Time-limited Loyalty Rewards

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Abstract

Most loyalty programs place tight restrictions on the timing of redemption of rewards. The goal of this paper is to understand the motivation behind these restrictions as well as its implications. I present an infinite-horizon model in which a monopolist can commit to sell the good to repeat customers at a reduced price. Such an option may or may not expire at a certain date. Consumer preferences are subject to temporary shocks, which implies that loyalty rewards may increase efficiency by fostering consumer participation. On the one hand, lifting time restrictions, raises the value of the rewards for consumers by allowing them to engage in intertemporal substitution. On the other hand, it induces consumers to excessively delay the redemption of rewards, and hence all future purchases. In most of the scenarios examined in this paper the second effect dominates and hence time-limited rewards are adopted. The interests of consumers and the firm tend to be aligned and hence there is little room for public intervention exclusively concerned with the time dimension of rewards.

Key words: loyalty rewards, repeat purchases, monopoly pricing

JEL codes: D43, L13

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1 Introduction

Loyalty programs have dramatically expanded during the past decades. According to the 2017 Colloquy Loyalty Census, in the United States alone consumers hold 3.8 billion loyalty program memberships (2.6 billion in 2012) that cover a wide array of industries, including retail, airlines, hotels, restaurants, car rentals, and financial services. The value of rewards earned in 2011 was $47.9 billion, although about one third of these claims remained unredeemed because of the time limits that are commonly used in most loyalty programs (Gordon and Hlavinka, 2011). In comparison, the penetration of loyalty rewards in Europe and other regions is much lower. Nevertheless, it is estimated that 80% of European shoppers belong to at least one loyalty program (Pearson, 2014).[1]

The design of loyalty programs varies considerably across sectors and firms. In the case of airlines (Frequent-Flyer programs), members earn points for purchases. They can redeem accumulated points to buy additional products or services, according to a complex set of rules. In other cases, the design is much simpler, such as in the case of "buy-ten-and-get-one-free" type policies (e.g., golf courts, car wash facilities), or fixed discounts (which are very popular in manufactured products).[2] These programs require sufficient commitment capacity to honor the rewards in the future. In fact, examples of partial default are not that rare. For instance, airlines, hotels and other providers of loyalty rewards can sometimes change the requirements for point redemption to reduce their liability (Chun et al, 2017). Thus, firms only have access to a limited commitment capacity, which places an upper bound on the size and scope of rewards. In this context, it is not surprising that consumers participating in loyalty programs typically exhibit cyclical behavior: a number of purchases at a regular price are followed by a purchase at a discount price, after which a new cycle begins. Hence, the time limits on the redemption of rewards may affect the frequency of regular purchases.

The theoretical literature on loyalty programs has examined the optimal design of rewards in a monopoly setting (Crémer, 1985; Caminal, 2012), its pro- or anti-competitive

[1] Bryan Pearson (2014), Four Big Differences between US and Europe Loyalty programs. Yahoo Small Business
https://smallbusiness.yahoo.com/advisor/four-big-differences-between-u-european-loyalty-programs-030144436.html

[2] Loyalty rewards are popular not only in consumer goods markets but also in markets for intermediate goods. The analysis of the latter case has mainly focused on the relative nature of the rewards (i.e., market share discounts) and the dominant position of firms offering the discounts. See, for instance, Calzolari and Denicolò (2013) or Majumdar and Shaffer (2009). In this paper, I will focus on markets for final consumption goods and will consider a monopoly market, and hence relative rewards can be ignored.

effects (Banerjee and Summers, 1987; Caminal and Matutes, 1990; Cairns and Gailbraith, 1990; Bulkley, 1992; Fudenberg and Tirole, 2000; Kim et al., 2001; Caminal and Claici, 2007; Chen and Pearcy, 2010; Fong and Liu, 2011; Brito and Vasconcelos, 2015)[3], or the role of loyalty programs in exploiting the agency problem between employers (who pay for the airline tickets) and the employees (who book travel and enjoy the benefits of loyalty rewards); such as in Basso et al. (2009).[4]

Most loyalty programs include various time restrictions on the collection and redemption of rewards. For example, frequent-flyer programs were introduced in the early-1980s and a few years later (United in 1989 and others followed) airlines introduced point (miles) expiration dates, which since then has become the industry standard.[5] Similarly, in other industries accumulated points or coupons also typically expire after a certain date.[6] In this paper, I will examine the motivation behind such time limits and its consequences. Why do firms place such time constraints? Are consumers harmed by them? Should time limits be regulated? To the best of my knowledge, this is the first paper that deals with these issues.

In this paper, I extend the existing literature on loyalty rewards by considering two new elements. First, I examine how different discount rates for consumers and firms can impact the design of loyalty programs. Second, I will study a truly dynamic model that captures the cyclical consumer behavior induced by loyalty programs. In this model, I can then focus on the main topic of this paper, which is the role of time limits on the redemption of rewards. I argue that these limits are indeed necessary to discipline consumers, who in their absence tend to excessively delay the redemption of rewards; and hence all future purchases. I also show that the interests of consumers and the firm are likely to be aligned, and hence there is little room for some kind of public intervention

[3] Armstrong (2006) and Fudenberg and Villas-Boas (2006) discuss how this literature fits into the broader theme of behavior-based price discrimination.

[4] The empirical literature on loyalty programs is not very conclusive. While some of the early research only found weak evidence for the positive effect of loyalty programs on the firm's profitability and market share (See, for instance, Sharp and Sharp, 1997; Bolton et al., 2000; Lal and Bell, 2003; and Lewis (2004), more recently Lederman (2007 and 2008) examined the airline industry and found evidence of the positive effect of the frequent flyer program (FFP) on the firm's sales and prices, whereas Liu and Yang (2009) reported positive effects of FFPs on aggregate demand. Finally, Behrens and McCaughey (2015) found small but positive effects of FFPs on both profits and consumer surplus.

[5] In most airlines the points expire after two years; see Eric Rosen; "A Brief History of Airline Frequent-Flyer Programs", The Points Guy, August 2019. Available at https://thepointsguy.com.

[6] Several empirical studies have documented the important effect of point expiration on consumer behavior and the effectiveness of loyalty programs; see, for instance, Dorotic et al. (2014) and Hartman and Viard (2008).

that is exclusively concerned with the time restrictions of loyalty programs.

More specifically, I present an infinite-horizon model in which a monopolist sells a non-durable good to a continuum of repeat buyers. An important ingredient of the model is that consumer preferences are subject to temporary shocks. As in most of the existing literature, this is the crucial feature that motivates the introduction of loyalty programs. Indeed, purchases at discount prices foster the participation of consumers with relatively low valuations. To enhance tractability, I will assume that the valuation of an individual consumer in a given period is the realization of a random variable, which is drawn from the same distribution, and is independent across consumers and time periods. Hence, consumers are homogeneous ex-ante but heterogeneous ex-post. At any point in time, the monopolist can offer a bundle that includes one unit of the current good plus the option of purchasing the good in the future at a reduced price (which is set equal to marginal cost).[7] Two different varieties of loyalty rewards are initially considered. In the first case, the option to purchase the good at the reduced price expires after one period (i.e., time-limited rewards). In the second case, this option never expires (i.e., time-unlimited rewards). Thus, the characterization of loyalty rewards is highly stylized but its limited scope can be justified by the default incentives discussed earlier. A crucial implication of this characterization is that purchases at regular prices are followed by the redemption of rewards (i.e., purchases at discount prices), after which a new cycle begins. Thus, both consumers and the firm must take into account how the timing of reward redemptions affects subsequent purchases at regular prices.

Consumers holding a current time-limited reward (which was acquired in the previous period) face a straightforward decision problem: since these rewards expire at the end of the period, it is optimal to purchase the good whenever their valuation exceeds the discount price. Since such a price is set equal to marginal cost, reward redemption implies ex-post efficiency. Hence, the redemption of loyalty rewards temporarily eliminates the inefficiency associated with monopoly power. However, the introduction of these rewards may or may not be optimal for the monopolist because they involve costs and benefits. The benefits materialize immediately: the consumers anticipate the additional future rents that are associated with the loyalty reward, and hence their willingness to pay for

[7] Repeat purchase coupons are another form of loyalty rewards. However, they do not imply a commitment to the future transaction price, only a better treatment of repeat buyers relative to newcomers. Coupons generate time inconsistency. More specifically, the presence of consumers with coupons leads firms to set regular prices above the ex-ante optimal level; see, for instance, Caminal (2012). In this paper, I will restrict my attention to a fixed price commitment.

the current purchase increases. In other words, thanks to the reward, the current demand function shifts upwards and this raises current profits. However, the reward also involves a liability. Some consumers will end up redeeming the reward when their valuation is sufficiently high so that they would have also purchased the good at the regular price (i.e., foregone monopoly rents). From an ex-ante point of view, costs and benefits are differently affected by the degree of the firm's and consumer's impatience. In particular, the increase in the consumer's willingness to pay depends on their discount factor, β. The more that the consumers value the future, the higher their willingness to pay for the bundle. However, the loss associated to the redemption of the reward depends on the firm's discount factor, δ. The more patient that the firm is, the higher the reduction in their continuation value. The first main result of the paper follows naturally: if $\frac{\beta}{\delta}$ is above some threshold (which is lower than 1), then benefits exceed costs and the firm prefers to introduce time-limited rewards relative to no reward.

An additional intuition can be obtained by considering two extreme scenarios. Let us fix $\delta > 0$ and consider different values of β in the range $[0, \delta]$. If consumers are myopic ($\beta = 0$), then they do not value the reward and hence the promise of a future discount does not affect their willingness to pay for the bundle (i.e., the firm obtains zero benefit). However, the firm will still lose future rents (i.e., incurs positive costs), and hence prefers not to introduce loyalty rewards. By continuity, the firm will still choose no rewards if β is a small, positive number. In the other extreme, both the consumer and the firm equally discount the future ($\beta = \delta$). For this case, consider the firm's behavior off-the-equilibrium path and suppose that the price of the bundle is equal to the optimal price in case of no reward plus the increase in the consumer's willingness to pay associated to the reward. In this case, the present value of consumer surplus is not affected by the reward because the extra consumer surplus generated by the redemption of the reward is exactly compensated by the increase in the regular price. However, the present value of the total surplus increases. First, the set of consumers purchasing the bundle remains unaffected, and hence the total surplus generated in the current period remains unchanged. Second, the total surplus generated in the next period, when consumers redeem the reward, will be higher (i.e., marginal cost pricing instead static monopoly pricing). Because these extra rents are equally discounted by consumers and the firm, the present value of total surplus increases and, given that the present value of consumer surplus remains constant, the present value of profits also increases. Clearly, the price considered here will not be (generically) optimal. Indeed, the shift in the current demand exceeds the future losses, and hence the equilibrium price is indeed lower. Consequently, the present value of both profits and consumer surplus increases. By continuity, the firm has also incentives to introduce these rewards if β is below but close to δ. Moreover, the interests of consumers and the firm are perfectly aligned and the consumers always benefit from loyalty rewards whenever the firm finds that these rewards are profitable.[8] In the benchmark model with linear demand, the threshold value of $\frac{\beta}{\delta}$ is $\frac{2}{3}$.[9] Thus, if $\frac{\beta}{\delta}$ is lower than $\frac{2}{3}$, and if the firm is forced to introduce time-limited rewards, then the consumers would also be unhappy because the firm would optimally charge a higher regular price to minimize the loss of future profits, and such a price increase would be higher than the extra consumer surplus generated by the reward.

If time limits are lifted, then the value of the rewards for consumers increases because they can then now engage in intertemporal substitution. More precisely, under time-unlimited rewards, consumers find it optimal to redeem the reward whenever their valuation exceeds a threshold, which is above the marginal cost. Indeed, if the valuation is only slightly above the marginal cost, then redemption of the reward would generates an insignificant currnt surplus. Instead, if the consumer waits, then they can redeem the reward later on when the valuation is higher, and thus obtain a higher present value of consumer surplus. Thus, lifting time constraints further raises the consumers' willingness to pay for the bundle (higher current profits). Furthermore, consumers will delay the redemption of the reward, and hence all future purchases at the regular price will take place after a longer time span (i.e., lower future profits). It turns out that the second effect always dominates. Indeed, consumers choose the timing of redemption to maximize the present value of consumer surplus, while ignoring the negative impact on the present value of profits; and hence they excessively delay the redemption of the reward relative to the total welfare maximizing benchmark. In fact, the magnitude of such a negative externality is so large that lifting time constraints necessarily reduces the present value of total surplus (even if $\frac{\beta}{\delta}$ is sufficiently close to 1); and hence there is no regular price that can make time-unlimited rewards profitable.

The interest of consumers and the firm regarding the adoption of time limits are also perfectly aligned. Thus, if $\frac{\beta}{\delta}$ is higher than $\frac{2}{3}$, and if the firm was forced to introduce

[8] This result is compatible with the evidence reported in Behrens and McCaughey (2015).

[9] The firm is indifferent between no rewards and time-limited rewards if the increase in consumers' willingness to pay, $\beta\left(\frac{1}{2}-\frac{1}{8}\right)$ is equal to the expected losses, $\delta\frac{1}{4}$; that is, $\beta=\frac{2}{3}\delta$.

time-unlimited results, then once again the firm would find it optimal to charge a higher regular price that would dominate the additional benefits generated by the option of freely choosing the timing of redemption. Consequently, the current model suggests that there is no room for a public intervention exclusively concerned with the time dimension of loyalty programs.

The optimality of the time limits is quite a robust result. In particular, if we generalize the linear demand function that is assumed in the benchmark model, and if we allow for lower monopoly mark ups, then lifting time constraints is still suboptimal. Under lower markups, the extra losses caused by the excessive delay of the reward redemption are reduced, which works in favor of time-unlimited rewards. However, the benefits of loyalty rewards are also reduced because the margin to foster consumer participation is lower. Hence, lifting time limits also generates a lower extra consumer surplus, which works against time-unlimited rewards.

Time-unlimited rewards may be optimal if the consumers intrinsically value time flexibility and both the consumer and the firm are less concerned about the impact of their actions on future purchases. In particular, I show that if consumers experience satiation (i.e., with a certain probability they leave the market after a consumption episode), then they more carefully choose the timing of their purchase and will care less about their future purchases. If the degree of satiation is sufficiently high, then lifting time restrictions allows consumers to redeem the reward whenever it is (approximately) efficient.

If we relax the assumption of ex-ante homogeneous consumers, then the interests of different types of consumers regarding the design of the loyalty program would differ and the welfare analysis would be much more complex. In particular, those consumers who benefit the most from loyalty programs (high frequency buyers) might prefer a more generous loyalty program (and with less time constraints) than the one chosen by the firm. However, those consumers who benefit very little from loyalty programs (occasional buyers) might prefer to scrap them all together. The effect of loyalty programs on aggregate consumer surplus is likely to depend on very specific details of the model; and hence a clear case in favor of public intervention will still be difficult to make.

An important element of the analysis is the discrepancy between the degree of the firm's and consumer's impatience. The belief that firms are relatively more patient than consumers is widespread. Meanwhile, the assumption of profit maximization implies that firms discount future cash flows at a rate equal to the cost of capital, which for firms with deep pockets may be fairly low. In contrast, consumers are more likely to be liquidity-constrained, and hence less willing to give up current money in exchange for future money. Moreover, the consumer's discount rate is typically thought of as a preference parameter, and hence is likely to vary across individuals. There is a fairly large literature that estimates the consumers' discount rates. For example, Frederick et al (2002) provides a survey of field and laboratory experiments. The estimates of the discount factors vary a great deal across studies, and even within studies. In any case, the results suggest that individuals are likely to discount the future at a higher rate than the firm's cost of capital.[10]

In the next section, I present the benchmark model. Section 3 studies the behavior of the consumer and the firm under the different designs of the loyalty program. Stationary equilibria are analyzed in Section 4. In Section 5, I discuss alternative designs of loyalty rewards. In Section 6 I consider a more general distribution of valuations to study the effect of lower mark-ups. In Section 7, I present the version of the model with consumer satiation. Finally, Section 8 offers some concluding remarks.

2 The benchmark model

An infinitely-lived monopolist sells a non-durable good to a continuum of infinitely-lived consumers. Time is discrete, and indexed by $t, t = 0, 1, 2, ...$ All agents are forward looking but they may discount the future differently; in particular, $\delta \in (0,1)$ is the firm's discount factor and $\beta \in (0,1)$ is the consumer's discount factor. It is natural to focus on the case $\beta \leq \delta$.

The firm produces the good at constant marginal cost, which is normalized to zero. Consumers can consume at most one unit of the good per period. Moreover, they are ex-ante alike but they experience idiosyncratic preference shocks. In particular, at the beginning of period t consumer i learns her valuation of the good, v_{it}, which is a realization of an i.i.d. random variable uniformly distributed on $[0, 1]$.[11] Hence, at each period the firm faces a downward sloping demand function. Moreover, past behavior provides no information on future behavior. Consequently, in the absence of commitment there is no

[10] Recent studies tend to confirm these findings (i.e., consumers are relatively more impatient) and they further emphasize that discount rates vary across individuals (Ching and Osborne, 2015), across cultures (Chen et al., 2005) and with the size of the delayed rewards (Myerson et al., 2003). In contrast, Chevalier and Goldsbee (2009) find evidence of very small consumer discount rates.

[11] In Section 6 I examine alternative distributions.

basis for history-based price discrimination. However, I allow the firm to commit to a particular type of loyalty reward. Thus, future transaction prices may depend on current behavior. I will be more specific about these arrangements later on. Consumer i's payoff in period t is the difference between her valuation, v_{it}, and the price, p_t, if consumer i purchases the good, and zero otherwise. The firm's payoff in period t is the current revenue. All agents are risk-neutral and hence they maximize the expected discounted value of their payoffs. The realization of v_{it} is the consumer's private information. All the other elements of the model are common knowledge.

Loyalty rewards. In the benchmark model I allow for three alternative pricing arrangements: (1) *static* pricing: the firm simply offers to sell one unit of the current good, with no commitment over future transaction prices. (2) *Time-limited* loyalty rewards: the firm offers a bundle that includes one unit of the current good, plus the option of purchasing one unit of the good in the following period at a discount price (which is set equal to marginal cost); that is, the loyalty reward expires after one period. (3) *Time-unlimited* loyalty rewards: the firm offers a bundle that includes one unit of the current good, plus the option of purchasing one unit of the good at marginal cost in any future period; that is, the reward never expires.

Thus, the scope of loyalty rewards is restricted to one-time purchases only. This is meant to reflect the firms' limited commitment capacity that was previously discussed. In particular, loyalty programs that induced consumers to accumulate multiple rewards would exacerbate the firms' incentives to default, and hence would cease to be credible. For the same reason, rewards cannot be renegotiated; hence, the next transaction of a consumer with a valid reward can only take place at the discount price (zero), after which she walks into the next period with no reward. Consequently, the optimal choices of consumers holding a reward are independent of the firm's current actions. The consequences of allowing for renegotiation as well as other elements of the design of loyalty programs are discussed in more detail in Section 5.

Timing. At the beginning of period t consumers learn the realization of their valuations, v_{it}. The firm offers (p_t, Σ_t), where p_t is the price of the bundle and Σ_t is the pricing arrangement selected: one of the options (1) to (3) described above. Consumers with some type of reward decide whether or not to redeem it. Consumers without a reward observe the firm's offer (p_t, Σ_t) and decide whether or not to accept it.

Strategies. At the beginning of a period consumers differ depending on the type of reward (if any) they are holding. Thus, the game is non-stationary: the distribution of consumer types evolves over time. However, since rewards cannot be renegotiated, the market is indeed segmented. In this context, it is natural to focus on stationary strategies that are independent of history. Thus, along the equilibrium path the firm always chooses the same price and type of arrangement in every period. In addition, each type of consumer (holding no reward, time-limited and time-unlimited reward) adopts the same decision rule over time.

3 Preliminaries

In this section I will study consumer and firm behavior in a particular period, for any arbitrary history of the game, and for given (stationary) expectations about the future. Because I restrict attention to strategies that are independent of history, I do not need to keep truck of the fraction of consumers holding each type of reward. The firm's payoff obviously depends on the fraction of consumers holding each type of reward, but for our analysis it is sufficient to focus on the firm's payoff with each segment of consumers.

I denote by U, U^R, and $\widetilde{U}^R$ the continuation value of a consumer with no reward, with a time-limited reward, and with a time-unlimited reward, respectively. Similarly, let Π, Π^R, and $\widetilde{\Pi}^R$ be the firm's continuation value with a consumer that holds no reward, a time-limited reward, and a time-unlimited reward, respectively.

3.1 No reward

Suppose that the firm offers no reward in period t, and sets a price p_t. Only consumers without a reward may react to such an offer. Note that their continuation value, U, is independent of both the current price and the consumer's current behavior. Hence, their optimization problem is static: they purchase the good if and only if $v_{it} \geq p_t$. Similarly, the firm's continuation value, Π, is independent of the current decisions. Hence, its optimization problem is also static. Thus, the firm faces a linear demand function per consumer, $x_t = 1 - p_t$, and sets the price that maximizes current profits: $p_t = \frac{1}{2}$. Hence, the expected present value of profits with a consumer holding no reward at the beginning of period t is:

$$\Pi_t = \frac{1}{4} + \delta\Pi. \tag{1}$$

In addition, a consumer holding no reward obtains a current expected payoff equal to

$\frac{1}{8}$. Hence, the present value of surplus at the beginning of period t is:

$$U_t = \frac{1}{8} + \beta U. \tag{2}$$

3.2 Time-limited rewards

Suppose that in period t the firm offers a bundle that includes time-limited loyalty rewards at a price p_t. A consumer that purchases the current bundle is expected to redeem the reward in period $t+1$ with probability one. The reason is that the reward expires at the end of the period, which implies that the consumer's continuation value at $t+2$ will be the same, U, independently of her decision at $t+1$. Since valuations are always above marginal cost this implies that the continuation value at the beginning of period $t+1$ of a consumer holding a time-limited reward is $U^R = \frac{1}{2} + \beta U$.

Thus, a consumer holding no reward in period t will make a purchase if and only if the value of the purchase, $v_{it} - p_t + \beta U^R$, is higher than the value of staying put, βU. In other words, if and only if $v_{it} \geq \overline{v}_t$, where:

$$\overline{v}_t = p_t - \beta \left(U^R - U \right). \tag{3}$$

Thus, a time-limited loyalty reward raises the consumer's willingness to pay by an amount $\Delta = \beta \left(U^R - U \right)$, which is precisely the present value of the reward. Since $U^R = \frac{1}{2} + \beta U$ we can rewrite Δ as follows:

$$\Delta = \beta \left[\frac{1}{2} - (1 - \beta) U \right]. \tag{4}$$

It is important to emphasize that Δ is independent of the current price, and only depends on the expectations of future prices as well as future consumer behavior. Therefore, from equation (3) $\frac{d\overline{v}_t}{dp_t} = 1$. Because the monopolist will be able to extract some rents, and the maximum level of rents that can be generated in a period is $\frac{1}{2}$, then $(1 - \beta) U < \frac{1}{2}$, which implies that $\Delta > 0$. Also note that Δ is lower when the continuation continuation value, U, is higher. In other words, when a consumer considers purchasing the bundle they anticipate the additional consumer surplus that can be obtained by redeeming the reward in the next period, $\frac{1}{2}$, but also the fact that all other future regular purchases will be delayed. As the value of these other purchases increases then the option value of waiting increases, and the current willingness to pay decreases.

Thus, with probability $\overline{v}_t$ the consumer does not purchase in period t and arrives at $t+1$ with no reward. In addition, if $v_{it} \geq \overline{v}_t$ the consumer makes a purchase and arrives at

$t+1$ with a time-limited reward. Hence, the expected value of a consumer with no reward at the beginning of period t is $U_t = \overline{v}_t \beta U + \int_{\overline{v}_t}^{1} \left(v - p_t + \beta U^R\right) dv$. Using the equation describing the consumer's optimal behavior, (3), we can write U_t as a function of the equilibrium value of $\overline{v}_t$:

$$U_t = \frac{1}{2}\left(1 - \overline{v}_t\right)^2 + \beta U. \tag{5}$$

Thus, comparing equations (2) and (5), consumers prefer a time-limited reward to no reward if and only if $\overline{v}_t \leq \frac{1}{2}$. The equations that determine $\overline{v}_t$ will be given later on.

I can now turn to the pricing decision. The monopolist sets the price of the bundle, p_t in order to maximize the expected value of profits that can obtain from consumers holding no reward. The firm correctly anticipates consumer behavior, which is given by equation (3). Thus, given p_t, a fraction $\overline{v}_t$ of consumers without a reward will abstain from purchasing, and hence in period $t+1$ will not hold a reward. However, a fraction $1-\overline{v}_t$ will now pay p_t and in the next period they will redeem the reward. More formally, in period t the firm sets the price p_t in order to maximize $\Pi_t = \left(1 - \overline{v}_t\right)\left(p_t + \delta \Pi^R\right) + \overline{v}_t \delta \Pi$, where $\overline{v}_t = p_t - \Delta$ and hence $\frac{d\overline{v}_t(p_t)}{dp_t} = 1$. Note that Π and Π^R are also independent of p_t. Hence, the optimal price is given by:

$$p_t = 1 - \overline{v}_t + \delta\left(\Pi - \Pi^R\right). \tag{6}$$

Moreover, given that the firm does not make any profits when the consumer redeems the reward, then $\Pi^R = \delta \Pi$, and hence $\delta\left(\Pi - \Pi^R\right) = \delta\left(1 - \delta\right)\Pi > 0$.

Thus, the firm engages in intertemporal price discrimination: sets a price above the level that maximizes short-run profits,$1 - \overline{v}$. Indeed, a price slightly above the short-run profit maximizing level generates a second order loss in current profits, but a first order future gain, by raising the fraction of consumers that will be ready to pay the regular price in the next period.

Using the optimal pricing formula, (6), we can write Π_t as a function of $\overline{v}_t$:

$$\Pi_t = \left(1 - \overline{v}_t\right)^2 + \delta \Pi \tag{7}$$

Hence, comparing equations (1) and (7), the firm prefers time-limited rewards over no rewards if and only if $\overline{v}_t \leq \frac{1}{2}$. Note that the interests of the consumer and the firm are perfectly aligned.

Solving the system of equations (3) and (6) we obtain the equilibrium values of $(\overline{v}_t, p_t)$. In particular,

$$\overline{v}_t = \frac{1}{2}\left[1 - \Delta + \delta\left(1 - \delta\right)\Pi\right] \tag{8}$$

Thus, time-limited rewards are preferred by both consumers and the firm if the increase in consumers' willingness to pay, Δ, is higher than the expected future losses, $\delta(1-\delta)\Pi$.

In summary, relative to the case of no rewards, the introduction of time-limited rewards has two effects. First, it increases consumers' willingness to pay, which shifts current demand upwards. This is a positive effect on the firm's profitability, and tends to raise both the price and consumer participation at time t. Second, the redemption of the rewards reduces the firm's future profits. This effect tends to raise the current price (in order to reduce the fraction of consumers with rewards) and hence reduce consumer participation. Thus, the introduction of time-limited rewards raises the regular price but the effect on consumer participation is in principle ambiguous.

3.3 Time-unlimited loyalty rewards

Suppose now that in period t the firm offers a bundle that includes time-unlimited loyalty rewards at a price p_t. The procedure to characterize the equilibrium behavior is analogous to the one used in the previous subsection. Let us first examine the behavior of consumers holding a time-unlimited reward. A consumer owning a time-unlimited reward at time $t+s$, $s>0$, will make a purchase if and only if $v_{i,t+s}+\beta U \geq \beta\widetilde{U}^R$; i.e., iff $v_{i,t+s} \geq \widetilde{\Delta} = \beta\left(\widetilde{U}^R - U\right)$. Such a continuation value satisfies: $\widetilde{U}^R = \int_0^{\widetilde{\Delta}} \beta\widetilde{U}^R dv + \int_{\widetilde{\Delta}}^1 (v+\beta U)\,dv$. Solving this equation for $\widetilde{U}^R$:

$$\widetilde{U}^R = \frac{1-\widetilde{\Delta}}{1-\beta\widetilde{\Delta}}\left[\frac{1+\widetilde{\Delta}}{2}+\beta U\right] \tag{9}$$

Note that $\widetilde{U}^R > U^R$ and hence $\widetilde{\Delta} > \Delta$.[12] Indeed, removing the time restriction on the redemption of the loyalty reward raises the consumer surplus generated by the reward, by allowing consumers to engage in intertemporal substitution. More specifically, if the current valuation is low enough, $v < \widetilde{\Delta}$, then the consumer is better off postponing the redemption of the reward, substituting a higher expected future surplus for a current low surplus.

It is important to note that, when choosing the optimal redemption period, the consumer does not take into account the impact of their decisions on the firms' profits, and hence chooses a level of $\widetilde{\Delta}$ that is too high relative to the threshold that maximizes the

[12] Let $\phi(x) = \int_0^x \beta\widetilde{U}^R dv + \int_x^1 (v+\beta U)\,dv$. First, $\phi(0) = U^R$. Second, if $x < \widetilde{\Delta}$ then $\frac{d\phi}{dx}$ >0. Consequently, $\phi\left(\widetilde{\Delta}\right) = \widetilde{U}^R > U^R$.

ex-ante total welfare. In fact, when v is below, but arbitrarily close to, $\widetilde{\Delta}$, and if these consumers were forced to redeem the reward, then they would suffer a second order loss. In contrast, the firm would enjoy a first order benefit because their future regular purchases would be brought forward.

I now turn attention to the decision of consumers without a reward. They will make a purchase in period t if and only if $v_{it} - p_t + \beta\widetilde{U}^R \geq \beta U$; that is, if and only if $v_{it} \geq \widetilde{v}_t$, where

$$\widetilde{v}_t = p_t - \widetilde{\Delta} \tag{10}$$

Note that the gap between the transaction price and the threshold value of v is the same for consumers with and without a loyalty reward. Indeed, using equation (9) in the definition of $\widetilde{\Delta}$ we can solve for $\widetilde{\Delta}$:

$$\widetilde{\Delta} = \frac{1 - \sqrt{1 - \beta^2 \left[1 - 2\left(1 - \beta\right) U\right]}}{\beta} \tag{11}$$

The ex-ante consumer utility of a consumer without a reward satisfies: $U_t = \int_0^{\widetilde{v}_t} \beta U dv + \int_{\widetilde{v}_t}^1 \left(v - p_t + \beta\widetilde{U}^R\right) dv$. Plugging equation (10) into this equation, we obtain an equation analogous to (5):

$$U_t = \frac{1}{2}\left(1 - \widetilde{v}_t\right)^2 + \beta U \tag{12}$$

Thus, comparing equations (2), (5) and (12), consumers will benefit from time-unlimited rewards if and only it fosters consumer participation when purchasing the bundle. In other words, if and only if $\widetilde{v}_t \leq \min\left\{\overline{v}_t, \frac{1}{2}\right\}$.

Turning to the pricing decision, the firm sets the price of the bundle, p_t, in order to maximize profits: $\Pi_t = \widetilde{v}_t\delta\Pi + (1 - \widetilde{v}_t)\left(p_t + \delta\widetilde{\Pi}^R\right)$. Hence, the optimal price will be analogous to equation (6):

$$p_t = 1 - \widetilde{v}_t + \delta\left(\Pi - \widetilde{\Pi}^R\right) \tag{13}$$

Hence, using equation (13) we obtain an expression analogous to equation (7):

$$\Pi_t = \left(1 - \widetilde{v}_t\right)^2 + \delta\Pi$$

Therefore, the firm prefers time-unlimited loyalty reward if and only if $\widetilde{v}_t \leq \min\left\{\overline{v}, \frac{1}{2}\right\}$. Once again, the interests of the consumer and the firm are perfectly aligned.

Now we need to compute the firm's continuation value with consumers holding a time-unlimited reward: $\widetilde{\Pi}^R = \widetilde{\Delta}\delta\widetilde{\Pi}^R + \left(1 - \widetilde{\Delta}\right)\delta\Pi$. Solving for $\widetilde{\Pi}^R$:

$$\widetilde{\Pi}^R = \frac{1 - \widetilde{\Delta}}{1 - \delta\widetilde{\Delta}}\delta\Pi \tag{14}$$

Note that $\widetilde{\Pi}^R < \delta\Pi = \Pi^R$. The reason is quite obvious at this point: under time-unlimited rewards consumers delay their regular purchases further.

Combining equations (10), (11), (13), and (14) we can compute the equilibrium values of $(p_t, \widetilde{v}_t)$. In particular:

$$\widetilde{v}_t = \frac{1}{2}\left(1 - \widetilde{\Delta} + \frac{(1-\delta)\,\widetilde{\Delta}}{1-\delta\widetilde{\Delta}}\delta\Pi\right) \tag{15}$$

Hence, comparing equations (8) and (15), both the firm and consumers prefer time-unlimited over time-limited rewards ($\widetilde{v}_t < \overline{v}_t$) if and only if $\widetilde{\Delta} - \Delta$ is higher than $\frac{\delta^2\widetilde{\Delta}}{1-\delta\widetilde{\Delta}}(1-\delta)\,\Pi$. Thus, relative to time-limited rewards, the introduction of time-unlimited rewards generates two effects. First, it raises the value of the rewards for consumers by letting them to engage in intertemporal price discrimination. This raises consumers' willingness to pay and pushes both the current price and consumer participation upwards. Second, it exacerbates the profit loss due to the fact that the redemption of rewards and hence all future purchases are delayed. This pushes the price further up (in order to reduce the fraction of consumers holding time-unlimited rewards) and hence consumer participation falls. Thus, under time-unlimited rewards the regular price is higher but consumer participation may in principle be higher or lower than in the case of time-limited rewards.

We can now summarize the results of this section regarding the choice of Σ_t:

Remark 1 *(1) The firm chooses the pricing scheme, Σ_t, that maximizes the participation of consumers with no reward; i.e., according to the lowest value of $\left\{\frac{1}{2}, \overline{v}_t, \widetilde{v}_t\right\}$. (2) The interests of consumers and the firm regarding Σ_t are perfectly aligned.*

In Section 4 I compare the values of $\left(\frac{1}{2}, \overline{v}_t, \widetilde{v}_t\right)$ in stationary equilibria, making use of equations (8) and (15).

3.4 Generalized time-unlimited rewards

It will be convenient to consider the introduction of time-limited rewards in a benchmark scenario in which consumers can commit to redeem the reward if and only if $v_{i,t+s} \geq \omega$, where ω is an exogenous parameter that can take values in the interval $\left[0, \widetilde{\Delta}\right]$. Thus, if $\omega = 0$ such reward is equivalent to a time-limited reward, and if $\omega = \widetilde{\Delta}$ we are back to the equilibrium in which consumers choose the timing of redemption optimally. Because individual consumer valuations are private information, consumers cannot actually commit to choose any value of ω different from $\widetilde{\Delta}$. Nevertheless, considering such an artificial scenario will contribute to our understanding of the relative merits of these two types of rewards.

Given ω, the continuation value of a consumer with a time-unlimited reward is analogous to equation (9): $\widetilde{U}^R(\omega) = \frac{1-\omega}{1-\beta\omega}\left[\frac{1+\omega}{2} + \beta U\right]$. Hence, the value of such a reward, denoted by $\Psi(\omega)$, can be written as:

$$\Psi(\omega) = \beta\left[\widetilde{U}^R(\omega) - U\right] = \frac{\Delta - \frac{\beta}{2}\omega^2}{1-\beta\omega} \tag{16}$$

Obviously, $\Psi(0) = \Delta$. In addition, $\Psi'(\omega) = \frac{\beta}{(1-\beta\omega)^2}\left(\Delta - \omega + \frac{\beta}{2}\omega^2\right)$. Hence, $\Psi'(\omega) > 0$ if $\omega < \widetilde{\Delta}$, and $\Psi'(\omega) = 0$ if $\omega = \widetilde{\Delta}$. Thus, consumer's valuation of the reward increases as ω increases in the interval $\left[0, \widetilde{\Delta}\right]$. In addition, a higher value of ω implies that the redemption of the reward, and hence all future purchases, are delayed. Indeed, the effect of ω on consumer participation (and hence on total welfare) is given by the equivalent to (15):

$$\widetilde{v}(\omega) = \frac{1}{2}\left[1 - \Psi(\omega) + \frac{\delta(1-\delta)\Pi}{1-\delta\omega}\right] \tag{17}$$

Thus, an increase in ω, on the one hand, raises consumers' willingness to pay, $\Psi(\omega)$, which reduces $\widetilde{v}(\omega)$ and, on the other hand, exacerbates future profit losses, which increases $\widetilde{v}(\omega)$. Note that, even thought the first effect is positive, it gets very small if ω is close to $\widetilde{\Delta}$. In contrast, the second effect, which is negative, increases in absolute value with ω.

4 Stationary equilibria

In this section I will characterize the stationary equilibria.

4.1 No rewards

Consider the existence of a stationary equilibrium in which the firm always offers no reward. In this case $U = \frac{1}{8(1-\beta)}$ and $\Pi = \frac{1}{4(1-\delta)}$.

I start by examining a one-shot deviation in which the firm offers time-limited rewards in period t. According to the discussion of Section 3.2, the consumer's valuation of a time-limited reward (equation (4)) is $\Delta = \frac{3\beta}{8}$. In other words, in period $t+1$ expected consumer surplus would be $\frac{1}{2}$, instead of $\frac{1}{8}$ in case of no rewards. Hence, the present value of the difference is $\frac{3}{8}\beta$. In addition, the firm's losses are $\delta(1-\delta)\Pi = \frac{\delta}{4}$. In period $t+1$ the firm would make zero profits with those consumers redeeming their reward, instead of $\frac{1}{4}$ in case of no reward. Hence, the present value of the difference is $\frac{\delta}{4}$. Consequently, using equation (8), a deviation to time-limited rewards is not profitable, $\overline{v}_t \geq \frac{1}{2}$, if and only if $\frac{3\beta}{8} \leq \frac{\delta}{4}$ or, equivalently, $\frac{\beta}{\delta} \leq \frac{2}{3}$.

Let us thus focus on the case $\frac{\beta}{\delta} \leq \frac{2}{3}$, and suppose that the firm deviates in period t and offers generalized time-unlimited rewards. From equation (17), $1 - 2\widetilde{v}_t(\omega) = \Omega(\omega) = \Psi(\omega) - \frac{\delta(1-\delta)\Pi}{1-\delta\omega}$, where $\Psi(\omega)$ is given by equation (16), $(1-\beta)U = \frac{1}{8}$ and $(1-\delta)\Pi = \frac{1}{4}$. Hence, the firm does not have incentives to deviate if and only if $\Omega\left(\widetilde{\Delta}\right) \leq 0$.

Clearly, $\Omega(0) \leq 0$; that is, the firm does not have incentives to intruduce time-limited results, as argued above. Moreover, in the Appendix I show that $\Omega'(\omega) < 0$. Indeed, as ω increases both the additional benefits (higher consumer willingness to pay) and the additional costs (future profit losses) increase. It turns out that the latter effect dominates for all ω. The reason is that as ω increases future transactions are delayed and hence costs and benefits are more intensively discounted. Since $\frac{\beta}{\delta}$ is small then consumer gains fall at a faster rate than the firm's profit losses. Consequently, total expected surplus decreases with ω.

All this discussion can be summarized as follows:

Proposition 2 *A stationary equilibrium with no rewards exists if and only if* $\frac{\beta}{\delta} \leq \frac{2}{3}$.

4.2 Time-limited rewards

Consider now the existence of a stationary equilibrium in which the firm offers time-limited rewards in every period. From the analysis of Section 3.2 this implies that $U = \frac{(1-\overline{v})^2}{2(1-\beta)}, \Pi = \frac{(1-\overline{v})^2}{(1-\delta)}$. Hence, the present value of the loyalty reward is $\Delta = \frac{\beta}{2}\left[1-(1-\overline{v})^2\right]$. Therefore, from equation (3), consumer behavior in stationary equilibrium is given by the following equation:

$$p = \overline{v} + \frac{\beta}{2}\left[1-(1-\overline{v})^2\right] \tag{18}$$

Given p, equation (18) implicitly determines the equilibrium value of $\overline{v}$. For the purpose of graphical illustration, it is worthwhile noticing that p is an increasing and concave function of $\overline{v}$. If $\overline{v} = 0$ then $p = 0$, and if $\overline{v} = 1$ then $p = 1 + \frac{\beta}{2}$. Also, the function shifts upwards as β increases.

Also, since $(1-\delta)\Pi = (1-\overline{v})^2$, the pricing formula (equation (6)) becomes:

$$p = 1 - \overline{v} + \delta \left(1 - \overline{v}\right)^2 \tag{19}$$

Equation (19) summarizes the firm's behavior in a stationary equilibrium and indicates the equilibrium price as a function of the consumer threshold. Note that p is a decreasing and convex function of $\overline{v}$ with $p = 1 + \delta$ if $\overline{v} = 0$, and $p = 0$ if $\overline{v} = 1$. Moreover, function (19) shifts upwards as δ increases.

Equations (18) and (19) determine the equilibrium values of $(p, \overline{v})$ in a stationary equilibrium with time-limited rewards. See Figure 1 for the case $\delta = 0.9, \beta = 0.8$. As β increases equation (18) shifts upwards, and hence $\overline{v}$ falls and p increases. Similarly, as δ increases, equation (19) shifts upwards, and hence both $\overline{v}$ and p increases. Analytically, the equilibrium value of $\overline{v}$ is implicitly given by:

$$\overline{v} = \frac{1}{2}\left\{1 - \frac{\beta}{2}\left[1 - \left(1 - \overline{v}\right)^2\right] + \delta \left(1 - \overline{v}\right)^2\right\} \tag{20}$$

We can now check the condition under which the firm does not want to deviate and offer no reward:

Lemma 3 $\overline{v} \leq \frac{1}{2}$ *if and only if* $\beta \geq \frac{2}{3}\delta$*. Moreover, for all* (β, δ)*, there is a lower bound on* $\overline{v}$ *such that* $\left(1 - \overline{v}\right)^2 < \frac{1}{3}$*.*

The proof is in the Appendix. Thus, a necessary condition for an equilibrium with time-limited rewards is that $\beta \geq \frac{2}{3}\delta$. Otherwise, the firm finds it optimal to deviate and cancel the loyalty program. The intuition is analogous to the one provided in relation to Proposition 1. In particular, if $\frac{\beta}{\delta} = \frac{2}{3}$, then the increase in the consumers' willingness to pay associated to the time-limited reward, $\frac{3\beta}{8}$, is equal to the present value of future losses, $\frac{\delta}{4}$, and hence $\overline{v} = \frac{1}{2}$. As β increases then $\overline{v}$ falls and hence the firm does not have incentives to deviate and offer no reward.

In this model, the threshold value of the ratio of discount factors is $\frac{2}{3}$. Obviously, the exact value has to do with the specific functional forms. However, it is important to emphasize that the fact that this threshold is lower than one is generic. More specifically, consider the case that $\beta = \delta$, and suppose that the firm sets (off-the-equilibrium path) a regular price $p = \frac{1}{2} + \Delta$. In this case, relative to no rewards, consumers' expected utility, U, remains unchanged, as the ex-ante value of the reward exactly offsets the higher regular price. However, the present value of total surplus increases: First, $\overline{v} = \frac{1}{2}$ and hence the rents generated when consumers pay the regular price remain constant. Second, total surplus during the redemption period increases by $\frac{1}{2} - \frac{3}{8}$. Since these extra rents are equally discounted by consumers and the firm, then $U + \Pi$ increases, whereas U, remains constant. Therefore, Π, must necessarily increase. If we now let the firm set the optimal price then profits will be even higher. Moreover, since the optimal price is below $\frac{1}{2} + \Delta$ then consumers also benefit from the introduction of the loyalty program.[13]

Let us now consider a one-shot introduction of time-unlimited rewards. The next result shows that if $\beta \geq \frac{2}{3}\delta$ then the firm does not have incentives to introduce this type of rewards.

Lemma 4 *If $\beta \geq \frac{2}{3}\delta$ then $\widetilde{v}_t > \overline{v}$.*

The complete proof is in the Appendix, but here I offer the bulk of the argument, which hopefully provides some additional intuition. From equations (8) and (15), $2\left[\overline{v} - \widetilde{v}_t(\omega)\right] = \Gamma(\omega) = \Psi(\omega) - \Delta - \frac{\delta^2\omega(1-\delta)\Pi}{1-\delta\omega}$. That is, the firm has incentives to deviate, $\overline{v} > \widetilde{v}_t(\omega)$, if and only if $\Gamma\left(\widetilde{\Delta}\right) > 0$. Considering arbitrary values of ω in the interval $\left[0, \widetilde{\Delta}\right]$ will provide further insights on the main forces at play. Using equation (16) we can write:

$$\Gamma(\omega) = \frac{\beta\omega}{1-\beta\omega}\left(\Delta - \frac{\omega}{2}\right) - \frac{\delta\omega}{1-\delta\omega}\delta(1-\delta)\Pi$$

It turns out that $\Gamma(\omega)$ may increase or decrease with ω, but it is definitely concave. It will be convenient to start by studying $\Gamma(\omega)$ around $\omega = 0$. Clearly, $\Gamma(0) = 0$. However, $\Gamma(\omega)$ may be positive or negative for small values of ω. It turns out that $\Gamma'(0) \geq 0$ if and only if

$$\left(\frac{\beta}{\delta}\right)^2 \geq \frac{2(1-\overline{v})^2}{1-(1-\overline{v})^2} \tag{21}$$

Since $\frac{\beta}{\delta} \geq \frac{2}{3}$ then $(1-\overline{v})^2 \geq \frac{1}{4}$. In addition, from the previous lemma, $(1-\overline{v})^2 < \frac{1}{3}$. Consequently, the right hand side of (21) is higher than $\frac{2}{3}$ and lower than one. In one extreme, if β is close to $\frac{2}{3}\delta$ ($\overline{v}$ is close to $\frac{1}{2}$), the right hand side of (21) is close to $\frac{2}{3}$, whereas the left hand side is close to $\frac{4}{9}$. Hence, condition (21) fails and therefore $\Gamma'(0) < 0$. Therefore, since $\Gamma(\omega)$ is concave, in this case time-unlimited rewards are inefficient for all values of ω. In the other extreme, if β is sufficiently close to δ then $\Gamma'(0) > 0$ and the firm has incentives to deviate provided ω is sufficiently low. The reason why $\Gamma'(0) > 0$

[13] Under time-limited rewards, the firm faces a time-inconsistency problem. In particular, the firm would like to commit to a higher future regular price. Such expectation would reduce consumers' continuation value, U, and hence their current willingness to pay, Δ, would increase. Thus, starting at the equilibrium price level, a small increase in the future price generates a second order loss in future profits, but a first order gain in current profits.

requires $\frac{\beta}{\delta}$ to be much higher than $\frac{2}{3}$ is again that with $\omega > 0$ it will take more than one period in expectation to redeem the reward and hence the relevant discount factors will be of the order of β^2 and δ^2.

Suppose now that $\frac{\beta}{\delta}$ is sufficiently high so that condition (21) is satisfied. The more favorable scenario is one where $\beta = \delta$. In this case $\Gamma(\omega) = \frac{\delta\omega}{1-\delta\omega}\left(\Delta - \frac{\omega}{2} - \delta(1-\delta)\Pi\right).$ Hence, $\Gamma(\omega) > 0$ if and only if $0 < \omega < \varpi = \Delta - \delta(1-\delta)\Pi$. I show in the Appendix that $\varpi \leq \frac{\delta}{4} < \widetilde{\Delta}$, and hence $\Gamma\left(\widetilde{\Delta}\right) < 0$.

In other words, if $\frac{\beta}{\delta}$ is sufficiently high, and ω is small then time-unlimited rewards generate higher total expected surplus than time-limited rewards and hence the firm would be willing to adopt them. The parameter ω affects total surplus through two alternative channels. A higher value of ω, on the one hand, allows consumers to engage in intertemporal substitution: avoid redeeming the reward when their valuations are extremely low and instead wait for higher valuations to materialize. On the other hand, all future purchases are further delayed, which reduces the present value of total welfare. If ω is sufficiently small then the first effect dominates. However, the value of ω chosen by consumers is so high such that the second effect dominates, which implies that lifting time-constraints on the redemption of loyalty rewards reduces the present value of total welfare.

It may be useful to illustrate the magnitude of the excessive delay problem (the gap between $\widetilde{\Delta}$ and ϖ.) If we let $\beta = \delta$ go to 1, then $\widetilde{\Delta}$ and ϖ go to 0.4648 and 0.1408, respectively.[14] Figure 2 plots $\Gamma(\omega)$.[15]

By combining Lemmas 3 and 4 we obtain the following result:

Proposition 5 *A stationary equilibrium with time-limited rewards exist if and only if* $\beta \geq \frac{2}{3}\delta$.

4.3 Time-unlimited rewards

Finally, consider the existence of a stationary equilibrium in which the firm offers in every period time-unlimited rewards. From the analysis of Section 3 this implies that $U = \frac{(1-\widetilde{v})^2}{2(1-\beta)}, \Pi = \frac{(1-\widetilde{v})^2}{(1-\delta)}$. Hence, $\widetilde{\Delta} = \frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - \left[1-(1-\widetilde{v})^2\right]}$; that is, the analogous

[14] The optimal threshold (the one that maximizes total welfare), ω^*, goes to 0.0729.

[15] The changes in the continuation value of consumers and the firm when time limits are lifted for one period provide a similar message: consumers' continuation value when holding a reward increases by 0.1080 ($= \widetilde{U}^R - U^R$), whereas the firm's continuation value with consumers holding a reward decreases by 0.2488 ($= \Pi^R - \widetilde{\Pi}^R$).

to equation (18) has shifted upwards as the increase in the consumer's willingness to pay is now higher. Similarly, the analogous of equation (19) has also shifted upwards as future purchases at the regular price are further delayed. Hence, in a possible equilibrium with time-unlimited rewards the price would be higher, but consumer participation can in principle go either way. In particular, the stationary version of equation (15) becomes:

$$\widetilde{v} = \frac{1}{2}\left[1 - \widetilde{\Delta} + \frac{\delta\,(1-\delta)\,\Pi}{1-\delta\widetilde{\Delta}}\right].$$

Given the previous results, it is not surprising that if $\beta \geq \frac{2}{3}\delta$ the firm finds it profitable to deviate and offer time-limited rewards. Also if $\beta < \frac{2}{3}\delta$ the firm finds it profitable to deviate and offer no rewards. Therefore:

Proposition 6 *There is no stationary equilibrium in which the firm offers time-unlimited rewards.*

The proof is in the Appendix.

4.4 Summary

If $\beta < \frac{2}{3}\delta$ there is a unique stationary equilibrium in which the firm offers no loyalty rewards. Instead, if $\beta > \frac{2}{3}\delta$ the unique stationary equilibrium is one in which the firm offers time-limited loyalty rewards. Consequently, the present model provides a theory of why firms place tight restrictions on the timing of redemption of rewards: to discipline consumers, who otherwise would delay the redemption of rewards, and hence future transactions, excessively. Also, the current model offers a necessary condition for the optimality of loyalty rewards: consumers must be sufficiently forward-looking. The latter insight will be qualified in the next section.

It is important to emphasize that the interests of consumers and the firm are perfectly aligned. In particular, in the case $\beta < \frac{2}{3}\delta$, if the firm is forced to introduce time-limited rewards then it will incur into losses. Moreover, consumers will also be unhappy: the extra value provided by the rewards will be more than compensated by the increase in the regular price. Similarly, in the case $\beta > \frac{2}{3}\delta$ if the firm is forced to introduce time-unlimited rewards, both consumers and the firm will be unhappy. In the case of consumers, the extra value of rewards with no constraints will be again more than compensated by the higher regular price. Consequently, there is no room for a type of public intervention exclusively concerned with the time dimension of loyalty rewards.

5 Alternative designs

5.1 The scope of rewards

In the previous analysis I focused on a very restrictive set of designs of loyalty programs: the reward is acquired by purchasing one unit of the good at the regular price, and it can be used in the future to purchase another unit at a discount price (which was set equal to marginal cost). It is, however, natural to think of more general designs. In particular, the firm could require n purchases at the regular price to earn the reward, which can be defined as the right to purchase the good m times at a discount price, q. In the popular buy-ten-and-get-one-free scheme $n = 10, m = 1, q = 0$. Instead, in the benchmark model I assumed that $n = m = 1$ and q equal to marginal cost. Now we need to scrutinize this assumption and offer some justification.

It is easy to show that in the stylized model laid out in Section 2 the optimal design consists on $n = 1, m = \infty$ and q equal to marginal cost. Thus, the optimal pricing policy is like a two-part tariff, in which future rents generated by the reward program are collected through an extremely high initial price. Clearly, the use of such scheme in the real world would be seriously challenged by at least three factors. First, consumers are likely to be liquidity-constrained and hence unable to pay up-front the present value of all future rents. Second, uncertainty (and asymmetric information) about future costs would be a serious obstacle to implement such a scheme. Third, and probably the most important argument, a firm that collected all future rents in the first period would be exposed to a colossal moral hazard problem: the incentives to maintain the quality of the good, or to introduce improvements would be drastically challenged. The firm would even be tempted to run away with the proceeds of the initial purchase and fail to deliver the good in the future.[16]

Thus, the assumption of $n = m = 1$ can be understood as a compromising modeling strategy. Loyalty rewards can still generate additional rents without triggering unmanageable incentives to default.

[16] As mentioned in the introduction, even after committing to relatively modest rewards, some firms have changed point requirements, or the exchange rate for converting points into cash, in an attempt to reduce their liabilities (Chun et al., 2017).

5.2 Renegotiation

In the benchmark model I assumed that loyalty rewards cannot be renegotiated. Such an assumption matters: the consumer and the firm do have incentives to renegotiate. In particular, consider the case of time-limited rewards. If a consumer redeems her reward in period t then their payoff is equal to $v_t + \beta U$. If she is offered to give away the reward and instead buy a new bundle (one unit of the good plus the option of purchasing one unit in period $t+1$ at zero price) at price p', then she obtains $v_t - p' + \beta U^R$. Hence, she will accept the deal if and only if $p' \leq \Delta$. It turns out that such willingness to pay for the new deal is profitable for the firm. If the consumer redeems the existing loyalty reward in period t then the firm obtains Π^R. Instead, after renegotiation the firm gets $p' + \delta \Pi^R$. Hence, by charging $p' = \Delta$, the firm obtains an extra profit: $\Delta - (1-\delta)\Pi^R = \Delta - \delta(1-\delta)\Pi > 0$. Allowing for this type of renegotiation would imply that the initial purchase leads to a sequence of purchases at a discount price. Hence, the initial increase in consumers' willingness to pay would be huge and therefore the incentives to default enormously magnified. Consequently, the firm needs to commit not to renegotiate the rewards in order to secure the credibility of the loyalty program.

5.3 More general discount prices

Even if we settle for the $n = m = 1$ design, we might still want to consider the possibility of setting a discount price, q, different from marginal cost. I now reexamine time-limited rewards, and let the firm choose $q, q \in \left[0, \frac{1}{2}\right)$. In this case, the continuation value of a consumer holding the reward is: $U^R = \frac{1}{2}(1-q)^2 + \beta U$. It is worth emphasizing that a fraction q of all rewards will remain unredeemed, as it happens in the real world. Since it is still the case that $\overline{v} = p - \beta\left(U^R - U\right)$ and $U = \frac{(1-\overline{v})^2}{2(1-\beta)}$, then $\Delta = \frac{\beta}{2}\left[(1-q)^2 - (1-\overline{v})^2\right]$. Note that Δ decreases with q. Plugging the value of Δ in equation (3), we obtain the equilibrium condition that generalizes equation (18):

$$p = \overline{v} + \frac{\beta}{2}\left[(1-q)^2 - (1-\overline{v})^2\right] \tag{22}$$

As q increases then equation (22) shifts downwards.

Similarly, given a fixed value of q, the firm's continuation value with a consumer holding a reward is $\Pi^R = q(1-q) + \delta\Pi$. Given that it is still the case that $\Pi = \frac{(1-\overline{v})^2}{(1-\delta)}$, then the pricing condition (which generalizes equation (19)) is

$$p = 1 - \overline{v} + \delta(1-\overline{v})^2 - \delta q(1-q) \tag{23}$$

As q increases then equation (23) also shifts downwards. Hence, the equilibrium value of $(p, \overline{v})$, for a given q, is determined by equations (22) and (23). Since Π is a decreasing function of $\overline{v}$, the optimal value of q is the one that minimizes the equilibrium value of $\overline{v}$:[17]

$$q^* = \frac{\delta - \beta}{2\delta + \beta}$$

The optimal value of q decreases with β. In addition, $q^* = \frac{1}{2}$ if $\beta = 0$, and $q^* = 0$ if $\beta = \delta$. Note that even in the case where β is arbitrarily low, the firm still chooses a (small but) positive reward. The intuition is as follows. If q is close to $\frac{1}{2}$ then the loss in short-run profits is of the second order (the price is around the optimal value), whereas the gain in consumer surplus is of the first order. Therefore, even if β is very low, the firm would still find it optimal to offer such a small reward. However, as q falls, the loss of profits increases at a faster rate than the increase in consumer surplus, which implies that the firm is willing to choose such an option only if β is higher. Similarly, if $\delta - \beta$ is positive but arbitrarily low, the firm will choose a positive value of q, but close to 0. The reason for this is that the extra consumer surplus generated by further decreasing q is similar to the additional loss of profits (the increase in total surplus is of second order); however, because $\beta < \delta$, the increase in the consumer's willingness to pay is not sufficient to cover the extra loss in profits.

Different industries are likely to be populated by consumers with different discount factors. According to the previous discussion, if consumers are very impatient then we should expect loyalty programs offering small rewards. Instead, if consumers are more patient then we should expect more generous loyalty rewards. Hence, the large observed variation in the size of rewards across industries may be explained by the different degree of impatience of consumers.

5.4 More general time restrictions

The other dimension of the loyalty program that deserves to be reconsidered is the time horizon to redeem the reward. In the benchmark model I focused on a simple alternative: either a one-period, or an infinite horizon. More generally, we could let the firm choose the horizon, T, and let T be any positive integer. Certainly, $T = 1$ might not be the

[17] By plugging equation (23) into equation (22) then we obtain an equation that can be expressed as $\Gamma(\overline{v}, q) = 0$. Then, using the implicit function, theorem (and the fact that $\frac{\partial \Gamma}{\partial \overline{v}} \neq 0$) then $\frac{d\overline{v}}{dq} = 0$ only if $\frac{\partial \Gamma}{\partial q} = 0$. The next equation in the main text is the solution to the latter condition.

optimal horizon. However, because $T = \infty$ is dominated by $T = 1$, the optimal value of T is finite. Hence, we should still expect the monopolist to set time limits on the redemption of rewards.

6 The degree of market power

The monopolist enjoyed a high degree of market power in the benchmark model (linear demand, constant marginal costs). In particular, in the case of no rewards, the ratio of profits over consumer surplus is equal to two. In that context, time-unlimited rewards induce the consumer to delay its redemption, so much that the total surplus is actually reduced. Thus, one may be tempted to attribute this result to the strong ability of the monopolist to extract rents. Perhaps, if the firms' profits are lower, then the negative externality diminishes and time unlimited rewards may then become efficient. In this section I argue that such an intuition is incorrect because it misses an alternative countervailing effect. More specifically, a lower mark-up (and hence a lower ratio of profits to total rents), on the one hand, reduces the losses infringed on the firm because of the delay in future purchases (like suggested above); and, on the other hand, it also reduces the gains in consumer surplus generated by lifting time restrictions on loyalty rewards. The reason why the gains in consumer surplus are also reduced is that the additional consumption episodes prompted by the redemption of rewards occur less frequently; they only appear in the case of sufficiently low consumer valuations. Consequently, time-limited loyalty rewards are likely to remain optimal for any degree of the firm's market power.

Thinking for the moment outside the limits of this model, it is evident that mark ups are determined by two main factors. The first is the degree of competition in the market. As discussed in Section 8, the presence of competing firms introduces additional effects into the analysis and their examination is left for future research. The second reason is the elasticity of the demand function. To illustrate the impact of the demand elasticity on the optimality of time limits, I will now consider a more general distribution of consumer valuations. More specifically, the density function of consumer valuations is now given by:

$$f(v) = \begin{cases} \frac{1-\alpha}{\alpha}, & \text{if } v \in [0, \alpha] \\ \frac{\alpha}{1-\alpha}, & \text{if } v \in (\alpha, 1] \end{cases} \tag{24}$$

where α is an exogenous parameter. Note that in the benchmark model $\alpha = \frac{1}{2}$. Here I will focus on values of α in the interval $\left(0, \frac{1}{2}\right]$. Thus, the average valuation is α, and if we let A be equal to $\frac{1-\alpha}{\alpha}$, then the static demand function is given by:

$$q(p) = \begin{cases} \frac{1}{A}(1-p), & \text{if } p \geq \alpha \\ 1 - Ap, & \text{otherwise} \end{cases}$$

Note that the demand function is piecewise linear and symmetric around $(p, q) = (\alpha, \alpha)$. Moreover, the static monopoly price is: $p = \frac{1}{2A}$; and hence it increases with α and p goes to marginal cost as α goes to zero.[18]

In other words, as α falls below $\frac{1}{2}$ the mark up is reduced and so is the ratio of profits over total surplus.

In the Appendix, I report the equilibrium conditions for the new functional form. I first characterize the stationary equilibrium with time-limited rewards, and then I consider a one-shot deviation in which the firm offers time-unlimited rewards in period t. I show that the sign of $\widetilde{v}_t - \overline{v}$ is equal to the sign of

$$-\left(\widetilde{\Delta} - \Delta\right) + \frac{\delta^2 A \widetilde{\Delta} (1-\delta) \Pi}{1 - \delta A \widetilde{\Delta}} \tag{25}$$

where $(1-\delta)\Pi = \frac{(1-A\overline{v})^2}{A}$. The first term, $\widetilde{\Delta} - \Delta$, is the increase in the consumer's willingness to pay caused the elimination of the time limits. The second term, $\frac{\delta^2 A\widetilde{\Delta}(1-\delta)\Pi}{1-\delta A\widetilde{\Delta}}$ is the reduction of future profits caused by the delay of future purchases. It turns out that $A\overline{v}$ and $A\widetilde{\Delta}$ are all independent of A. In other words, as α falls, and hence A increases, both $\overline{v}$ and $\widetilde{\Delta}$ fall. However, the fraction of consumers with a time-unlimited reward that do not redeem it in the current period, $A\widetilde{\Delta}$, as well as the fraction of consumers without a reward that do not make a regular purchase, $A\overline{v}$, both stay constant. Similarly, $A\Delta$ is also independent of A. We can rewrite expression (25) as

$$\frac{1}{A}\left\{-\left(A\widetilde{\Delta} - A\Delta\right) + \frac{\delta^2 A\widetilde{\Delta}(1 - A\overline{v})^2}{1 - \delta A\widetilde{\Delta}}\right\}$$

Consequently, the magnitude of $(\widetilde{v}_t - \overline{v})$ decreases with A, but its sign does not change. Therefore, time-limited rewards dominate time-unlimited results for all $\alpha < \frac{1}{2}$.

7 The value of flexible timing

Time flexibility had no intrinsic value in the benchmark model. In that setup, consumption valuations were independent over time, and hence the optimal consumption rule (first

[18] Given the exact symmetry of the demand function there is another static monopoly price, $p = \frac{1}{2}$, that can be eliminated by slightly perturbing the demand function.

best) was static: buy the good if and only if the valuation exceeds the marginal cost (no intertemporal substitution). Loyalty rewards did induce consumers to engage in intertemporal substitution, but only because of the fluctuations of transaction prices. In any case, loyalty rewards improved efficiency, at least under some conditions. In particular, under time-limited rewards, consumers followed the optimal rule during redemption periods. Instead, the consumer holding time-unlimited rewards used an intermediate threshold (in between marginal cost and the threshold used for regular purchases), which implied that all future transactions were inefficiently delayed. Thus, providing time flexibility (lifting time constraints) was not enough to increase total welfare.

In this section I consider an alternative specification of the model that makes time flexibility intrinsically valuable, whereas it also reduces the importance of future purchases. In particular, I introduce satiation in consumer preferences: after a consumption episode there is a probability $1-\mu$ that the consumer becomes satiated and leaves the market for ever. In this context, the consumer must choose the timing of their purchases more carefully, even under constant prices. Indeed, in the first best the consumer engages in intertemporal substitution.

For simplicity I focus on the case $\beta = \delta$.[19] Obviously, if $\gamma = 1$ then we are back to the benchmark model. Thus, it is not surprising that the interesting analytical results are obtained in the other extreme, when γ is low.

7.1 The optimal consumption plan

Let W be the expected total at the beginning of a period generated by a consumer who stays in the market. Consumer i should purchase the good in period t if and only if $v_{it} + \gamma\delta W \geq \delta W$; that is if and only if $v_{it} \geq \overline{v}^*$. The optimal threshold value is given by:

$$v^* = (1-\gamma)\,\delta W^* \tag{26}$$

where W^* is the optimal level of total surplus, which satisfies: $W^* = \int_0^{v^*} \delta W^* dv + \int_{v^*}^1 (v + \gamma\delta W^*)\, dv$. Using equation (26):

$$W^* = \frac{(1-v^*)^2}{2(1-\delta)} \tag{27}$$

Thus, in the optimal plan the consumer engages in intertemporal substitution. Facing the risk of satiation, the consumer prefers to abstain when valuations are low ($v_{it} < v^*$). The optimal plan is thus given by equations (26) and (27).

[19] As argued earlier, if $\frac{\beta}{\delta} < 1$ then time limits become more attractive.

7.2 Time-unlimited rewards

I now characterize a stationary equilibrium with time-unlimited rewards. Using the notation introduced in the previous sections, a consumer with a reward will redeem it if and only if $v_{it} + \gamma\delta U \geq \delta \widetilde{U}^R$. That is, the threshold value is:

$$\widetilde{v}^R = \delta \widetilde{U}^R - \gamma \delta U. \tag{28}$$

In period t a consumer who holds no reward purchases the bundle if and only if $v_{it} - p_t + \gamma\delta\widetilde{U}^R \geq \delta U$. Therefore the threshold is given by:

$$\widetilde{v}_t = p_t + \delta \widetilde{U} - \gamma\delta\widetilde{U}^R. \tag{29}$$

As usual, these continuation values can be written as functions of the thresholds:

$$\widetilde{U} = \frac{(1-\widetilde{v})^2}{2(1-\delta)} \tag{30}$$

$$\widetilde{U}^R = \frac{\left(1-\widetilde{v}^R\right)^2}{2(1-\delta)} \tag{31}$$

The firm's continuation value with a consumer holding a time-unlimited reward must satisfy: $\widetilde{\Pi}^R = \widetilde{v}^R\delta\widetilde{\Pi}^R + \left(1-\widetilde{v}^R\right)\gamma\delta\Pi$. Hence, solving for $\widetilde{\Pi}^R$:

$$\widetilde{\Pi}^R = \frac{\left(1-\widetilde{v}^R\right)\gamma\delta\Pi}{1-\widetilde{v}^R\delta} \tag{32}$$

Hence, as γ goes to zero, $\widetilde{\Pi}^R$ also goes to zero. In words, as the probability of a future purchase at the regular price vanishes the firm's continuation value with a consumer holding a time-unlimited reward goes to zero.

In period t the firm chooses the current price, p_t, in order to maximize $\widetilde{v}_t\delta\Pi + (1-\widetilde{v}_t)\left(p_t + \gamma\delta\widetilde{\Pi}^R\right)$, where $\widetilde{v}_t$ is given by equation (29) (and hence $\frac{d\widetilde{v}_t}{dp_t} = 1$). From the first order condition:

$$p_t = 1 - \widetilde{v}_t + \delta\Pi - \gamma\delta\widetilde{\Pi}^R \tag{33}$$

Plugging this condition into the objective function, and evaluating it at a stationary equilibrium:

$$\Pi = \frac{(1-\widetilde{v})^2}{(1-\delta)} \tag{34}$$

The equilibrium values of $(p_t, \widetilde{v}_t)$ are given by equations (29) and (33). Thus, if we let W and $\widetilde{W}^R$ be the present value of total surplus generated by a consumer with no reward, and a time-limited reward respectively ($W = U + \Pi, \widetilde{W}^R = \widetilde{U}^R + \widetilde{\Pi}^R$), then in a stationary equilibrium:

$$\widetilde{v} = \frac{1}{2}\left(1 + \delta W - \gamma\delta\widetilde{W}^R\right) \tag{35}$$

It is important to note that, since $\widetilde{\Pi}^R$ goes to zero as γ goes to zero, then $\widetilde{U}^R$ goes to $\widetilde{W}^R$. In addition, from equations (28) and (35), $\left(\widetilde{v}^R, \widetilde{W}^R\right)$ goes to $(\overline{v}^*, W^*)$. In words, if γ small, and conditional on having survived the regular purchase, a consumer with a time-unlimited reward behaves approximately as in the optimal plan, and hence achieves a continuation value arbitrarily close to the maximum potential level. On the other hand, $\widetilde{v}$ converges to a value higher than $\overline{v}^*$, due to the price distortion, and hence W converges to a value lower than W^*. Thus, $\widetilde{W}^R - W$ converge to a positive number as γ goes to zero.

7.3 One-shot deviation: no reward

Suppose now that the firm deviates in period t and offers no reward. All players expect that in future periods the firm will again offer time-unlimited rewards. A consumer with no reward will purchase the good in period t if and only if $v_{it} \geq \overline{v}_t^n = p_t + (1-\gamma)U$. Then, the firm chooses p_t in order to maximize $\Pi_t = \overline{v}_t^n \delta\Pi + (1 - \overline{v}_t^n)(p_t + \gamma\delta\Pi)$. Hence, from the first order condition we have that $p_t = 1 - \overline{v}_t^n + (1-\gamma)\delta\Pi$. As usual, $\Pi_t = \frac{(1-\overline{v}_t^n)^2}{1-\delta}$. By plugging the definition of $\overline{v}_t^n$ into the pricing equation:

$$\overline{v}_t^n = [1 + \delta(1-\gamma)W] \tag{36}$$

Hence, from equations (35) and (36):

$$2(\overline{v}_t^n - \widetilde{v}) = \gamma\delta\left(\widetilde{W}^R - W\right).$$

As mentioned above, if γ is sufficiently small $\widetilde{W}^R > W$, and hence $\overline{v}_t^n > \widetilde{v}$. For those consumers who survive the purchase at the regular price, time-unlimited rewards almost generate the first best level of utility, certainly higher than in the absence of rewards. The prospect of enjoying those rewards stimulate consumer participation and increase efficiency.

7.4 One-shot deviation: time-limited rewards

Suppose now that the firm deviates in period t and offers a time-limited reward. All players still expect that in future periods the firm will again offer time-unlimited rewards.

Consumers with a time-limited reward in period $t+1$ will be expected to purchase the good if and only if $v \geq \overline{v}^R = \delta(1-\gamma)U$. In period t a consumer without a reward will purchase the bundle if and only if $v \geq \overline{v}_t$, where

$$\overline{v}_t = p_t + \delta U - \gamma \delta U^R \tag{37}$$

Such a continuation value satisfies: $U^R = \left[\overline{v}^R + \gamma\left(1-\overline{v}^R\right)\right]\delta U + \frac{1}{2}\left(1-\left(\overline{v}^R\right)^2\right)$.

The firm's continuation value with a consumer that stays in the market at time $t+1$ and holds a time-unlimited reward, Π^R, must satisfy that $\Pi^R = \overline{v}^R\delta\Pi + \left(1-\overline{v}^R\right)\gamma\delta\Pi$. In period t the firm sets p_t in order to maximize $\Pi_t = \overline{v}_t\delta\Pi + (1-\overline{v}_t)\left(p_t + \gamma\delta\Pi^R\right)$. Hence, from the first order condition:

$$p_t = 1 - \overline{v}_t + \delta\Pi - \gamma\delta\Pi^R. \tag{38}$$

Once again, plugging the first order condition on the objective function: $\Pi_t = \frac{(1-\overline{v}_t)^2}{(1-\delta)}$.

By combining equations (37) and (38):

$$\overline{v}_t = \frac{1}{2}\left(1 + \delta W - \gamma\delta W^R\right). \tag{39}$$

Note that as γ goes to zero $\overline{v}^R$ goes to $\delta U < \delta W < \delta W^*$. Hence, $\overline{v}^R < \overline{v}^*$. That is, consumers holding a time-limited reward tend to consume excessively, because of the expectation of higher future prices. Moreover, these rewards only last for one period, and after expiration they will insufficiently consume because of the positive prices. Therefore, $W^R < W^*$.

From equations (35) and (39):

$$2\left(\overline{v}_t - \widetilde{v}\right) = \gamma\delta\left(\widetilde{W}^R - W^R\right)$$

Hence, as γ goes to zero $\widetilde{W}^R - W^R$ converges to a positive number, and hence a deviation to a time-limited reward is not profitable.

Summarizing, when γ is sufficiently small, delays of future purchases cease to be relevant, whereas the demand for time flexibility is magnified. In particular, the redemption of time-limited rewards induce excessive consumption during the redemption period. In contrast, time-unlimited rewards offer sufficient time flexibility to almost replicate the first best consumption plan. Hence, by continuity there is a positive threshold value of $\gamma, \overline{\gamma} > 0$, such that if $0 < \gamma < \overline{\gamma}$ lifting time limits on the redemption of rewards raises total surplus.

8 Concluding remarks

8.1 Competition

It is well-known that competition introduces new channels through which loyalty rewards affect consumer behavior. In particular, they generate endogenous switching costs (Caminal and Matutes, 1990) and hence distort the efficient allocation of consumers across firms. The extent to which these new channels may alter the main insights of the paper is, of course, an open issue. In any case, lifting time limits can have additional costs and benefits for a competing firm. On the one hand, offering a more valuable (time-unlimited) reward may enhance the firms' business stealing effect. On the other hand, if the consumer accumulates rewards with different time restrictions, introduced by different firms, then they will be induced to use the time-limited reward immediately (i.e., before it expires) and postpone the redemption of the time-unlimited reward, which may actually become a less effective switching cost. Whether or not the enhanced business stealing effect is strong enough to revert the main results of this paper remains to be seen. In any case, the introduction of competition indeed alters the optimistic view of loyalty programs portrayed by the current model, by recognizing the allocative inefficiency caused by the endogenous switching costs that they generate.[20]

8.2 Heterogeneous consumers

In the benchmark model, I assumed that consumers were ex-ante identical, and hence they equally valued loyalty rewards. A natural extension is to introduce heterogeneous consumers. For instance, a fraction of consumers might be infinitely-lived, as in the benchmark model, and the rest might live for a single period (occasional consumers). Obviously, only the former would value loyalty rewards. If the firm cannot keep track of the individual purchase histories, then the regular price would be a linear combination of the price computed above (targeting long-lived consumers) and the optimal static price (targeting occasional consumers). Clearly, the firm's ability to appropriate some of the rents generated by loyalty rewards will be undermined. As a result, the interests of long-lived consumers and the firm concerning the introduction of loyalty rewards will no longer be aligned. In particular, long-lived consumers may prefer time-limited rewards when the firm prefers not to introduce them, because the increase in the regular price

[20] In addition, as pointed out by Fong and Liu (2011), loyalty rewards may facilitate collusion. However it remains to be seen whether or not time-constraints reinforce the collusive effect.

generates losses with occasional consumers. However, occasional consumers always prefer no rewards to avoid higher regular prices. Similar considerations can be made regarding the time dimension of loyalty rewards. Long-lived consumers may prefer time-unlimited rewards even though the firm chooses to maintain time limits (once again, the price increase associated to lifting time limits would be moderated by the presence of occasional consumers). However, occasional consumers will always prefer time limits because they imply lower regular prices. Consequently, the effect of loyalty programs on total consumer surplus is likely to depend on very specific details of the model, and hence a clear case in favor of public intervention will still be difficult to make.

8.3 Personalized pricing

This paper provides a favorable view of loyalty programs. It argues that the commitment to discount prices on repeat purchases may help to reduce the inefficiency generated by market power. Moreover, it shows that consumers also benefit from loyalty programs, which is compatible with the evidence provided in Behrens and McCaughey (2015). However, it is frequently noted that loyalty programs allow firms to track down the purchase histories of individual consumers, which may facilitate personalized pricing, which in turn may result in lower consumer surplus. This may well be the case, and hence it might be an important element in an overall assessment of loyalty programs.[21] However, such an information gathering role does not seem to interfere much with the main predictions of the present model. Naturally, it would expand the region of parameter values for which the firm finds loyalty programs to be optimal, but it would not affect the choice on the time restrictions on the redemption of rewards. Moreover, if thanks to the loyalty program the firm can classify consumers in different categories, then this would contribute to validate the assumption of ex-ante identical consumers used in the analysis.

9 Appendix

9.1 Proof of Proposition 2

The only result that remains to be proved is that $\Omega\left(\widetilde{\Delta}\right) < 0$. First, $\Omega(0) = \frac{3}{8}\beta - \frac{1}{4}\delta \leq 0$. Second, $\Omega'(\omega) = \frac{\beta\left(\Delta - 2\omega + \beta\omega^2\right)}{(1-\beta\omega)^2} - \frac{\frac{\delta^2}{4}}{(1-\delta\omega)^2} \leq \frac{1}{(1-\delta\omega)^2}\left[\beta\left(\Delta - 2\omega + \beta\omega^2\right) - \frac{\delta^2}{4}\right] < 0$. The first

[21] In some industries firms have already access to very rich information on the history of purchases of individual consumers, even if they are not members of any loyalty program. In contrast, in some other industries, like grocery retailing, loyalty programs may be a more useful source of information.

inequality holds because $(1-\beta\omega)^2 \geq (1-\delta\omega)^2$. In order to check the last inequality, note that the first term in the square bracket is lower than $\beta\Delta$, which in turn is lower than $\frac{\delta^2}{4}$. Therefore, $\Omega\left(\widetilde{\Delta}\right) < 0$ and $\widetilde{v}_t > \frac{1}{2}$. Summarizing if $\frac{\beta}{\delta} \leq \frac{2}{3}$ a deviation offering time-unlimited rewards is not profitable.

9.2 Proof of Lemma 3

Let $\chi(\overline{v})$ be the right hand side of equation (20): $\chi(\overline{v}) = \frac{1}{2}\left\{1 - \frac{\beta}{2}\left[1-(1-\overline{v})^2\right] + \delta(1-\overline{v})^2\right\}$. χ is a continuos and decreasing function of $\overline{v}$. Also, $\chi\left(1-\frac{1}{\sqrt{3}}\right) = \frac{1}{2} + \frac{\delta-\beta}{6} > 1 - \frac{1}{\sqrt{3}}$, $\chi(1) = \frac{1}{2} - \frac{\beta}{4} \geq \frac{1}{4}$. Hence, there is a unique value of $\overline{v} > 1 - \frac{1}{\sqrt{3}}$ that satisfies equation (20). This shows the second part of the Lemma.

As for the first part, note that $\chi(\overline{v})$ shifts downwards with β and upwards with δ. Moreover, if $\beta = \frac{2}{3}\delta$ then $\chi(\overline{v}) = \frac{1}{2} - \frac{\delta}{6}\left[1 - 4(1-\overline{v})^2\right]$. Therefore, $\overline{v} = \frac{1}{2}$. Consequently, $\overline{v} \leq \frac{1}{2}$ if and only if $\beta \geq \frac{2}{3}\delta$.

9.3 Proof or Lemma 4

Since $\beta \leq \delta$, then $\Gamma(\omega) \leq \frac{\delta\omega}{1-\delta\omega}\left[\Delta - \frac{\omega}{2} - \delta(1-\delta)\Pi\right]$. Let $\overline{\omega}$ be the value of ω that equalizes the right hand side of this inequality to zero; i.e., $\overline{\omega} = 2\left[\Delta - \delta(1-\delta)\Pi\right]$. Hence, $\Gamma(\omega) < 0$ for all $\omega > \overline{\omega}$. Also, using the equilibrium values of U and Π we have that $\overline{\omega} = 2\left\{\frac{\beta}{2}\left[1-(1-\overline{v})^2\right] - \delta(1-\overline{v})^2\right\} \leq \delta\left[1 - 3(1-\overline{v})^2\right] \leq \frac{\delta}{4}$. The first inequality is due to the assumption that $\beta \leq \delta$, and the second holds because $\beta \geq \frac{2}{3}\delta$, and hence $(1-\overline{v})^2 \geq \frac{1}{4}$. Also, $\widetilde{\Delta}$ increases with β, and if $\beta = \frac{2}{3}\delta$, then $\widetilde{\Delta} = \frac{1-\sqrt{1-\frac{\delta^2}{3}}}{\frac{2}{3}\delta} > \frac{\delta}{4}$. We conclude that for any value of $\frac{\beta}{\delta} \in \left[\frac{2}{3}, 1\right]$, $\widetilde{\Delta} > \frac{\delta}{4} \geq \overline{\omega}$. Hence, $\Gamma\left(\widetilde{\Delta}\right) > 0$.

9.4 Proof of Proposition 6

Consider the case $\beta < \frac{2}{3}\delta$ and suppose that $\widetilde{v} \leq \frac{1}{2}$. We will take a detour and consider the case that at time $t = 0$ the firm introduces generalized time-unlimited rewards for an exogenous value of ω, and from their onwards time-unlimited rewards for $\omega = \widetilde{\Delta}$.Using the notation introduced in section 4.1, let $\Omega(\omega) = \frac{\Delta - \frac{\beta}{2}\omega^2}{1-\beta\omega} - \frac{\delta(1-\delta)\Pi}{1-\delta\omega}$, where $\Delta = \beta\left[\frac{1}{2} - (1-\beta)U\right]$, and $(1-\beta)U = \frac{(1-\widetilde{v})^2}{2}$, $(1-\delta)\Pi = (1-\widetilde{v})^2$. From equation (17) $\widetilde{v} \leq \frac{1}{2}$ if and only if $\Omega\left(\widetilde{\Delta}\right) \geq 0$. First, $\Omega(0) = \frac{\beta}{2} - \beta(1-\beta)U - \delta(1-\delta)\Pi \leq \frac{3\beta}{8} - \frac{\delta}{4} < 0$. Second, $\Omega'(\omega) = \frac{\beta\left(\Delta - 2\omega + \beta\omega^2\right)}{(1-\beta\omega)^2} - \frac{\delta^2(1-\delta)\Pi}{(1-\delta\omega)^2} \leq \frac{1}{(1-\delta\omega)^2}\left[\beta\left(\Delta - 2\omega + \beta\omega^2\right) - \delta^2(1-\delta)\Pi\right] \leq \frac{1}{(1-\delta\omega)^2}\left[\frac{\beta^2}{2} - \beta^2(1-\beta)U - \delta^2(1-\delta)\Pi\right] \leq \frac{1}{(1-\delta\omega)^2}\left(\frac{3\beta^2}{8} - \frac{\delta^2}{4}\right) < 0$. Therefore, $\Omega\left(\widetilde{\Delta}\right) < 0$

and $\widetilde{v} > \frac{1}{2}$. We have reached a contradiction. Hence, $\widetilde{v} > \frac{1}{2}$ and the firm has incentives to deviate and cancel the loyalty program.

Consider now the case $\beta \geq \frac{2}{3}\delta$ and suppose that $\widetilde{v} \leq \min\left\{\overline{v}_t, \frac{1}{2}\right\}$. Using the notation introduced in section 4.2, let $\Gamma(\omega) = \frac{\beta\omega}{1-\beta\omega}\left(\frac{\beta}{2} - \beta(1-\beta)U - \frac{\omega}{2}\right) - \frac{\delta\omega}{1-\delta\omega}\delta(1-\delta)\Pi$, where $(1-\beta)U = \frac{(1-\widetilde{v})^2}{2}$, and $(1-\delta)\Pi(1-\widetilde{v})^2$. $\Gamma(\omega)$ is a concave function. Since $\beta \leq \delta$, $\Gamma(\omega) \leq \frac{\delta\omega}{1-\delta\omega}\left(\Delta - \frac{\omega}{2} - \delta(1-\delta)\Pi\right)$. Hence, if $\omega \geq \overline{\omega} = 2\Delta - 2\delta(1-\delta)\Pi$ then $\Gamma(\omega) \leq 0$. Note that $\widetilde{\Delta} > \Delta > \overline{\omega}$. To check the last inequality, note that $\Delta = \frac{\beta}{2} - \frac{\beta}{2}(1-\widetilde{v})^2 < 2\delta(1-\widetilde{v})^2$ since $(1-\widetilde{v})^2 \geq \frac{1}{4}$. We have reached a contradiction.

9.5 The degree of market power

Under the alternative distribution of consumer valuations (24), in the absence of rewards the price is equal to $\frac{1}{2A}$ (and hence total demand per period is equal to $\frac{1}{2}$), profits per period are equal to $\frac{1}{4A}$, and consumer surplus per period is equal to $\frac{(5-8\alpha)}{8A}$.

In the case of time-limited rewards, the equilibrium conditions that replace (18) and (19) are:

$$p = (1+\beta)\overline{v} - \frac{A\beta}{2}\overline{v}^2 \tag{40}$$

$$p = \frac{1}{A}\left[1 - A\overline{v} + \delta(1-A\overline{v})^2\right] \tag{41}$$

and profits and consumer surplus per period are:

$$(1-\delta)\Pi = \frac{(1-A\overline{v})^2}{A} \tag{42}$$

$$(1-\beta)U = \alpha - \overline{v} + \frac{A}{2}\overline{v}^2 \tag{43}$$

Thus, equation (40) can also be written as $p = \overline{v} + \Delta$, where $\Delta = \frac{\beta}{2}\overline{v}(2 - A\overline{v})$. Thus, the equilibrium value of $\overline{v}$ must satisfy:

$$2A\overline{v} = 1 - A\Delta + \delta(1-A\overline{v})^2$$

Since $A\Delta$ is independent of A, then $A\overline{v}$ is also independent of A. In other words, as α falls (and hence A increases) $\overline{v}$ falls, but the fraction of consumers that do not purchase the bundle, $A\overline{v}$, stay constant.

As in the benchmark model, both consumers and the firm prefer time-limited rewards over no rewards, provided $\overline{v} \leq \frac{1}{2A}$, which holds if and only if $\beta \geq \frac{2}{3}\delta$.[22] The intuition

[22] First, if $\beta = \frac{2}{3}\delta$, then $\overline{v} = \frac{1}{2A}$ is the unique solution to equations (40) and (41). Second, if β increases equation (40) shifts upwards and hence $\overline{v}$ decreases.

is still the same. The increase in consumer surplus in the next period associated to the redemption of the reward is $\alpha - \frac{(5-8\alpha)}{8A} = \frac{3}{8A}$. Hence, the increase in consumers' willingness to pay associated to the value of the redemption is equal to $\frac{3\beta}{8A}$. Also, the present value of the loss of profits is $\frac{\delta}{4A}$. Therefore the former is higher than the latter if and only if $\beta \geq \frac{2}{3}\delta$. Thus, for all α, the interests of consumers and the firms are aligned as far as the decision between no rewards and time-limited rewards is concerned.

Next, in the case $\beta \geq \frac{2}{3}\delta$, I consider a one-shot deviation that consists of offering time-unlimited rewards in period t, under the expectation that the firm will offer again time-limited rewards in all future periods. The equilibrium conditions that determine $(p_t, \widetilde{v}_t)$, and replace equations (10) and (13), are:

$$p_t = \widetilde{v}_t + \widetilde{\Delta} \tag{44}$$

$$p_t = \frac{1}{A}\left[1 - A\widetilde{v}_t + \frac{\delta A (1-\delta)\Pi}{1-\delta A\widetilde{\Delta}}\right] \tag{45}$$

where $(1-\delta)\Pi$ is given by equation (42). Also, $\widetilde{\Delta}$ solves the following equation:

$$\widetilde{\Delta} = \frac{\beta}{1 - A\beta\widetilde{\Delta}}\left[\alpha - \frac{A}{2}\widetilde{\Delta}^2 - (1-\beta)U\right] \tag{46}$$

where $(1-\beta)U$ is given by equation (43).

Profits and consumer surplus after the deviation are given by:

$$\Pi_t = \frac{1}{A}(1 - A\widetilde{v}_t)^2 + \delta\Pi \tag{47}$$

$$U_t = \alpha - \widetilde{v}_t + \frac{A}{2}\widetilde{v}_t^2 + \beta U \tag{48}$$

Hence, once again, the interest of consumers and the firm are perfectly aligned. They prefer time-unlimited over time-limited results if and only if $\widetilde{v}_t \leq \overline{v}$. Thus, the equilibrium value of $\widetilde{v}_t$ must satisfy:

$$2A\widetilde{v}_t = 1 - A\widetilde{\Delta} + \frac{\delta(1 - A\overline{v})^2}{1 - \delta A\widetilde{\Delta}}$$

Since $A\widetilde{\Delta}$ is independent of A, then $A\widetilde{v}_t$ is also independent of A. Consequently, $A\widetilde{v}_t - A\overline{v}$ is also independent of A. Hence, since $\widetilde{v}_t > \overline{v}$ for $\alpha = \frac{1}{2}$ (Lemma 4), then this inequality also holds for any $\alpha < \frac{1}{2}$.

10 References

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