite theorem (see Gibbard, 1973 and Satterthwaite, 1975). This is an example in which our Theorem 1 also applies, since the universal set of strict preferences is partially knit (see Proposition 2) and dictatorships are strongly group strategy-proof, but we use it here as a warning sign that the implications of Theorem 1, as already explained may or may not be of interest depending on the environments.

We now present our second result that shows that only constant mechanisms can be ex post incentive compatible and respectful in knit environments.

Theorem 2 *Let* $(\Theta, \mathfrak{R})$ *be any knit environment and* $f : \Theta \to A$ *be any mechanism. If* f *is ex post incentive compatible and respectful, then* f *is constant.*

Similarly to our previous theorem, since respectfulness does not have bite for strict environments, the following Corollary 3 holds straightforwardly.

Corollary 3 *Let* $(\Theta, \mathfrak{R})$ *be any strict knit environment and* $f : \Theta \to A$ *be any mechanism. If* f *is ex post incentive compatible, then* f *is constant.*

Some comments on other papers with related results and on implications of the result are in order. The conclusion of Theorem 2 is in the same vein than the one that Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006) obtain under completely different premises. These authors focus on environments where preference profiles will be represented by n-tuples of money-separable utility functions, and where the preference function is smooth, among other assumptions, while our restrictions apply to environments that do not have such characteristics. Other papers arrive at the same conclusion than Theorem 2 in the context of more specific models. These include the models considered by Che, Kim, and Kojima (2015) (see their Theorem 1), Austen-Smith and Feddersen (2006) (see their Theorem and Corollary), Dasgupta and Maskin (2000) (see their Example 4), and also Examples 2, 4, and 6 in our Subsection 4.2 below.

Observe also that there is no contradiction between our result in Theorem 2 that only constant mechanisms are strategy-proof and that of the Gibbard-Satterthwaite theorem which also admits dictatorship, since the universal set of preferences where the latter applies is not knit, as shown in Proposition 1, and thus Theorem 2 does not apply.

Finally notice that, since we work with single valued direct mechanisms, our environments are separable in the sense of Bergemann and Morris (2005), and their Corollary 1 applies: no mechanism is interim incentive compatible unless it is ex post incentive compatible. Because of that, Theorem 2 and Corollary 3 have direct implications on the weaker interim notion, with no need to be explicit about agents' beliefs.

4 Applications

In this section we present examples of environments satisfying our properties and to which our main results apply. First, we concentrate on well-known frameworks in private values environments. Then, we present and analyze some interdependent values environments in voting, in allocation problems, and in auctions where our theorems have bite. The proofs of all the results presented in this section are placed in Appendix B.

4.1 Private values environments

In private values environments each individual preferences is obtained through the preference function from her own type. That is, changes in j's type do not affect i's preferences if $i \neq j$.

In the following Proposition 1 we state that no private values environment can be knit. Thus, our Theorem 2 has no bite for those environments.

Proposition 1 *No private values environment* $(\Theta, \mathfrak{R})$ *for which there exist* $\theta_i, \widetilde{\theta}_i \in \Theta_i$ *such that* $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$ *for some* $i \in N$ *can be knit.*

Propositions 2, 3, 4, and 5 below state that partial knitness is satisfied by several well-known private values environments. To avoid extra notation at this point, we define each one of the environments with detail at the beginning of the proof of the corresponding proposition.

These four private values environments have an additional common characteristic: types and preferences coincide. In particular, for each agent i, $R_i(\theta_i) = \theta_i$ and $\Theta_i = \mathcal{R}_i$. Thus, each component of the preference function $\mathfrak{R}$ is the identity and we write the environment simply as the Cartesian product of individual preferences $\times_{i\in N}\mathcal{R}_i$.

We begin by the universal domain of strict preferences.

Proposition 2 *The Cartesian product of the set of all strict preferences in the classical social choice problem is partially knit.*

Note that a particular case encompassed in Proposition 2 is the one with two alternatives at stake.

Another interesting case is provided by the set of strict single-peaked preferences on a finite set of alternatives. We know that it is not knit by Proposition 1, but as stated in Proposition 3, it is partially knit.

Proposition 3 *The Cartesian product of the set of all strict single-peaked preferences on a finite set of alternatives in the classical social choice problem is partially knit.*

In the housing problem, agents' admissible preferences over their individual assignment are strict. And, again, they define a partially knit environment, as stated in Proposition 4.

Proposition 4 *The Cartesian product of the set of all preferences in the housing problem is partially knit.*[13]

The last example we include is one where there are only three alternatives at stake and the set of preferences of each agent is any subset of strict ones. Any domain of such preferences is partially knit.

Proposition 5 *The Cartesian product of any subset of individual preferences over* $A = \{x, y, z\}$ *is partially knit.*

4.2 Interdependent values environments

The situations we describe in interdependent and non-private values environments are simple, as examples must be, but chosen to highlight essential contributions to several fields of application: voting, allocations, and auctions. The examples come in pairs, to show that, with the same sets of type profiles, but depending on the associated preference functions, one can cross the line between positive and negative results. Examples 2 and 3 refer to deliberative juries and are inspired in our reading of Austen-Smith and Feddersen (2006) who build on the classical Condorcet jury problem and add the possibility that agents share (true or false) information. Our second pair of examples, 4 and 5, refer to house allocation problems and are this time inspired by the analysis of Che, Kim, and Kojima (2015), regarding the existence of Pareto efficient and ex-post incentive compatible mechanisms in that context.. The last two, Examples 6 and 7, refer to auctions, following the trail of Dasgupta and Maskin (2000) and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006).

These examples are framed in the language we have developed in our paper, and they allow us to clarify several of the points we try to make all along. In particular, we can provide blood and flesh to the general and rather abstract notion of a preference function, by exhibiting how it is defined to fit the particulars of the case at hand.

4.2.1 Deliberative Juries

Example 2 A three-person jury $N = \{1, 2, 3\}$ must decide over two alternatives: whether to acquit (A) or to convict (C) a defendant under a given mechanism. The defendant is either guilty (g) or innocent (i). Each juror j gets a signal $s_j = g$ or $s_j = i$.

Jurors's preferences arise from combining the different signals they obtain from the deliberation, according to their bias in favor of acquittal in view of their observed signals and of those declared by others. In this example, jurors are either high-biased (h) or low-biased (l). High-biased jurors (h) prefer to convict if and only if all other jurors declare the guilty signal and they have also observed it ($s = (g, g, g)$), whereas low-biased ones (l) prefer to

[13]The same result would hold in the one-to-one matching problem where admissible preferences over individual assignments are strict and different for each agent: those of each woman are defined on all men and on herself, while those of each man are defined on all women and himself.

convict if and only if they have observed the guilty signal or at least one other committee member has declared it ($s \neq (i, i, i)$).

Each juror j's type is $\theta_j = (b_j, s_j) \in \Theta_j = B \times S$ where $B = \{h, l\}$ and $S = \{g, i\}$. A type profile $\theta \in \Theta = (B \times S)^n$. Let CA denote the preference to convict rather than to acquit and AC be the converse order. The preference function is defined such that for each type profile $\theta \in \Theta$ and for each juror $j \in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j, s_j), \theta_{N \setminus \{j\}}\right) = \left\{ \begin{array}{l} CA \text{ if either } b_j = h \text{ and } s = (g, g, g) \text{ or } b_j = l \text{ and } s \neq (i, i, i), \\ AC, \text{ otherwise.} \end{array} \right\}$$

The environment $(\Theta, \mathfrak{R})$ in this example is knit (see Proposition 6). Hence we know by Theorem 2 that it will be impossible to design non-constant, ex post incentive compatible, and respectful mechanisms in such framework.

Proposition 6 *The environment* $(\Theta, \mathfrak{R})$ *in Example 2 is knit.*

We provide the reader with some hints on the techniques that we use to check for our restrictions on environments in this example and subsequent ones.[14]

To check knitness for a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$, we must show that there are passages to a third type profile θ' which are A-satisfactory from θ and C-satisfactory from $\widetilde{\theta}$, respectively.

Consider the following three type profiles, $\theta = (\theta_1, \theta_2, \theta_3) = ((l, g), (h, g), (l, i))$, $\widetilde{\theta} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3) = ((l, g), (h, g), (l, g))$ and $\theta' = (\theta'_1, \theta'_2, \theta'_3) = ((l, i), (h, i), (l, i))$. The profiles of preferences they induce are shown in Table 2.

$R(\theta) = R((l, g), (h, g), (l, i))$	$R(\widetilde{\theta}) = R((l, g), (h, g), (l, g))$	$R(\theta') = R((l, i), (h, i), (l, i))$
$C \quad A \quad C$	$C \quad C \quad C$	$A \quad A \quad A$
$A \quad C \quad A$	$A \quad A \quad A$	$C \quad C \quad C$

Table 2: Agents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

As shown in Table 3, it is possible to sequentially move from θ to θ' by successively changing, one by one, the type of the agents as follows. First, agent 1 from (l, g) to (h, i), then agent 2 from (h, g) to (h, i) and finally agent 1 from (h, i) to (l, i). According to our notation, $S = ((h, i), (h, i), (l, i))$ and $I(S) = \{1, 2, 1\}$. Likewise, as shown in Table 4, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. First, agent 1, then agent 3 and finally agent 2, all from signal g to i, while their b's remain fixed. That is, $\widetilde{S} = ((l, i), (l, i), (h, i))$ and $I(\widetilde{S}) = \{1, 3, 2\}$. In Table 3, alternative A either does not change its relative position (an A-reshuffling), or improves it (an A-monotonic transform). Similarly, in Table 4, the same requirements are satisfied but this time for alternative C.

[14] The reader that finds the following argument useful to better understand our condition may also find a similar one regarding partial knitness in the text preceding the proof of Proposition 7 in Appendix B.

$R(\theta)$	$R^1(\theta, S)$	$R^2(\theta, S)$	$R^3(\theta, S) = R(\theta')$
$R((l,g),(h,g),(l,i))$	$R((\mathbf{h},\mathbf{i}),(h,g),(l,i))$	$R((h,i),(h,\mathbf{i}),(l,i))$	$R((\mathbf{l},i),(h,i),(l,i))$
C A C	A A C	A A A	A A A
A C A	C C A	C C C	C C C

Table 3: Induced agents' preferences given the specified type changes from θ to θ'.

$R(\widetilde{\theta})$	$R^1(\widetilde{\theta}, \widetilde{S})$	$R^2(\widetilde{\theta}, \widetilde{S})$	$R^3(\widetilde{\theta}, \widetilde{S}) = R(\theta')$
$R((l,g),(h,g),(l,g))$	$R((l,\mathbf{i}),(h,g),(l,g))$	$R((l,i),(h,g),(l,\mathbf{i}))$	$R((l,i),(h,\mathbf{i}),(l,i))$
C C C	C A C	C A C	A A A
A A A	A C A	A C A	C C C

Table 4: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

Example 3 Consider the framework of Example 2 and change the jurors' attitude to convict versus acquit as follows. Each juror may now be either unswerving or median. Unswerving jurors (u) prefer to convict if and only if they have observed the guilty sign and have also received such a sign from at least another juror. Median jurors (m) again prefer to convict under the same circumstances but also if they receive two guilty signals from other jurors.

For instance, if juror 1 is unswerving she will prefer to convict if either (g,g,g), (g,g,i), or (g,i,g) but if juror 2 is unswerving she will convict if either (g,g,g), (g,g,i), or (i,g,g)). Yet being median is the same for both agents, they will prefer to convict if either (g,g,g), (g,g,i), (g,i,g), or (i,g,g).

Each juror j's type is $\theta_j = (b_j, s_j) \in \Theta_j = B \times S$ where $B = \{u, m\}$ and $S = \{g, i\}$. A type profile $\theta \in \Theta = (B \times S)^n$. The preference function is defined such that for each type profile θ and for each juror $j \in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j, s_j), \theta_{N\setminus\{j\}}\right) = \left\{ \begin{array}{ll} CA & \text{if either } b_j = u,\ s_j = g \text{ and } s_l = g \text{ for some } l \neq j, \\ & \text{or } b_j = m \text{ and } \#\{l \in N : s_l = g\} \geq 2, \text{ and} \\ AC & \text{otherwise.} \end{array} \right\}$$

This environment $(\Theta, \mathfrak{R})$ is partially knit (see Proposition 7) but not knit.

Proposition 7 *The environment* $(\Theta, \mathfrak{R})$ *in Example 3 is partially knit.*

To show that it is not knit, we present a family of mechanisms, the quota rules, that are non-constant, respectful, and ex post incentive compatible in $(\Theta, \mathfrak{R})$ which is stated in Remark 1.

Let $q \in \{1, 2, 3\}$. A *voting by quota* q mechanism, f, chooses C for a type profile θ if and only if at least q agents have induced preferences from θ such that C is preferred to A.[15] Formally, for each type profile $\theta = (b, s) \in \Theta$,

$$f(\theta) = C \text{ if and only if } \#\{i \in N : R_i(\theta) = CA\} \geq q.$$

[15] See Austen-Smith and Feddersen (2006) and Barberà and Jackson (2004) for papers where these rules are analized.

Remark 1 *A voting by quota q mechanism is non-costant, ex post incentive compatible, and respectful in the environment* $(\Theta, \mathfrak{R})$ *in Example 3.*

Now, Theorem 1 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, full efficiency is satisfied in this example because the range of the mechanism is the set of alternatives.

4.2.2 Private goods without money

Example 4 Let $N = \{1, 2\}$ be a set of agents, $O = \{a, c\}$ be a set of objects. Each agent must be assigned one and only one object. Thus, the set of alternatives is $A = \{x = (a, c), z = (c, a)\}$, where the first component refers to the object that agent 1 gets. There is no money in this economy.

The type $\theta_i \in \Theta_i$ of each agent i is given by a signal θ_i in $\Theta_i = [0, 1]$. Each individual $i \in N$ is endowed with a given auxiliary function $g_i : \Theta \rightarrow \mathbb{R}$ *increasing* in both signals.[16] The preference function $\mathfrak{R}$ is such that for each agent $i \in N$ and for each type profile $\theta \in \Theta = [0, 1] \times [0, 1]$, $R_i(\theta)$ is as follows: x is at least as good as z if and only if $g_i(\theta) \geq 0$.

The environment in Example 4 is knit (see Proposition 8). Therefore by Theorem 2 only constant mechanisms can be ex post incentive compatible and respectful in this context.

Proposition 8 *The environment* $(\Theta, \mathfrak{R})$ *in Example 4 is knit.*

Example 5 We consider the framework of Example 4, except that we change agents' preference functions to be induced by $g_1(\theta) = \min\left(median\left\{\frac{1}{4}, \theta_1, \theta_1, \theta_2\right\}\right) - \frac{1}{4}$ and $g_2(\theta) = \min\left(median\left\{\frac{1}{4}, \theta_2, \theta_2, \theta_1\right\}\right) - \frac{1}{4}$, respectively. That is, for each agent $i \in N$ and for each type profile $\theta \in \Theta$, $R_i(\theta)$ is as follows: x is at least as good as z if and only if $g_i(\theta) \geq 0$.

The main but significant difference between this example and the preceding one is that now the functions g_i are just weakly increasing.

Like in Example 3 above, the environment in this example is partially knit (see Proposition 9) but not knit.

Proposition 9 *The environment* $(\Theta, \mathfrak{R})$ *in Example 5 is partially knit.*

To prove non-knitness, we consider the veto mechanisms defined below. Before introducing them we need the following definition: consider a partition of the type (signal) space and a useful graphical representation of it which is similar to the one defined in Che, Kim, and Kojima (2015).

Let $\{S_{ac}, S_{ca}, S_{aa}, S_{cc}, S^0\}$ be the partition of Θ where:
S^0 is the set of type profiles for which both agents are indifferent between a and c,
S_{ac} is the set of type profiles for which agent 1 prefers a to c, agent 2 prefers c to a, and the

[16] Che, Kim, and Kojima (2015) also impose the following property which they call the *single-crossing property*: $\frac{\partial u_i(\theta)}{\partial s_i} > \frac{\partial u_j(\theta)}{\partial s_i}$ for any $\theta \in \Theta$. However, as they already mention, this condition is not required for the impossibility result to hold.

preferences are strict for at least one agent,
S_{ca} is equally defined after changing the roles of c and a,
S_{aa} is the set of type profiles for which both agents prefer a to c, and
S_{cc} is equally defined after changing the roles of c and a.

In terms of alternatives, when the type profiles are in S_{ac} both agents prefer x to z, when they are in S_{ca} both prefer z to x, in S_{aa}, 1 prefers x over z and 2 prefers z over x, in S_{cc}, 1 prefers z over x and 2 prefers x over z, and in S^0 both are indifferent between x and z.

Figure 1 provides a generic representation of these sets whose frontiers correspond to the pairs of signals leading to agents' indifference curves over alternatives: $\{\theta \in \Theta = [0,1]\times[0,1] : xI_i(\theta)y\}$. Since we have assumed that g_i is increasing in both types, agents' indifference curves are strictly decreasing, and since S_{ac} and S_{ca} are non-empty the two curves will have an interior intersection.[17]

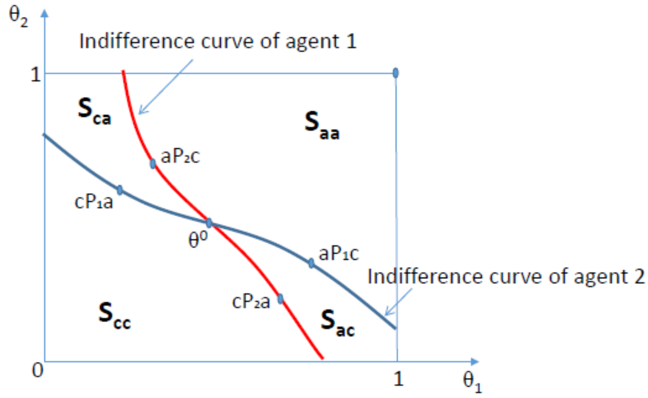


Figure 1. Examples of the partition of S in Example 4.

Now we say that a mechanism $f_{veto\ x}$ is a *veto rule for* x if for any type profile the outcome is agent 1's best alternative when it is unique, and it is agent 2's best alternative otherwise. Formally, for $\theta \in \Theta = [0,1] \times [0,1]$,

$$f_{veto\ x}(\theta) = \left\{ \begin{array}{c} x = (a,c) \text{ if } \theta \in S_{ca}, \text{ and} \\ z = (c,a) \text{ if } \theta \in S_{aa} \cup S_{ac} \cup S_{cc} \cup S^0 \end{array} \right\}.$$

In view of Theorem 2 the existence of these non-constant, ex post incentive compatible, and respectful mechanisms implies that the environment is no longer knit. In Remark 2 we show that veto rules satisfy the three properties.

Remark 2 *$f_{veto,x}$ is non-constant, ex post incentive compatible, and respectful in the environment $(\Theta, \mathfrak{R})$ in Example 5.*

Now, Theorem 1 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, full efficiency is obtained in this example since the range is the whole set of alternatives.

[17] Although in all pictures corresponding to this example the indifference curves only intersect once, our formal arguments apply to the multiple intersection case.

4.2.3 Auctions

There is one unit of an indivisible good to be auctioned. Let N be the set of buyers (agents). An alternative in this model tells us which single agent, if any, gets the good and what positive price she pays for it, meaning then that the rest of agents do not get the good and pay zero. If no agent gets the good, no one pays anything. Formally, an alternative x is written as $x = (x_1, ..., x_n) \in A = (\{0,1\} \times \mathbb{R}_+)^n$, with $x_i = (a_i, p_i)$ where $a_i = 1$ and $p_i > 0$ if and only if agent i gets the good, and $p_l = 0$ for all agents l that do not get it.

We assume that agents' preferences are selfish. Agents only care about whether or not they are awarded the good and, if so, about how much they must pay for it. Therefore, we can define their preferences on the part of the alternative that concerns them and then naturally extend such preferences to alternatives.

Let θ_i be agent i's type, which is her signal as usually called in the auctions' literature. Suppose that $\theta_i \in \Theta_i \subseteq \mathbb{R}$, where Θ_i has a minimum, say $\underline{\theta}_i$. Each individual $i \in N$ is endowed with a given auxiliary function $g_i : \Theta \to \mathbb{R}$ which we assume to satisfy the following standard condition: (a) g_i is *non-decreasing* in her own signal θ_i. The preference function $\mathfrak{R}$ is such that for each agent $i \in N$ and for each type profile $\theta \in \Theta$, $R_i(\theta)$ is as follows:

(1) $(1, p_i)P_i(\theta)(1, q_i)$ for all $q_i > p_i$ (agent i strictly prefers paying less than more), and

(2) $(1, g_i(s))I_i(\theta)(0, 0)$ (agent i is indifferent between not getting the good and paying nothing or receiving the good and paying $g_i(\theta)$).

Notice that $g_i(\theta)$ is buyer i's valuation of the good, g_i has a minimum in Θ_i, and that the preference relation of i is fully determined once we know which alternative $(1, g_i(\theta))$ is indifferent to $(0, 0)$.

Before introducing our pair of examples for auctions, let us first make an important point related to the auctions literature. We state in Proposition 10 that any one-dimensional and single-crossing environment is not knit. The single-crossing condition requires that the change in the valuation function of agent i when she changes her type is greater than the change of the valuation function of any other agent. Formally, for any $i, j \in N$, $\theta \in \Theta$ and $\theta'_i \in \Theta_i$, such that $\theta'_i > \theta_i$ then $g_i(\theta'_i, \theta_{-i}) - g_i(\theta_i, \theta_{-i}) > g_j(\theta'_i, \theta_{-i}) - g_j(\theta_i, \theta_{-i})$.[18]

Proposition 10 *For any $i \in N$, let $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$ and let g_i be weakly increasing in the type of each agent and satisfy the single-crossing condition. Then, $\Theta = \times_{i \in N} \Theta_i$ is not knit.*

Let us know introduce our pair of examples.

Example 6 Let us assume that, in addition to condition (a), for any agent i, the evaluation will be the lowest possible if all other agents but i receive the lowest signal. This is formally expressed by condition (b) $g_i(\theta) = g_i(\underline{\theta})$ for θ such that $\theta_j = \underline{\theta}_j$ for all $j \in N \backslash \{i\}$.[19]

[18] Several different versions have been used in the literature. A common one is: $\frac{\delta g_i(\theta)}{\delta \theta_i} > \frac{\delta g_j(\theta)}{\delta \theta_i}$ for all $i, j \in N$, where $\frac{\delta g_j}{\delta \theta_i}$ denotes the partial derivative of g_j with respect to θ_i. We have adapted this condition to non-necessarily differentiable valuation functions.

[19] An example of a g_i function satisfying these properties is presented by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006). In our notation, they consider the case where $g_i(\theta) = \beta_i + \alpha \prod_{j \in N} \theta_j$, $\beta_i \in [0, 1]$, $\alpha > 0$ and the signal/type space is $\theta_i = [0, 1]$. Note that by fixing β_i and α, we have a unique preference formation rule for each agent.

Under conditions (a) and (b), the environment in this example is knit[20] (see Proposition 11). Note that condition (b) is incompatible with the single-crossing condition (see Proposition 10 above). Hence, again by Theorem 2 we know that it will be impossible to design non-constant, ex post incentive compatible and respectful mechanisms in such framework. This negative result parallels those in Examples 2 and 5 above, where Theorem 2 also applies.

Proposition 11 *The environment* $(\Theta, \mathfrak{R})$ *in Example 6 is knit.*

Now, Example 7 will explore the positive consequences of apparently small changes in the set of types and the preference function.

Example 7 For simplicity, let $N = \{1, 2\}$, $\Theta_i = \{0, 1\}$ for all $i \in N$ and $l, m, h \in \mathbb{R}_+$ with $0 = l < m < h$. The agent's preference function is defined as in the general framework but will now be based on a different auxiliary function that takes three possible values, low, medium and high.

More formally,

$$g_i(\theta) = \begin{cases} l & \text{if} \quad \theta_i = 0 \\ m & \text{if} \quad \theta_i = 1 \text{ and } \theta_j = 1, \\ h & \text{if} \quad \theta_i = 1 \text{ and } \theta_j = 0. \end{cases}$$

Observe that for each agent i, g_i satisfies (a) and the following condition (c) which establishes that the valuation of the good by agent i depends negatively on other agents' signals:

(c) g_i is non-increasing in θ_j, for all $j \in N \backslash \{i\}$.

We assert that the environment in this example is not knit, but is partially knit (see Proposition 12).

Proposition 12 *The environment* $(\Theta, \mathfrak{R})$ *in Example 7 is partially knit.*

Therefore, we can apply Theorem 1 and conclude that any ex post incentive compatible and respectful mechanism on that environment will also be ex post group incentive compatible, and therefore, Pareto efficient on the range.

In view of Theorem 2, to prove that is not knit, it is enough to show that the environment in Example 7 admits a non-constant, ex post incentive compatible, and respectful mechanism. Here is such a mechanism whose properties are proved in Remark 3. Let $l < p < m$ and $l < p' < m$. Let $f_{p,p'}$ be such that no agent gets the good if both signals are 0, agent 1 gets the good and pays p if her signal is 1, and agent 2 gets the good and pays p', otherwise. Formally, for $\theta \in \Theta = \{0, 1\} \times \{0, 1\}$,

[20] Our examples are chosen to illustrate our points, and the readers may want to create additional ones or to use them for comparison with alternative results. Take, for instance, the function $g_i(\theta) = \max\{\theta_1, ..., \theta_n\}$, that is used in Ivanov, Levin, and Niederle (2010), for other purposes. Such auxiliary function g_i satisfies condition (a) but not (b), and it could be used to define a knit (hence, also partially knit) environment. Since our purpose is only to provide some examples, we leave the possibility of constructing new ones based on this g_i to the interested readers.

$$f_{p,p'}(\theta) = \left\{ \begin{array}{c} ((0,0),(0,0)) \text{ if } \theta_1 = \theta_2 = 0, \\ ((1,p),(0,0)) \text{ if } \theta_1 = 1, \text{ and} \\ ((0,0),(1,p')) \text{ if } \theta_1 = 0,\ \theta_2 = 1. \end{array} \right\}$$

Remark 3 *$f_{p,p'}$ is non-constant, ex post incentive compatible, and respectful in the environment (Θ, R) in Example 7.*

5 Further comments

We have studied the possibility of designing non-trivial mechanisms satisfying ex post group incentive compatibility when agents are characterized by possibly interdependent types. In these general contexts, we have stressed the important role played by preference functions, which associate a profile of agents' preferences to each type profile. We have defined environments as combinations of types and preference functions, identified two classes of environments that we call partially knit and knit, and shown that they lie on different sides of the frontier between possibility and impossibility. For partially knit environments there may exist non-trivial ex post group incentive compatible and respectful mechanisms, while for knit environments only constant mechanisms can be ex post incentive compatible and respectful.

Let us discuss the content and implications of our two conditions and results.

We first offer a discussion of the meaning of our conditions of partial knitness and knitness in the light of Arrow's seminal idea that fruitful domain restrictions are based on similarities in the way how agents perceive the alternatives they face, even if their preferences may be widely different (see Chapter VII in Arrow 1963).[21] We elaborate on the role of similarities for our analysis in several steps. Remember that our two conditions on environments, partial knitness and knitness, rely on the possibility to connect admissible pairs of type profiles through sequences of changes in individual types.

Suppose that a strict environment $(\Theta, \mathfrak{R})$ is such that there exist an agent i and two pairs, say (x, θ) and $(z, \widetilde{\theta})$; for which i prefers x over z at θ and z over x at $\widetilde{\theta}$: that is, her preferences over both alternatives differ on that pair under the two type profiles.[22] Then, for such two pairs to be pairwise knit, there must exist some agent, say j, whose change in type in one of the sequences of types, S or $\widetilde{S}$, induces an improvement (maybe weak) of the corresponding reference alternative for agent j and a strict worsening for agent i. For this environment to be knit we have to check if the two pairs are pairwise knit.

Let us point out that some papers in the literature, like Dasgupta and Maskin (2000) and Che, Kim, and Kojima (2015), obtain negative results assuming conditions on the environments allowing that when an agent changing her type induces an improvement of the corresponding reference alternative for herself and also a worsening for another agent. Dasgupta and Maskin (2000) in Proposition 3 show that there is no efficient auction with

[21] The example considered by Arrow is that of single-peaked preferences. Obviously, the preferences of agents in single-peaked profiles may have very different preferences than others, but they all must classify alternatives according to a common linear order.

[22] Otherwise, each agent would have the same preferences over x and z for any type profile.

regular equilibria if there exist a type profile $\theta \in \Theta$ and $\widehat{\theta}_j \in \Theta_j$ such that the valuation function for agent j does not change and that of some other agent changes. Che, Kim, and Kojima (2015) in Section 3 in the context of two agents and two alternatives, assume that there exist a type profile $\theta \in \Theta$ and $\widehat{\theta}_j \in \Theta_j$ such that agent j's ordinal preferences are the same but the preferences the other agent are different. Then, they show that any ex post incentive compatible mechanism must be constant.

If we were checking for partial knitness, the above two pairs do not have to be checked because the set of agents whose type changes when going from θ to $\widetilde{\theta}$ are such that all of them prefer z to x: note that any agent i preferring x to z under θ should belong to $N \backslash C(\theta, z, x)$ and her type in $\widetilde{\theta}$ coincide with the one in θ.

Let us now clarify why no private values environment can be knit. In these environments there will always exist two pairs like the ones defined above that are not pairwise knit since changes in agent j's types do not affect agent i's preferences. Therefore private values environments are not knit (see the proof of Proposition 1 in Appendix B), however, they can be partially knit or not.

In this section we have tried to establish a connection between our efforts to extract the common features of different environments and the seminal but not always remembered notion of similarity proposed in Arrow's. For both types of environments, knit and partially knit, the emphasis is not on the fact that preferences are similar, but that agents perceive the structure of types in a similar manner. This is the case for some partially knit environments, and not for those that are knit.

We now discuss about the existing gap between our two general theorems. We go from our condition of knitness that precipitates impossibility to another property, partial knitness, which ensures that all ex post incentive compatible mechanisms are also ex post group. In the middle, we have environments for which there do exist ex post incentive compatible mechanisms (for instance, under the single-crossing in auctions) but not all of them are ex post group (like the second price auction).

A final comment is related to our Theorems 1 and 2 where we show that partial knitness and knitness, respectively, are sufficient conditions for the respective result to hold. As shown by Example 2 in Appendix C, partial knitness is not necessary for the equivalence between ex post individual and group incentive compatibility. The necessity of knitness in Theorem 2 is still an open question for the case of three or more alternatives. In a companion paper Barberà, Berga, and Moreno (2019), we show that for the case of two alternatives at stake, knitness is not only sufficient but also necessary.

Acknowledgements

We owe a great debt to Matthew Jackson and Arunava Sen, who corrected serious mistakes in previous versions of this paper, and did it in the most friendly manner. Claude d'Aspremont, Carmen Beviá, Luis Corchón, Matthias Dahm and Jordi Massó also helped us through. None of them is responsible for remaining flaws. Salvador Barberà acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2015-0563) and grant ECO2017-83534-P and FEDER, and from the Generalitat de Catalunya, through grant

2017SGR-0711. Dolors Berga and Bernardo Moreno acknowledge the support from the Spanish Ministry of Economy, Industry and Competitiveness through grants ECO2016-76255-P and ECO2017-86245-P, respectively, and thank the MOMA network.

References

Arrow, K.: Social Choice and Individual Values. 2nd edition New York: Wiley (1st edition 1951) (1963)

Arrow, K.: The Property Rights Doctrine and Demand Revelation under Incomplete Information. In M. Boskin, Economies and Human Welfare, Academic Press NY (1979)

Ausubel, L.: An Efficient Ascending-Bid Auction For Multiple Objects. Discussion Paper, University of Maryland (1997)

Austen-Smith, D., Feddersen, T.J.: Deliberation, Preference Uncertainty, and Voting Rules. American Political Science Review 100, 209–217 (2006)

Barberà, S., Berga, D., Moreno, B.: Individual versus group strategy-proofness: When do they coincide?. Journal of Economic Theory 145, 1648–1674 (2010)

Barberà, S., Berga, D., Moreno, B.: Group Strategy-Proof Social Choice Functions with Binary Ranges and Arbitrary Domains: Characterization Results. International Journal of Game Theory 41, 791–808 (2012a)

Barberà, S., Berga, D., Moreno, B.: Two necessary conditions for strategy-proofness: On what domains are they also sufficient?. Games and Economic Behavior 75, 490–509 (2012b)

Barberà, S., Berga, D., Moreno, B.: Group strategy-proofness in private good economies. American Economic Review 106, 1073–1099 (2016)

Barberà, S., Berga, D., Moreno, B.: Domains admitting ex post incentive compatible and respectful mechanisms: a characterization for the two alternatives case. In Trockel,W. (Ed.): Social Design - Essays in Memory of Leonid Hurwicz. Springer (2019)

Barberà, S., Jackson, M.O.: Choosing How to Choose: Self-Stable Majority Rules and Constitutions. Quaterly Journal of Economics 119 (3), 1011–1048 (2004)

Bergemann, D., Välimäki, J.: Information Acquisition and Efficient Mechanism Design. Econometrica 70: 1007–1035 (2002)

Bergemann, D., Morris, S.: Robust mechanism design. Econometrica 73, 1771–1813 (2005)

Che, Y-K., Kim, J., Kojima, F.: Efficient Assignment with Interdependent Values. Journal of Economic Theory 158, 54–86 (2015)

Copic J.: Robust efficient decision rules, mimeo, October (2017)

Crémer, J., McLean, R.: Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands Are Interdependent. Econometrica 53, 345–361 (1985)

D'Aspremont, C., Gérard-Varet L.-A.: Incentives and Incomplete Information. Journal of Public Economics 11, 25–45 (1979)

D'Aspremont, C., Crémer, J., and Gérard-Varet L.-A.: Incentives and the existence of Pareto-optimal revelation mechanisms. Journal of Economic Theory 51 (2), 233–254 (1990)

Dasgupta, P., Maskin, E.: Efficient Auctions. The Quarterly Journal of Economics 115, 341–388 (2000)

Dizdar, D., Moldovanu, B.: On the importance of uniform sharing rules for efficient matching. Journal of Economic Theory 165: 106–123 (2016)

Gibbard, A.: Manipulation of Voting Schemes: A General Result. Econometrica 41, 587–601 (1973)

Grandmont, J.-M.: Intermediate preferences and the majority rule. Econometrica 46: 317–330 (1978)

Holmstrom, B., Myerson, R.B.: Efficient and Durable Decision Rules with Incomplete Information. Econometrica 51, 1799–1819 (1983)

Jackson, M.O.: Mechanism Theory. In Optimization and Operations Research. Edited by Ulrich Derigs, in the Encyclopedia of Life Support Systems, EOLSS Publishers: Oxford UK,[http://www.eolss.net] (2003)

Jehiel P., Meyer-Ter-Vehn, M., Moldovanu, B., Zame, W.: The limits of ex post implementation. Econometrica 74, 585–610 (2006)

Jehiel P., Moldovanu, B.: Efficient design with interdependent valuations. Econometrica 69 (5): 1237–1259 (2001)

Ivanov, A., Levin D., Niederle, M.: Can relaxation of beliefs rationalize the winner's curse?: An experimental study. Econometrica 78, 1435–1452 (2010)

Larsson, B., Svensson L.-G.: Strategy-proof voting on the full preference domain. Mathematical Social Sciences 52: 272–287 (2006)

Manjunath, V.: Group strategy-proofness and social choice between two alternatives. Mathematical Social Sciences 63, 239–242 (2012)

Maskin, E.: Auctions and Privatizations. In Privatization, ed. by H. Siebert. Kiel: Institut fuer Weltwirtschaft der Universitaet Kiel: 115–136 (1992)

Moulin, H.: On strategy-proofness and single-peakedness. Public Choice 35, 437–455 (1980)

Perry, M., Reny P.J.: An efficient auction. Econometrica 70, 3: 1199–1212 (2002)

Pourpouneh, M., Ramezanianz, R., Sen, A.: The Marriage Problem with Interdependent Preferences, mimeo (November 2018)

Saporiti, A.: Strategy-proofness and single-crossing. Theoretical Economics 4: 127–163 (2009)

Satterthwaite, M.: Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions. J. Econ. Theory 10, 187–217 (1975)

Satterthwaite, M., Sonnenschein, H.: Strategy-Proof Allocation Mechanisms at Differentiable Points. The Review of Economic Studies 48, 587–597 (1981)

Shapley, L., Scarf, H.: On Cores and Indivisibility. Journal of Mathematical Economics 1, 23–37 (1974)

Shenker, S.: Some Technical Results on Continuity, Strategy-Proofness, and Related Strategic Concepts. Mimeo (1993)

Thomson, W.: Non-bossiness. Social Choice and Welfare 47: 665–696 (2016)

Vickrey, W.: Utility, Strategy and Social Decision Rules, Quarterly Journal of Economics 74, 507–535 (1960)

Yamashita, T., Zhu, S.: On the Foundations of Ex Post Incentive Compatible Mechanisms. TSE Working Paper, n. 18-938, Toulouse (2018)

Appendix A. Proofs of results in Section 3

Proof of Theorem 1. Let $(\Theta, \mathfrak{R})$ be a partially knit environment and let f be a respectful mechanism. By definition, ex post group incentive compatibility implies ex post incentive compatibility. To prove the converse, suppose, by contradiction, that there exist $\theta \in \Theta$, $C \subseteq N$, $\#C \geq 2$, $\widetilde{\theta}_C \in \times_{i \in C}\Theta_i$ such that for any agent $i \in C$, $f(\widetilde{\theta}_C, \theta_{N\backslash C}) R_i(\theta) f(\theta)$ and $f(\widetilde{\theta}_C, \theta_{N\backslash C}) P_j(\theta) f(\theta)$ for some agent $j \in C$. Let $z = f(\widetilde{\theta}_C, \theta_{N\backslash C})$ and $x = f(\theta)$. Note that (i) $z \neq x$, (ii) $\overline{C}(\theta, z, x) \neq \varnothing$, $\#C(\theta, z, x) \geq 2$ since $C \subseteq C(\theta, z, x)$, and (iii) $\widetilde{\theta}_j = \theta_j$ for any $j \in N\backslash C(\theta, z, x)$ again since $C \subseteq C(\theta, z, x)$.

Since (Θ, R) is partially knit and conditions in Definition 6 are satisfied, (x, θ) and $(z, \widetilde{\theta})$ are pairwise knit. Thus, there exist $\theta' \in \Theta$ and two sequences of types $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Now, we will show the following:

(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and

(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$.

Statements in (a) and (b) yield to a contradiction. By definition of the sequences S and $\widetilde{S}$, we know that $m^{t_S}(\theta, S) = m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S}) = \theta'$. However, $f(\theta') = f(m^{t_S}(\theta, S)) = x$ by (a) while $f(\theta') = f(m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S})) = z$ by (b).

We prove (a) in steps, from $h = 1$ to $h = t_S$. The proof of (b) is identical and omitted.

Step 1. Let $h = 1$. By Definition 3, $R^1_{i(S,1)}(\theta, S)$ is an x-monotonic transform of $R^0_{i(S,1)}(\theta, S) = R_{i(S,1)}(\theta)$. (1)

Observe that $f(m^1(\theta, S)) \in U\left(R^1_{i(S,1)}(\theta, S), x\right)$. (2)

(otherwise, if $f(m^1(\theta, S)) \notin U\left(R^1_{i(S,1)}(\theta, S), x\right)$, we would get a contradiction to ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under f at $(\theta^S_{i(S,1)}, (m^0(\theta, S))_{N\backslash\{i(S,1)\}})$ via $\theta_{i(S,1)}$).

By (1) and (2) we have that $f(m^1(\theta, S)) \in U\left(R^0_{i(S,1)}(\theta, S), x\right)$. (3)

By ex post incentive compatibility of f, $f(m^1(\theta, S)) \notin \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$. (4)

(otherwise, if $f(m^1(\theta, S)) \in \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$, $f(m^1(\theta, S)) P^0_{i(S,1)}(\theta) x$ contradicting ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under f at θ via $\theta^S_{i(S,1)}$).

Thus, by (3) and (4) we have that $f(m^1(\theta, S))$ is indifferent to x according to preference $R^0_{i(S,1)}(\theta, S)$ (that is, $f(m^1(\theta, S)) I^0_{i(S,1)}(\theta, S) x$). (5)

Then, by respectfulness, we get that $f(m^1(\theta, S)) = f(m^0(\theta, S)) = f(\theta) = x$ which ends the proof of (a) for $h = 1$.

Step $h \in \{2, ..., t_S\}$. By repeating the same argument than in Step 1 on the recursive fact that $f(m^{h-1}(\theta, S)) = x$, we obtain that $f(m^h(\theta, S)) = f(m^{h-1}(\theta, S)) = x$. ■

Part of the proof of Theorem 2 follows an identical reasoning used in the proof of Theorem 1. We write down the first part of the proof, which is the one that differs, and specify from where on the argument is the same.

Proof of Theorem 2. Let $(\Theta, \mathfrak{R})$ be a knit environment and let f be an ex post incentive compatible and respectful mechanism. Assume, by contradiction, that f was not constant. Then, there will be $x, z \in A$, $x \neq z$ such that $x = f(\theta)$ and $z = f(\widetilde{\theta})$ for some θ and $\widetilde{\theta}$ in Θ. Since $(\Theta, \mathfrak{R})$ is knit, the two pairs formed by an alternative and a type profile, (x, θ) and $(z, \widetilde{\theta}) \in A \times \Theta$, are pairwise knit. Thus, there exist $\theta' \in \Theta$ and two sequences $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Although these sequences are not necessarily the same than the ones we used in the proof of Theorem 1, from this point on, we can use the same reasoning as there, and show that
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$,
again leading to a contradiction. Adding the arguments we have already used in the proof of Theorem 1 we would complete the one for the present theorem. ■

The proof of Corollaries 1 and 3 is straightforward from the corresponding theorems: it is obtained just applying the first part in each step where respectfulness is not used.

Appendix B. Proofs of the results in the Applications section

Private values

Proof of Proposition 1. Let $i \in N$ and $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $\theta_i \neq \widetilde{\theta}_i$ be such that $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$. That is, $R_i(\theta_i, \theta_{N\setminus\{i\}}) \neq R_i(\widetilde{\theta}_i, \theta_{N\setminus\{i\}})$ for all $\theta_{N\setminus\{i\}} \in \times_{j\in N\setminus\{i\}}\Theta_j$ since $(\Theta, \mathfrak{R})$ is a private values environment. Then, there will be a pair of alternatives, say x and z, such that $xP_i(\theta_i)z$ and $zR_i(\widetilde{\theta}_i)x$ (otherwise, for $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $R_i(\theta_i) = R_i(\widetilde{\theta}_i)$). To show that the environment $(\Theta, \mathfrak{R})$ is not knit, we prove that the two pairs $(x, (\theta_i, \theta_{N\setminus\{i\}}))$, $(z, (\widetilde{\theta}_i, \theta_{N\setminus\{i\}}))$, whatever $\theta_{N\setminus\{i\}}$, are not pairwise knit. That is, there does not exist any θ', S, and $\widetilde{S}$ such that the passage from θ to θ' through S be x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ be z-satisfactory. We prove it by contradiction. Suppose otherwise that there exist θ^*, S^*, $\widetilde{S}^*$, such that the passages $\{m^h(\theta, S^*)\}_{h=0}^{t_{S^*}}$ and $\left\{m^h(\theta, \widetilde{S}^*)\right\}_{h=0}^{t_{\widetilde{S}^*}}$ from θ to θ^* through S^* and $\widetilde{\theta}$ to θ^* through $\widetilde{S}^*$ are x and z-satisfactory, respectively.
Since we are in a private values environment, changes in the type of agent j never affect the induced preferences of other agents, in particular never affect i's induced preferences if $j \neq i$. Moreover, we know that $xP_i(\theta_i, \theta_{N\setminus\{i\}})z$ and $zR_i(\widetilde{\theta}_i, \theta_{N\setminus\{i\}})x$. These two observations imply that agent i must belong to $I(S^*) \cup I(\widetilde{S}^*)$. That is, i will appear in at least one of these two sequences.
We concentrate on the steps of the passage where agent i changes her type and we show that there is no θ^* compatible with x-satisfactory and z-satisfactory passages from θ to θ^*

and from $\widetilde{\theta}$ to θ^*.
Without loss of generality, by the remark just after Definition 3, we can assume that all types of agent i in S^* and $\widetilde{S}^*$ appear in the first positions in these sequences. Let's define $I_{S^*,i} \equiv \{h \in \{1,2,...,i_{S^*}\} : i(S^*,h) = i\}$ and $I_{\widetilde{S}^*,i} = \left\{h \in \{1,2,...,i_{\widetilde{S}^*}\} : i(\widetilde{S}^*,h) = i\right\}$.
Take $1 \in I_{S^*,i}$. Since $R_i^1(\theta, S^*)$ is an x-monotonic transform of $R_i(\theta_i, \theta_{N\setminus\{i\}})$, we have that $xP_i(m_i^1(\theta,S^*))z$. By repeating the same argument for each $h \in I_{S^*,i}$ we finally obtain that $xP_i(m_i^{i_{S^*}}(\theta,S^*))z$ where $m_i^{i_{S^*}}(\theta,S^*) = \theta_i^*$.
Now, take $1 \in I_{\widetilde{S}^*,i}$. Since $R_i^1(\widetilde{\theta}^*, \widetilde{S}^*)$ is a z-monotonic transform of $R_i(\widetilde{\theta}_i^*, , \theta_{N\setminus\{i\}})$, we have that $zR_i(m_i^1(\widetilde{\theta}^*, \widetilde{S}^*))x$. By repeating the same argument for each $h \in I_{\widetilde{S}^*,i}$ we finally obtain that $zR_i(m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*))z$ where $m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*) = \theta_i^*$.
As mentioned above, changes in types of agents different from i will not change agent i's preferences. Thus, we have obtained the desired contradiction. On the one hand that $xP_i(\theta^*)z$ and on the other hand, that $zR_i(\theta^*)x$. ■

For the private values environments in Propositions 2, 3, 4, and 5, the following two relevant observations hold and are used in their proofs: types are preferences in these cases, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.

Proof of Proposition 2. Let $\mathcal{U}$ denote the universal set of strict preferences in the classical social choice problem. Thus, $\mathcal{R}_i = \mathcal{U}$. To prove partial knitness, take any (x,R), $(z,\widetilde{R}) \in A \times \mathcal{U}^n$ such that $\overline{C}(R,z,x) = C(R,z,x) \neq \varnothing$, $\#C(R,z,x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N\setminus C(R,z,x)$. Without loss of generality, let $C(R,z,x) = \{1,2,...,c\}$ where c also denotes its cardinality. Now, we must show that (x,R), $(z,\widetilde{R})$ are pairwise knit. To do that, we construct S, $\widetilde{S}$ and R' satisfying the condition in pairwise knitness.
For each $R_i \in \mathcal{U}$, let us denote by R_i^z the preference obtained by lifting z to the first position and keep the relative position of all other alternatives.
Now, start from R and define $S = \{R_1^z, R_2^z, ..., R_c^z\}$ where $t_S = c$. Note that for each $h \in \{1,...,c\}$, $R_j^h(R,S) = R_j^{h-1}(R,S)$ for all $j \in N\setminus i(S,h)$ and $R_{i(S,h)}^h(R,S) = R_{i(S,h)}^z \in \mathcal{U}$. That is, for all i, $R_i^h(R,S)$ is an x-reshuffling of i's previous preferences $R_i^{h-1}(R,S)$. Then, $R' = R^c(R,S) = (R_{C(R,z,x)}^z, R_{N\setminus C(R,z,x)}) \in \mathcal{U}^n$.
Now, start from $\widetilde{R}$ and define $\widetilde{S} = \left\{\widetilde{R}_1^z, \widetilde{R}_2^z, ..., \widetilde{R}_c^z, R_1^z, R_2^z, ..., R_c^z\right\}$ where $t_{\widetilde{S}} = 2c$. For each $h \in \{1,...,c\}$, $R_j^h(\widetilde{R},\widetilde{S}) = R_j^{h-1}(\widetilde{R},\widetilde{S})$ for all $j \in N\setminus i(\widetilde{S},h)$ and $R_{i(\widetilde{S},h)}^h(\widetilde{R},\widetilde{S}) = \widetilde{R}_{i(\widetilde{S},h)}^z$. That is, for all i, $R_i^h(\widetilde{R},\widetilde{S})$ is a z-monotonic transform or a z-reshuffling (if z was already the top or does not change preferences) of i's previous preferences $R_i^{h-1}(\widetilde{R},\widetilde{S})$. Moreover, for $h \in \{c+1,...,2c\}$, $R_j^h(\widetilde{R},\widetilde{S}) = R_j^{h-1}(\widetilde{R},\widetilde{S})$ for all $j \in N\setminus i(\widetilde{S},h)$ and $R_{i(\widetilde{S},h)}^h(\widetilde{R},\widetilde{S}) = R_{i(\widetilde{S},h)}^z$, which for each agent is a z-reshuffling of her previous preferences $\widetilde{R}_i^z$. Then, $R' = R^{2c}(\widetilde{R},\widetilde{S}) = (R_{C(R,z,x)}^z, R_{N\setminus C(R,z,x)})$. ■

Proof of Proposition 3. Let A be a finite and ordered set of alternatives in $\mathbb{R}$, the real line. For all $i \in N$, let $\mathcal{R}_i = \mathcal{S}$ be the set of strict single-peaked preferences on A according to the established real numbers order. We introduce some notation: given $R_j \in \mathcal{S}$, $p(R_j)$

denotes the peak, that is, the best alternative, of R_j in A. Let $\overline{L}(R_i, x) = \{y \in A : xP_i y\}$ be the *strict lower contour set of* R_i *at* x. Given $R_j \in \mathcal{S}$ and $x \in A$, define $r(R_j, x)$ as the first alternative in $\overline{L}(R_j, x)$ in the opposite side of alternative x with respect to $p(R_j)$.

To prove partial knitness, take any (x, R), $(z, \widetilde{R}) \in A \times \mathcal{S}^n$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$ and show that (x, R), $(z, \widetilde{R})$ are pairwise knit. Without loss of generality, let $x < z$, which implies that $p(R_j) > x$. Also without loss of generality, let $C(R, z, x) = \{1, 2, ..., c\}$ where c denotes its cardinality. Now define $I(S) = I(\widetilde{S}) = C(R, z, x) = \{1, 2, ..., c\}$ and construct for each agent $j \in \{1, 2, ..., c\}$, R'_j depending on the cases below.

Take any $j \in C(R, z, x)$ and consider the following cases.

<u>Case 1</u>. $\widetilde{R}_j$ is such that $x\widetilde{P}_j z$. Take $R'_j \in \mathcal{S}$ such that $p(R'_j) \in [x, z)$, $r(R_j, x) = z$, and $zP'_j y$ for all $y < x$. Notice that such R'_j exists, and the two following set inclusions hold: $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$, $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$. Thus, R'_j is both an x-monotonic transform *of* R_j and a z-monotonic transform of $\widetilde{R}_j$ (observe that with strict preferences, the above inclusion of strict lower contour sets is equivalent to Definition 1).

<u>Case 2</u>. $\widetilde{R}_j$ is such that $z\widetilde{P}_j x$. Consider several subcases.

<u>Case 2.1</u>. $\overline{L}(R_j, x) \subseteq \overline{L}(\widetilde{R}_j, x)$. Let $R'_j = \widetilde{R}_j$ and observe that R'_j is an x-monotonic transform of R_j (obviously, R'_j is a z-monotonic transform of $\widetilde{R}_j$ since $R'_j = \widetilde{R}_j$).

<u>Case 2.2</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$. We distinguish additional subcases which require different definitions of R'_j.

<u>Case 2.2.1</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j, z)$. Let $R'_j = R_j$ and observe that R'_j is an x-monotonic transform of R_j (obviously since $R'_j = R_j$) and R'_j is also a z-monotonic transform of $\widetilde{R}_j$.

<u>Case 2.2.2</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ *and* $\overline{L}(R_j, z) \subsetneqq \overline{L}(\widetilde{R}_j, z)$. This implies that either *(a)* $p(R_j), p(\widetilde{R}_j) \in (x, z)$ or else *(b)* $p(R_j), p(\widetilde{R}_j) > z$.

If *(a)* holds, then let R'_j be such that $p(R'_j) \in \left[\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\right]$, $r(R'_j, x) = r(R_j, x)$ and $r(R'_j, z) \geq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.

If *(b)* holds, then let R'_j be such that $p(R'_j) \in \left[z, \min\{p(R_j), p(\widetilde{R}_j)\}\right]$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.

Then, observe that R'_j defined in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.

<u>Case 2.2.3</u>: $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $z \in \left(\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\right)$. Assume that $p(R_j) < z < p(\widetilde{R}_j)$, otherwise, a similar argument would work.

This implies that either *(a)* $r(R_j, x) \in \left(z, p(\widetilde{R}_j)\right]$ or *(b)* $r(R_j, x) \in \left(p(\widetilde{R}_j), r(\widetilde{R}_j, x)\right)$ holds.

If *(a)* holds, then let R'_j be such that $p(R'_j) \in [z, r(R_j, x))$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.

If *(b)* holds, then let R'_j be such that $p(R'_j) \in \left[z, \min\{r(R_j, x), r(\widetilde{R}_j, z)\}\right)$, $r(R'_j, x) \leq r(R_j, x)$

and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$.
Then, observe that R'_j in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.
Finally, for each $j \in C(R, z, x)$ we repeat the same argument. ■

Proof of Proposition 4. The proof follows the same argument as the one in Proposition 2, given that agents have all possible strict preferences over individual assignments and preferences are selfish. As in Barberà, Berga, and Moreno (2016), just note that although preferences over individual assignments are strict, preferences over alternatives allow for indifferences, by selfishness: all alternatives with the same individual assignment are indifferent for such individual agent. Thus, in the case of housing $C(R, z, x) \supseteq \overline{C}(R, z, x)$ holds and R_i^z are the preferences obtained by lifting z and also all alternatives with the same individual assignment z_i to the first position and keep the relative position of all other alternatives. ■

Proof of Proposition 5. Let $A = \{x, y, z\}$ be the set of alternatives. Let $\widetilde{\mathcal{L}}$ be the set of all strict preferences on A and for each agent $i \in N$, let $\mathcal{D}_i \subseteq \widetilde{\mathcal{L}}$ be the set of i's preferences. It is worth noting that for each $i \in N$ and each pair of alternatives $a, b \in A$ there exist at most three individual preferences in $\mathcal{D}_i$ such that aP_ib, two of them with b as the worst alternative and another one with b in the middle position. To show that $\times_{i \in N}\mathcal{D}_i$ is partially knit, take any pair (x, R), $(z, \widetilde{R}) \in A \times (\times_{i \in N}\mathcal{D}_i)$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$ and show that (x, R), $(z, \widetilde{R})$ are pairwise knit. Let $S(R)$ and $\widetilde{S}(R)$ be the partition of $\overline{C}(R, z, x)$ such that $S(R) = \left\{i \in \overline{C}(R, z, x) : x \text{ is bottom according to } R_i\right\}$ and $\widetilde{S}(R) = \left\{i \in \overline{C}(R, z, x) : x \text{ is second according to } R_i\right\}$ (well-defined by the note above).
Let $R'_i = \widetilde{R}_i$ for each $i \in S(R)$, let $S = S(R)$, and observe that for each $i \in S(R)$ and each possible $\widetilde{R}_i$, $R'_i = \widetilde{R}_i$ is an x-monotonic transform of R_i since x is bottom of R_i.
Let $R'_i = R_i$ for each $i \in \widetilde{S}(R)$, let $\widetilde{S} = \widetilde{S}(R)$, and observe that for each $i \in \widetilde{S}(R)$ and each possible $\widetilde{R}_i$, $R'_i = R_i$ is an z-monotonic transform of $\widetilde{R}_i$ since z is top of R_i. ■

Deliverative juries

Proposition 6 *The environment* $(\Theta, \mathfrak{R})$ *in Example 2 is knit.*

Proof of Proposition 6. To prove knitness we just need to combine the following two results.
(1) Consider a pair formed by (A, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. Let $\theta' \in \Theta$ be such that $\theta'_1 = (l, i)$ and $\theta'_j = (h, i)$ for any $j \in N \backslash \{1\}$. We now define the sequence S to sequentially go from type profile θ to type profile θ' by successively changing the type of the agents in S while preserving A-satisfactoriness. First change, one by one and in any order, agents' signals from $s_j \neq i$ to i. By definition of l and h, in each of the above changes, the induced preferences of the agent changing her type is an A-monotonic transform of her previous preferences (sometimes an A-reshuffling).
Observe that by definition of the preference functions, the following condition is satisfied: if $\widehat{s}_j = i$ for all $j \in N$, all jurors prefer A to C for any $\widehat{b}_j \in B$.

We now change, one by one and in any order, each agent's $b_j \neq h$ from b_j to h for any $j \in N\backslash\{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these changes, the induced preferences of each agent is the same and therefore they are an A-reshuffling of their previous preferences. Then, we have defined S such that θ leads to θ' through S and the passage from θ to θ' is A-satisfactory.
(2) Consider a pair (C, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. We now define the sequence S to go from type profile θ to θ' above by successively changing the type of the agents in S while preserving C-satisfactoriness. First change, one by one and in any order, agents from $s_j \neq g$ to g. By definition of l and h, in each of the above changes, the induced preferences of the agent changing her type is a C-monotonic transform of her previous preferences (sometimes a C-reshuffling).
Observe that by definition of the preference function, the following property is satisfied: if $\widehat{s}_j = g$ for all $j \in N$, all jurors prefer C to A for any $\widehat{b}_j \in B$.
We now change one by one, and in any order, each agent's $b_j \neq h$ from b_j to h for any $j \in N\backslash\{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these steps, the preferences of the agents stay the same and therefore they are a C-reshuffling of their previous ones. After that, we change the signal of the agent 1 from g to i. This implies that the preferences of agent 1 remain identical, but those of all others go from C preferred to A, to A preferred to C, given that $b_j = h$ for any $j \in N\backslash\{1\}$. Finally, we change the type of the rest of the agents one by one from g to i. In each one of these steps the preferences of the agent that moves is still A preferred to C. The passage from θ to θ' is C-satisfactory by construction. ■

Before engaging in the proof that the environment in Example 3 is partially knit (see Proposition 7), we develop the argument for a particular example as mentioned in Footnote 14.

Consider a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$ where $\theta = ((u, g), (u, i), (m, g))$ and $\widetilde{\theta} = ((m, i), (u, i), (u, g))$. Let $\theta' = ((m, i), (u, i), (m, g))$. The profiles of preferences they induce are shown in Table 5.

$R(\theta) = R((u,g),(u,i),(m,g))$	$R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$	$R(\theta') = R((m,i),(u,i),(m,g))$
C A C	A A A	A A A
A C A	C C C	C C C

Table 5: Agents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

We can check that $\overline{C}(\theta, C, A) = C(\theta, C, A) = \{1, 3\}$ and $\widetilde{\theta}_2 = \theta_2$ (that is, requirements in Definition 6 are satisfied). As shown in Table 6 below, it is possible to move from θ to θ' by successively changing, one by one, the type of the agents. In this case, agent 1 from (u, g) to (m, i). According to our notation, $I(S) = \{1\}$. Likewise, as shown in Table 7 below, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. In this case, agent 3 from (u, g) to (m, g), that is, $I(\widetilde{S}) = \{3\}$. In Table 6, note that the preferences $R_1(\theta')$ of agent 1 are an A-monotonic transform of her previous ones, which also involve a change of those for agent 3. Similarly, notice that the preferences $R_3(\theta')$ of 3 in Table 7 are a C-reshuffling of her previous ones.

$R(\theta) = R,((ug),(u,i),(m,g))$	$R(\theta') = R((\mathbf{m},\mathbf{i}),(u,i),(m,g))$
$C \quad A \quad C$	$A \quad A \quad A$
$A \quad C \quad A$	$C \quad C \quad C$

Table 6: Induced agents' preferences given the specified type changes from θ to θ'.

$R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$	$R(\theta') = R((m,i),(u,i),(\mathbf{m},g))$
$A \quad A \quad A$	$A \quad A \quad A$
$C \quad C \quad C$	$C \quad C \quad C$

Table 7: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

In Tables 6 and 7, we have illustrated the idea of partial knitness for two given type profiles. We now show that any relevant pair of type profiles are connected through two appropriate sequences.

Proposition 7 *The environment $(\Theta, \mathfrak{R})$ in Example 3 is partially knit.*

Proof of Proposition 7. Take two pairs (A,θ), $(C,\widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta,C,A) = C(\theta,C,A) \neq \varnothing$, $\#C(\theta,C,A) \geq 2$, and for $j \in N \backslash C(\theta,C,A)$, $\widetilde{\theta}_j = \theta_j$. By definition, for all $j \in N$, $\theta_j = (b_j, s_j)$ and $\widetilde{\theta}_j = (\widetilde{b}_j, \widetilde{s}_j)$. We have to show that there exist $\theta' \in \Theta$ and sequences of types S and $\widetilde{S}$ such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$, and the passages from θ and $\widetilde{\theta}$ to θ' are, respectively, A and C-satisfactory.
Let $\theta' \in \Theta$ be such that $\theta'_j = (b_j, g)$ for any $j \in C(\theta,C,A)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta,C,A)$. Define the sequence $S = \{(b_k, g)\}$, where $k \in C(\theta,C,A)$ and $s_k = i$. Note that $I(S)$ is either a singleton or empty. If the latter, let θ' be θ.
By definition of the preference function in the example, if some agent j prefers C to A, the signal profile must be such that at most one agent k has signal i: $s_k = i$. Thus, S is well-defined. Moreover, $b_k = m$ since for unswerving jurors to have C over A their signal must be g. And by definition of m increasing the support for g implies that preferences remain C over A for agent k (i.e. and A-reshuffling) and will be C over A for the other agents.
Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory.
We now go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta,C,A)$, one by one in any order, from to $\widetilde{s}_j \neq g$ to g. This set of agents are those in $I(\widetilde{S})$.
By definition of the preference function, if one agent changes her signal by increasing the support for a guilty verdict, then each agents' induced preferences remain either the same as before or change in favor of C. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a C-monotonic transform of her previous ones (sometimes a C-reshuffling).
Now, take any two pairs (C,θ), $(A,\widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta,A,C) = C(\theta,A,C) \neq \varnothing$, $\#C(\theta,A,C) \geq 2$, and for $j \in N \backslash C(\theta,A,C)$, $\widetilde{\theta}_j = \theta_j$, a similar argument would work but

defining $\theta' \in \Theta$ to be such that $\theta'_j = (b_j, i)$ for any $j \in C(\theta, A, C)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta, A, C)$. Define the sequence $S = \{(b_k, i)\}$, where $k \in C(\theta, A, C)$ and $s_k = g$. Note that $I(S)$ is either a singleton or empty. If the latter, let θ' be θ.
Again, by definition of the preference function in the example, if some agent j prefers A to C, the signal profile must be such that only one single agent, or at most two, have signal g. In the latter case, none of the two are agent j, and both have preferences C over A. Thus, S is well-defined. Moreover, by definition of m and u increasing if the single agent with signal g says i, that preferences of this agent and those of all other agents will be A over C.
Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory. We now sequentially go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta, A, C)$, one by one in any order, from to $\widetilde{s}_j \neq i$ to i. This set of agents are those in $I(\widetilde{S})$. By definition of agents' preference function, if one agent changes her signal by increasing the support for verdict of innocence, then each agents' induced preferences remain either the same as before or change in favor of A. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a A-monotonic transform of her previous ones (sometimes a A-reshuffling). ■

Remark 1 *A voting by quota q mechanism is non-costant, ex post incentive compatible, and respectful in the environment in Example 3.*

Proof of Remark 1. In Table 8 below we describe all possible results of voting by quota for different values of q in Example 3. We have four matrices, one for each type of agent 3. In the rows of each matrix we write the four types of agent 1 and in the columns the four types of agent 2. In each cell, we write each agent's best alternative according to their preferences at a given type profile, followed by the outcome of a quota mechanism. When two outcomes appear in a cell, the one in the left stands for the outcome of voting by quota 3 and the right one is the outcome for both quota 1 and 2, which in this example are always the same.
Given Table 8, it is easy to check that these rules are ex post incentive compatible. In addition, they also satisfy anonymity. Note that respectfulness is trivially satisfied in these environments where preferences are strict and alternatives have no private component.

$\theta_3 = (m, i)$	$\theta_2 = (m, i)$	$\theta_2 = (m, g)$	$\theta_2 = (u, i)$	$\theta_2 = (u, g)$
$\theta_1 = (m, i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (m, g)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (u, i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (u, g)$	AAA A	CCC C	AAA A	CCC C
$\theta_3 = (u, i)$	$\theta_2 = (m, i)$	$\theta_2 = (m, g)$	$\theta_2 = (u, i)$	$\theta_2 = (u, g)$
$\theta_1 = (m, i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (m, g)$	AAA A	CCA A/C	AAA A	CCA A/C
$\theta_1 = (u, i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (u, g)$	AAA A	CCA A/C	AAA A	CCA A/C

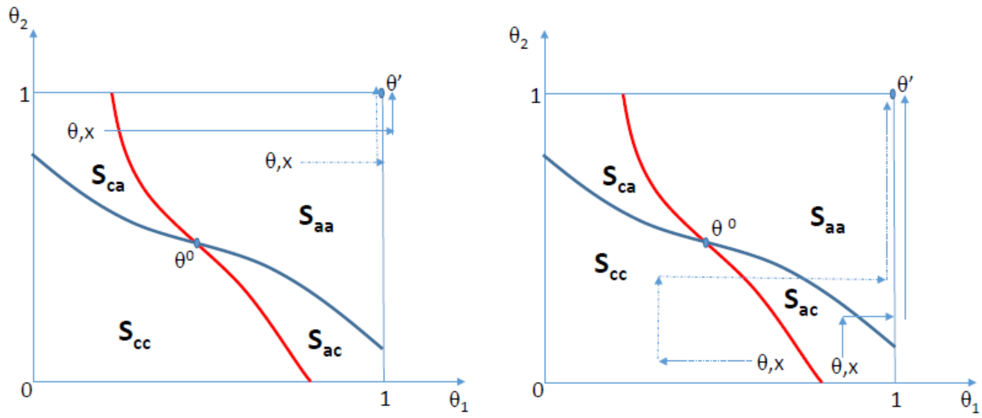
$\theta_3 = (m, g)$	$\theta_2 = (m, i)$	$\theta_2 = (m, g)$	$\theta_2 = (u, i)$	$\theta_2 = (u, g)$
$\theta_1 = (m, i)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (m, g)$	CCC C	CCC C	CAC A/C	CCC C
$\theta_1 = (u, i)$	AAA A	ACC A/C	AAA A	ACC A/C
$\theta_1 = (u, g)$	CCC C	CCC C	CAC A/C	CCC C
$\theta_3 = (u, g))$	$\theta_2 = (m, i)$	$\theta_2 = (m, g)$	$\theta_2 = (u, i)$	$\theta_2 = (u, g)$
$\theta_1 = (m, i)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (m, g)$	CCC C	CCC C	CAC A/C	CCC C
$\theta_1 = (u, i)$	AAA A	ACC A/C	AAA A	ACC A/C
$\theta_1 = (u, g)$	CCC C	CCC C	CAC A/C	CCC C

Table 8. Each agent's best alternative and outcomes of all voting by quota mechanisms. ■

Private goods without money

Proposition 8 *The environment* $(\Theta, \mathfrak{R})$ *in Example 4 is knit.*

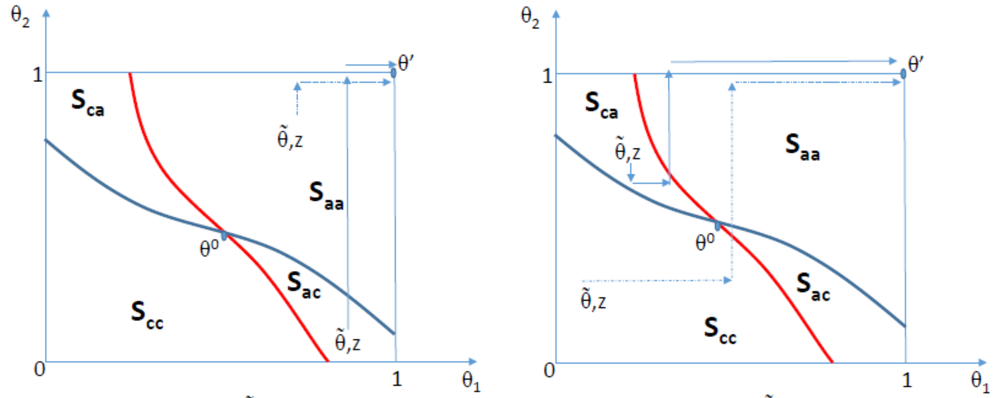
Proof of Proposition 8. Given any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ we will show that there exist θ', S, $\widetilde{S}$ such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passages are x and z-satisfactory. We choose $\theta' = (1, 1)$ independently of the two chosen pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. In defining the sequence S from θ to θ' with x as reference alternative, we distinguish two cases where we will end up analyzing all possible $\theta \in \Theta$. In particular, we cover the case where θ and $\widetilde{\theta}$ are the same. Note that in Example 4, we assume that the sets S_{ac} and S_{ca} are non-empty.
<u>Case 1</u>. $\theta \in S_{ca} \cup S_{aa} \cup S^0$. First change the type of agent 1 from $\theta_1 \neq 1$ to 1. Since the function g_1 is increasing in type 1**,** the preferences of agent 1 induced by this change are either an x-reshuffling (if $\theta \in S_{aa}$) or an x-monotonic transform ($\theta \in S_{ca} \cup S^0$) of her original ones. Then change the type of agent 2 from θ_2 to 1. Again, since the function g_2 is increasing in type 2, the preferences of agent 2 induced by this change are an x-reshuffling of her original ones (see Picture 2.a in Figure 2).
<u>Case 2</u>. $\theta \in S_{ac} \cup S_{cc}$. In this case we may not be able to change types of agents from $\theta_i \neq 1$ to $(1, 1)$ as directly as above.
If θ is a type profile from which we could reach another one in S_{aa} by letting the type of the first agent to be 1, we use the same argument as in Case 1: first change the type of agent 1 from $\theta_1 \neq 1$ to 1. The preferences of agent 1 induced by this change are either an x-reshuffling (if $\theta \in S_{ac}$) or an x-monotonic transform (if $\theta \in S_{cc}$) of her original ones. Then change the type of agent 2 from θ_2 to 1. The preferences of agent 2 induced by this change are an x-reshuffling of her original ones.
If not, before reaching this situation, the sequence S must start by previous changes of signals, at most one for each agent, as shown in Picture 2.b in Figure 2, that keep us within the element of the partition where θ belongs to. The induced preferences resulting from these previous type changes remain unchanged.



Picture 2.a. Case 1: $\theta \in S_{ca} \cup S_{aa} \cup S^0$ Picture 2.b. Case 2: $\theta \in S_{ac} \cup S_{cc}$

Figure 2. Changes of agents' types in Cases 1 and 2, proof of Proposition 7.

To define the sequence $\widetilde{S}$ from $\widetilde{\theta}$ to θ' with z as reference alternative, we would follow a parallel construction to Cases 1 and 2 above. The relevant cases would now be Case 3: $\widetilde{\theta} \in S_{ac} \cup S_{aa} \cup S^0$ and Case 4: $\widetilde{\theta} \in S_{ca} \cup S_{cc}$ where we would consider all possible type profiles $\widetilde{\theta} \in \Theta$ including θ. The proof for the existence of the sequence $\widetilde{S}$ would require a similar argument to those of Cases 1 and 2, respectively, but changing first agent 2's signal to 1 when required to get to S_{aa}. See the graphical representation in Figure 3.
The construction of these passages proves that our environment is knit as we wanted to



Picture 3.a. Case 3: $\tilde{\theta} \in S_{ac} \cup S_{aa} \cup S^0$ Picture 3.b. Case 4: $\tilde{\theta} \in S_{ca} \cup S_{cc}$

Figure 3. Changes of agents' types in Cases 3 and 4, proof of Proposition 7.

show. ■

Before engaging in the proof that the environment in Example 5 is partially knit, observe that the changes in the functions g_i imply that the sets $\overline{S}_{ca} = \{\theta \in \Theta : zP_1x \text{ and } zP_2x\}$ and $\overline{S}_{ac} = \{\theta \in \Theta : xP_1z \text{ and } xP_2z\}$ are empty, and that S^0 is not a singleton. Due to the specific form of g_i the indifference set is L-shaped and thick, as shown in Figure 4.

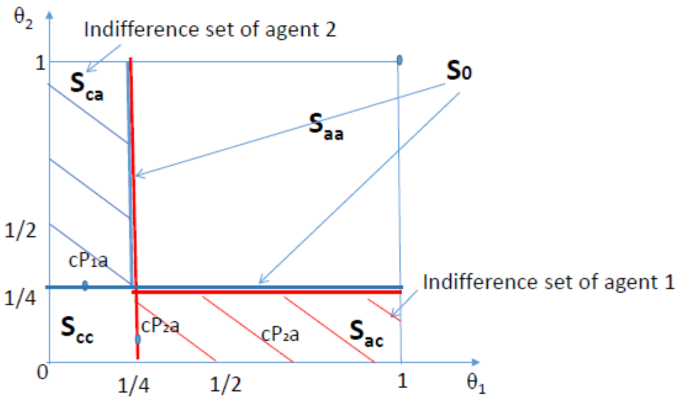


Figure 4. Partition of S in Example 5.

Proposition 9 *The environment $(\Theta, \mathfrak{R})$ in Example 5 is partially knit.*

Proof of Proposition 9. Remember that type profiles are signal profiles. Thus, we identify s with θ. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \varnothing$ and $\#C(\theta, z, x) \geq 2$. These two conditions on θ imply that we must only consider $\theta \in S_{ca}$, i.e. where agent 1 strictly prefers z to x and agent 2 is indifferent between x and z. Define $\theta' = \widetilde{\theta}$.

We have to define S such that θ leads to $\theta' = \widetilde{\theta}$ through S and the passage is x-satisfactory. We distinguish two cases. See the graphical representation of both cases in Figure 5.

Case 1. $\widetilde{\theta} \in S_{aa} \cup S_{ca}$. Define $S = \{\widetilde{\theta}_1, \widetilde{\theta}_2\}$ and $I(S) = \{1, 2\}$. Note that if $\widetilde{\theta}, \theta \in S_{ca}$ the proof is obvious since we move along the same set S_{ca} and no agent preferences change.

Suppose that $\widetilde{\theta} \in S_{aa}$. We first increase the signal of agent 1 to $\theta'_1 = \widetilde{\theta}_1$. The induced preferences of agent 1 are an x-monotonic transform of her previous ones. Agent 2 turns to strictly prefer z to x, that is, $zR_2(\theta'_1, \theta_2)x$. Decrease or increase now agent 2's signal to $\theta'_2 = \widetilde{\theta}_2$. Note that agent 2's induced preferences are identical to her previous ones, thus, are obviously an x-reshuffling of them. So we have gone from θ to θ' through adequate types changes with respect to x.

Case 2. $\widetilde{\theta} \in S_{cc} \cup S_{ac}$. Define $S = \{\widetilde{\theta}_2, \widetilde{\theta}_1\}$ and $I(S) = \{2, 1\}$. We first decrease the signal of agent 2 to $\theta'_2 = \widetilde{\theta}_2$. The induced preferences of agent 2 are an x-monotonic transform of her previous ones $R_2(\theta)$ (since $zP_2(\theta)x$ while $xP_2(\theta_1, \theta'_2)z$). Agent 1 turns to have the same preferences as before, that is, $zR_1(\theta_1, \theta'_2)x$. Now, we decrease or increase agent 1's signal to $\theta'_1 = \widetilde{\theta}_1$. Note that agent 1's induced preferences are either identical to her previous ones (thus, obviously an x-reshuffling of those) or an x-monotonic transform of $R_1(\theta_1, \theta'_2)$ (since $zP_1(\theta_1, \theta'_2)x$ while $zI_1(\theta')x$). So, we have gone from θ to θ' through adequate changes of types with reference x.

It remains to consider any two pairs where (z, θ), $(x, \widetilde{\theta}) \in A \times \Theta$ are such that $\overline{C}(\theta, x, z) \neq \varnothing$ and $\#C(\theta, x, z) \geq 2$, for which a symmetric and similar argument would work.

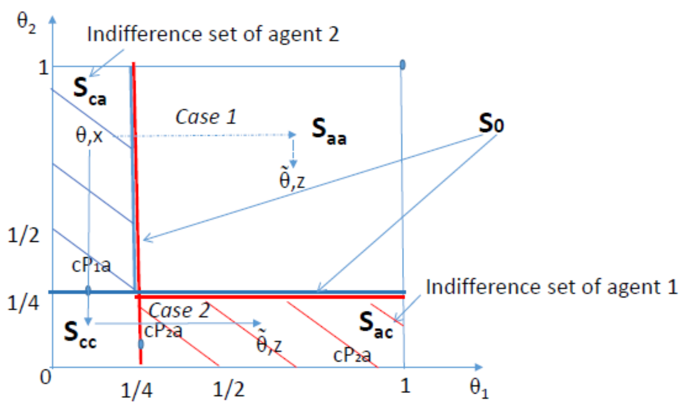


Figure 5. Changes of agents' types, proof of Proposition 8.

■

Remark 2 *$f_{veto,x}$ is non-constant, ex post incentive compatible, and respectful in the environment (Θ, R) in Example 5.*

Proof of Remark 2. Observe that, by definition, $f_{veto\ x}$ is non-constant and no agent can gain by changing her individual types, since she will either obtain the same or an indifferent one, when deviating, or else obtain her best outcome through by being truthful. Ex post group incentive compatibility is straightforward since changing both types it is impossible to weakly improve both agents, and at least on of them strictly: note that either agent 1 or 2 strictly lose (we need to check 6 cases: $\theta \in S_{aa}$ and $\theta' \in S_{ca}$ or vice versa; $\theta \in S_{ac}$ and $\theta' \in S_{ca}$ or vice versa; and $\theta \in S_{cc}$ and $\theta' \in S_{ac}$ or vice versa). To show that $f_{veto\ x}$ is respectful, note that the only way for agent 1 to remain indifferent according to her initial preferences $R_1(\theta)$ and get a different outcome when changing her type is when $\theta \in S_{ac}$ and $\theta'_1 < \frac{1}{4}$ such that $(\theta'_1, \theta_2) \in S_{cc}$. However, $R_1(\theta'_1, \theta_2)$ is not an $x = f_{veto\ x}(\theta)$-monotonic transform of $R_1(\theta)$. Similarly, for agent 2, to remain indifferent and get a different outcome when changing her type $\theta \in S^0$ and $\theta_2 \geq \frac{1}{4}$, $\theta'_2 < \frac{1}{4}$. However, $R_2(\theta_1, \theta'_2)$ is not a $z = f_{veto\ x}(\theta)$-monotonic transform of $R_2(\theta)$. ■

Auctions

Proposition 10 *For any $i \in N$, let $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$ and let g_i be weakly increasing in the type of each agent and satisfy the single-crossing condition. Then, $\Theta = \times_{i \in N} \Theta_i$ is not knit.*

Proof of Proposition 10. Take the following two pairs (alternative and profile of types): (x, θ) where $\theta = (\overline{\theta}_1, \underline{\theta}_2, \underline{\theta}_3, .., \underline{\theta}_n)$ and $x = ((1, g_1(\theta), (0,0), .., (0,0)), (z, \widetilde{\theta})$ where $\widetilde{\theta} = (\underline{\theta}_1, \overline{\theta}_2, \underline{\theta}_3, .., \underline{\theta}_n)$ and $z = ((0,0), (1, g_2(\widetilde{\theta}), .., (0,0))$. By Claims 1 and 2 below, we show that these two pairs can not be pairwise knit.

For (x, θ) and $(z, \widetilde{\theta})$ to be pairwise knit, there should exist θ' and sequences of type S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory, the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

<u>Claim 1.</u> There is no θ' and sequence of types S such that the passage from θ to θ' through S is x-satisfactory.

Claim 2. There is no θ' and sequence of types $\widetilde{S}$ such that the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Proof of Claim 1. Suppose, by contradiction, that there exist θ' and a sequence of types S such that the passage from θ to θ' through S is x-satisfactory. We distinguish two cases:
Case 1. Suppose first that agent 1 is the first agent changing her type in the sequence S, that is, $1 = i(S,1)$. Since agent 1's type in θ is $\overline{\theta}_1$, the highest possible in Θ_1, we have that $\theta^S_{i(S,1)} < \overline{\theta}_1$. By weakly increasingness of the g_j's, we have that for any $j \in N$

$$g_j(\theta) \geq g_j(m^1(\theta, S)). \tag{1}$$

Since $R^1_1(\theta, S)$ has to be a x-monotonic transform of $R_1(\theta)$, we have that $g_1(m^1(\theta, S)) = g_1(\theta)$ and $R^1_1(\theta, S) = R_1(\theta)$. By single-crossing, for any $j \in N\backslash\{1\}$,

$$g_1(\theta) - g_1(m^1(\theta, S)) > g_j(\theta) - g_j(m^1(\theta, S)). \tag{2}$$

Since $g_1(m^1(\theta, S)) - g_1(\theta) = 0$, we get to a contradiction between Equations (1) and (2). Therefore, agent 1 can not be the first agent changing types in S.
Case 2. Suppose that any agent $i \in N\backslash\{1\}$ is the first agent changing her type in the sequence S, that is, $i = i(S,1)$. Since agent i's type is the lowest possible in Θ_i, we have that $\theta^S_{i(S,1)} > \underline{\theta}_i$. By weakly increasingness of the g_j's, for any $j \in N$

$$g_j(m^1(\theta, S)) \geq g_j(\theta). \tag{3}$$

Since $R^1_i(\theta, S)$ has to be a x-monotonic transform of $R_i(\theta)$, we have that $g_i(m^1(\theta, S)) = g_i(\theta)$ and $R^1_i(\theta, S) = R_i(\theta)$. By the single-crossing condition, for any $j \in N\backslash\{i\}$,

$$g_i(m^1(\theta, S)) - g_i(\theta) > g_j(m^1(\theta, S)) - g_j(\theta). \tag{4}$$

Since $g_i(m^1(\theta, S)) - g_i(\theta) = 0$, we get to a contradiction between Equations (3) and (4). Therefore, agent $i \in N\backslash\{1\}$ can not be the first agent changing types in S.
From Cases 1 and 2 above, we obtain that there is no sequence of types S such that the passage from θ to θ' through S is x-satisfactory.
This ends the proof of Claim 1.
The *proof of Claim 2* is similar to that of Claim 1 and, therefore, it is omitted. ■

The following Lemma 1 is used in the proof of Proposition 11 (also for the one of Proposition 12 stated below).

Lemma 1 *Let g_k be non-decreasing in θ_k. For all $\theta \in \Theta$, $R_k(\theta'_k, \theta_{-k})$ is a y-monotonic transform of $R_k(\theta)$ for all $\theta'_k < \theta_k$, $k \in N$ and $y \in A$ such that $y_k = (0,0)$.*

Proof. Take $\theta \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (0,0)$ and $\theta'_k < \theta_k$. Since g_k is non-decreasing in θ_k, $g_k(\theta'_k, \theta_{N\backslash\{k\}}) \leq g_k(\theta)$ which means that agent k values the good in signal profile $(\theta'_k, \theta_{N\backslash\{k\}})$ at most as under profile θ. Thus, $(0,0)$ weakly improves its position in $R_k(\theta'_k, \theta_{N\backslash\{k\}})$ compared to its position in $R_k(\theta)$. Formally, $R_k(\theta'_k, \theta_{N\backslash\{k\}})$ is a y-monotonic transform of $R_k(\theta)$. ■

Proposition 11 *The environment $(\Theta, \mathfrak{R})$ in Example 6 is knit.*

Proof of Proposition 11. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. We must find θ', sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Consider $\theta' = \left(\widetilde{\theta}_i, \underline{\theta}_{N \setminus \{i\}}\right)$. We first propose a sequence of types $S = \underline{\theta}$ $(t_S = n)$ with $I(S)$ defined as follows. We initially change, one by one, the signal of agents k that do not get the good in x from θ_k to $\underline{\theta}_k$ following the order of natural numbers. If there is one agent i left who was getting the good in x change her signal from θ_i to $\underline{\theta}_i$. In each step $h \in \{1, ..., n-1\}$, by Lemma 1, we obtain that $R_{i(S,h)}(m^h(\theta, S))$ is an x-monotonic transform of $R_{i(S,h)}(m^{h-1}(\theta, S))$ since no agent $i(S, h)$ gets the good in x.
As for the last agent in the sequence, her preferences will not change when her signal goes from θ_i to $\underline{\theta}_i$ due to assumption (b) of function g_i.
This completes our argument that the passage from θ to θ' through S is x-satisfactory.
We could repeat exactly the same argument to show that the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory after replacing the roles of θ by $\widetilde{\theta}$ and x by z. ■

The following Lemma 2 is used in the proof of Proposition 12.

Lemma 2 *Let g_k be non-decreasing in θ_k. For all $\theta \in \Theta$, $R_k(\theta'_k, \theta_{-k})$ is a y-monotonic transform of $R_k(\theta)$ for all $\theta'_k > \theta_k$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$.*

Proof. Take $\theta \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$ and $\theta'_k > \theta_k$. Since g_k is non-decreasing in θ_k, $g_k(\theta'_k, \theta_{N \setminus \{k\}}) \geq g_k(\theta)$ which means that agent k values the good in signal profile $(\theta'_k, \theta_{N \setminus \{k\}})$ at least as under profile s. Thus, $(1, p)$ weakly improves its position in $R_k(\theta'_k, \theta_{N \setminus \{k\}})$ compared to its position in $R_k(\theta)$. Formally, $R_k(\theta'_k, \theta_{N \setminus \{k\}})$ is a y-monotonic transform of $R_k(\theta)$. ■

Proposition 12 *The environment $(\Theta, \mathfrak{R})$ in Example 7 is partially knit.*

Proof of Proposition 12. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \emptyset$, $\#C(\theta, z, x) = 2$. Some agent must get the good either in x or in z, otherwise $\overline{C}(\theta, z, x) = \emptyset$.
First, assume that the same agent i gets the good both in x and in z. Define $\theta' = \left(\max\{\theta_i, \widetilde{\theta}_i\},\ \min\{\theta_j, \widetilde{\theta}_j\}\right)$, $S = \widetilde{S} = \{\max\{\theta_i, \widetilde{\theta}_i\}, \min\{\theta_j, \widetilde{\theta}_j\}\}$ where $I(S) = I(\widetilde{S}) = \{i, j\}$. Note that for step $h = 1$, either $\theta_{i(S,1)} = \theta_{i(\widetilde{S},1)} = \theta_i$ if $\theta_i > \widetilde{\theta}_i$ or $\theta_{i(S,1)} = \theta_{i(\widetilde{S},1)} = \widetilde{\theta}_i$ if $\theta_i < \widetilde{\theta}_i$. Thus, either because there is no signal change or by Lemma 2, we obtain that $R_i(m^1(\theta, S))$ is an x-monotonic transform of $R_i(m^0(\theta, S))$ and $R_i(m^1(\widetilde{\theta}, \widetilde{S}))$ is an z-monotonic transform of $R_i(m^0(\widetilde{\theta}, \widetilde{S}))$. Note that for step 2, either $\theta_{i(S,h)} = \theta_{i(\widetilde{S},h)} = \theta_j$ if $\theta_j < \widetilde{\theta}_j$ or $\theta_{i(S,h)} = \theta_{i(\widetilde{S},h)} = \widetilde{\theta}_j$ if $\theta_j > \widetilde{\theta}_j$. Thus, either because there is no signal change or by Lemma 1, we obtain in step 2 that $R_j(m^2(\theta, S))$ is an x-monotonic transform of $R_j(m^1(\theta, S))$ and $R_j(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_j(m^1(\widetilde{\theta}, \widetilde{S}))$. Thus, the passage from θ to θ' through S is x-satisfactory, and that from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Second, suppose that different agents get the good in x and z. Without loss of generality, say that agent 1 gets the good in x while agent 2 gets it in z. Thus, alternatives x and z are such that $x_1 = (1, p_x)$, $z_1 = (0, 0)$, $x_2 = (0, 0)$, $z_2 = (1, p_z)$.

Now, we consider three cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Case 1. $\theta = (0, 1)$.

The conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ are satisfied since $p_x > l$ and $p_z > l$. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1, 1)$, let $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{1\}$, if $\widetilde{\theta} = (0, 0)$, let $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and if $\widetilde{\theta} = (1, 0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$, $I(S) = \{1, 2\}$. By applying Lemma 2, Lemma 1 or both, respectively, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.

Case 2. $\theta = (1, 1)$.

For conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ to hold we must have either $p_x > m$ and $p_z \leq m$, or $p_z < m$ and $p_x \geq m$. Suppose that the former holds. Otherwise, a similar proof would follow.

If $\widetilde{\theta} = (0, 1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$.

If $\widetilde{\theta} = (1, 0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta, S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 1.

If $\widetilde{\theta} = (0, 0)$, let $\theta' = (0, 1)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 1\}$, $I(\widetilde{S}) = \{2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$. Moreover, $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ since $l < p_z \leq m$.

Case 3. $\theta = (0, 0)$ and $\theta = (1, 0)$.

For both θ, $g_2(\theta) = l$. Since $2 \in C(\theta, z, x)$ then $p_z \leq l$, contradicting our hypothesis.

Third, the last remaining possibility is that in only one of the two alternatives, x or z, some agent gets the good. Without loss of generality, suppose that agent 1 gets the good in x. Note that for conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ to hold, for any $\theta \in \Theta$, $1 \in \overline{C}(\theta, z, x)$ since $2 \in C(\theta, z, x)$.

Now, we consider four cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Case 4. $\theta = (0, 1)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1, 1)$, let $S = \{\theta_{i(S,1)} = 1\}$ and $I(S) = \{1\}$, if $\widetilde{\theta} = (0, 0)$, let $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{2\}$, and if $\widetilde{\theta} = (1, 0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$ and $I(S) = \{1, 2\}$. By applying either Lemma 2, Lemma 1 or both consecutively in this order, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.

Case 5. $\theta = (1, 1)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > m$ must be satisfied.

If $\widetilde{\theta} = (0, 1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$.

If $\widetilde{\theta} = (1,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta, S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 1.
If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_2(m^2(\theta, S))$ is an x-monotonic transform of $R_2(m^1(\theta, S))$ by Lemma 1.
Case 6. $\theta = (0,0)$.
Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied.
If $\widetilde{\theta} = (0,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{2\}$, and observe that $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ by Lemma 1.
If $\widetilde{\theta} = (1,0)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{1\}$, and observe that $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ by Lemma 1.
If $\widetilde{\theta} = (1,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{2,1\}$, and observe that, by Lemma 1, $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ and $R_1(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$.
Case 7. $\theta = (1,0)$.
Since $1 \in \overline{C}(\theta, z, x)$, $p_x > h$ must be satisfied.
If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta} = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$.
If $\widetilde{\theta} = (0,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$ and $I(\widetilde{S}) = \{2\}$. Observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$. Moreover, $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 1.
If $\widetilde{\theta} = (1,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$ and $I(\widetilde{S}) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ and $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 1. ■

Remark 3 *$f_{p,p'}$ is non-constant, ex post incentive compatible, and respectful in the environment $(\Theta, \mathfrak{R})$ in Example 7.*

Proof of Remark 3. By definition $f_{p,p'}$ is not constant. To show that $f_{p,p'}$ is ex post incentive compatible we first observe that agent 1 can never strictly gain by deviating from any $\theta \in \Theta$. For any $\theta_2 \in \Theta_2$, since $g_1(0, \theta_2)) = l$, $g_1(1, \theta_2) \in \{m, h\}$, and $p \in (l, m)$, then $f_1(0, \theta_2) P_1(0, \theta_2) f_1(1, \theta_2)$ and $f_1(1, \theta_2) P_1(1, \theta_2) f_1(0, \theta_2)$ where $f_1(0, \theta_2) = (0, 0)$ and $f_1(1, \theta_2) = (1, p)$. Similarly, we can show that agent 2 can never strictly gain by deviating from any $\theta \in \Theta$. For any $\theta_1 \in \Theta_1$, since $g_2(\theta_1, 0) = l$, $g_2(\theta_1, 1) \in \{m, h\}$, and $p' \in (l, m)$, then $f_2(\theta_1, 0) R_2(\theta_1, 0) f_2(\theta_1, 1)$ and $f_2(\theta_1, 1) R_2(\theta_1, 1) f_2(\theta_1, 0)$ where $f_2(\theta_1, 0) = (0, 0)$ and $f_2(\theta_1, 1) \in \{(0,0), (1, p')\}$. To check respectfulness, observe that agent 1 is not indifferent between any pair of outcomes obtained when she is the only one changing types. As for agent 2, observe that the same holds if $\theta_1 = 0$, for which we use condition c of g_i's. For $\theta_1 = 1$, $f(1,0) = f(1,1)$. Thus, respectfulness holds. ■

Appendix C. Illustrative examples

In this Appendix we first show satisfactoriness and knitness using Example 1. Then, Example 2 shows that partial knitness is not necessary for the result in Theorem 1 to hold.

Remark 4 *The first passage defined in Example 1 is a-satisfactory. The second is not.*

Proof of Remark 4. Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \underline{\theta}_2)$, and $S = \{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\}$ be a sequence of individual types. Note that, $I(S) = \{2, 1, 2\}$ and $t_S = 3$. The passage from θ to θ' through S is a-satisfactory. To show it, we have to check that for each $h \in \{1, 2, t_S = 3\}$, $R^h_{i(S,h)}(\theta, S)$ is an a-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.
For that, observe first that $R^0_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \underline{\theta}_2)$, $R^1_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \overline{\theta}_2)$, $R^1_{i(S,2)}(\theta, S) = R_1(\underline{\theta}_1, \overline{\theta}_2)$, $R^2_{i(S,2)}(\theta, S) = R_1(\overline{\theta}_1, \overline{\theta}_2)$, $R^2_{i(S,3)}(\theta, S) = R_2(\overline{\theta}_1, \overline{\theta}_2)$, and $R^3_{i(S,3)}(\theta, S) = R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Then, using the table in Example 1, note that the following three facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2) = a(bc)$ is an a-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2) = b(ac)$ since $U(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \{a\} \subseteq U(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \varnothing \subseteq \overline{U}(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{b\}$.
Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2) = bca$ since $U(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\} \subseteq U(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{c\} \subseteq \overline{U}(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{b, c\}$.
Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ is an a-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ since both preferences coincide.
Now, let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, and $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ be a sequence of individual types. Note that, $I(S) = \{1, 2\}$ and $t_S = 2$. The passage from θ to θ' through S is not a-satisfactory. To show it, observe that for $h = 1$, $R^h_{i(S,h)}(\theta, S)$ is not an a-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$. By definition, $R^0_{i(S,1)}(\theta, S) = R_1(\theta)$ and $R^1_{i(S,1)}(\theta, S) = R_1(\overline{\theta}_1, \underline{\theta}_2)$.
Moreover, $R_1(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ is not an a-monotonic transform of $R_1(\theta) = acb$ since $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\} \nsubseteq U(R_1(\theta), a) = \{a\}$ (in fact, $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{c\} \nsubseteq \overline{U}(R_1(\theta), a) = \varnothing$). ■

Remark 5 *The environment $(\Theta, \mathfrak{R})$ in Example 1 is knit.*

Proof of Remark 5. To check that the environment $(\Theta, \mathfrak{R})$ is knit for $\Theta = \{(\underline{\theta}_1, \underline{\theta}_2), (\underline{\theta}_1, \overline{\theta}_2), (\overline{\theta}_1, \underline{\theta}_2), (\overline{\theta}_1, \overline{\theta}_2)\}$, we must prove that all pairs of alternatives and types are pairwise knit, that is, can be connected through satisfactory sequences. To do that, we will show how to choose the appropriate ones for two specific cases, and then argue that all others can be reduced essentially to one of the patterns we shall follow.
Case 1. $(x, \theta) = (a, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (b, (\overline{\theta}_1, \underline{\theta}_2))$.
Define $\theta' = \widetilde{\theta} = (\overline{\theta}_1, \underline{\theta}_2)$, $S = \{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\}$ (thus, $I(S) = \{2, 1, 2\}$ and $t_S = 3$), $\widetilde{S} = \varnothing$ (thus, $I(\widetilde{S}) = \varnothing$ and $t_{\widetilde{S}} = 0$). Note that since $\theta' = \widetilde{\theta}$, then $\widetilde{\theta}$ trivially leads to θ' through $\widetilde{S}$ and this passage from $\widetilde{\theta}$ to θ' is b-satisfactory. We need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the three (t_S) following facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$. Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2)$ is an a-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2)$.
Case 2. $(x, \theta) = (c, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (a, (\underline{\theta}_1, \overline{\theta}_2))$.

Define $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), $\widetilde{S} = \{\overline{\theta}_1\}$ (thus, $I(\widetilde{S}) = \{1\}$ and $t_{\widetilde{S}} = 1$). As above, first we need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the two (t_S) following facts hold: $R_1(\overline{\theta}_1, \underline{\theta}_2)$ is a c-monotonic transform of $R_1(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_2(\overline{\theta}_1, \overline{\theta}_2)$ is a c-reshuffling of $R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Second, we need to show that $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passage is a-satisfactory. For that we need to observe using the table that $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$.
To finish the proof of knitness we should consider all remaining combinations of (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. Observe that each one of those cases can be embedded in either Case G1 or Case G2 below, which generalize Cases 1 and 2, respectively.
Case G1. (x, θ) and $(z, \widetilde{\theta})$ such that $x \in \{a, b\}$.
Case G2. (x, θ) and $(z, \widetilde{\theta})$ such that $x = c$.
To prove knitness for Case G1, consider $\theta' = \widetilde{\theta}$, $\widetilde{S} = \varnothing$, and S will depend on θ and $\widetilde{\theta}$. Similarly, to prove knitness for Case G2, consider $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), and $\widetilde{S}$ will depend on θ and $\widetilde{\theta}$. ■

Example 2 *Consider a private values environment with a finite set of agents N, six alternatives $A = \{a_1, a_2, a_3, a_4, y, z\}$, and each agent i has only two strict preferences $\mathcal{R}_i = \{R^2, R^4\}$:*

R^2	R^4
a_2	a_4
y	y
$\mathbf{a}_3$	a_1
a_4	a_2
z	z
a_1	$\mathbf{a}_3$

To show that the environment $\times_{i \in N} \mathcal{R}_i$ is not partially knit, take the two pairs (a_3, R) and $(y, \widetilde{R})$, where $R = (R^2)^n$ and $\widetilde{R} = (R^4)^n$ (note that $C(R, y, a_3) = N$). These two pairs are not pairwise knit since there is no agent's preference $\widehat{R} \neq R^2$ such that $\widehat{R}$ be an a_3-monotonic transform of R^2 and no agent's preference $\overline{R} \neq R^4$ such that $\overline{R}$ be an y-monotonic transform of R^4. Thus, we can not construct R'.
However, by Theorem 1 in Barberà, Berga, and Moreno (2010) we know that any strategy-proof mechanism on $\times_{i \in N} \mathcal{R}_i$ is strong group strategy-proof since $\times_{i \in N} \mathcal{R}_i$ satisfies sequential inclusion (by their Example 3).[23]

[23]Another example following the same reasoning would consist of considering two agents, the same six alternatives, and enlarging the set of individual preferences to be the four preferences in Example 3 in Barberà, Berga, and Moreno (2010).