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BSE Working Paper 1329| March 2022

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On the optimal design of a Financial Stability Fund*

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March 12, 2022

Abstract

We develop a model of a Financial Stability Fund (Fund) for a union of sovereign countries. By contract design, the Fund never has expected undesired losses while, being default-free, a participant country has greater ability to borrow and share risks than using sovereign debt financing. The Fund contract also provides better incentives for the country to reduce endogenous risks. These efficiency gains arise from the ability of the Fund to offer long-term contingent financial contracts, subject to limited enforcement (LE) and moral hazard (MH) constraints as part of the contingencies. We develop the theory (welfare theorems, with a new price decentralization) and quantitatively compare the constrained-efficient Fund economy with an incomplete markets economy with default. In particular, we characterize how prices and allocations differ, when the two economies are subject to exogenous productivity and endogenous government expenditure shocks. In our economies, calibrated to the euro area 'stressed countries', substantial welfare gains are achieved, particularly in times of crisis. The Fund is, in fact, a risk-sharing, crisis prevention and resolution mechanism, which transforms participant countries' defaultable sovereign debts into union's safe assets. In sum, our theory can help to improve current official lending practices and, eventually, to design an *European Fiscal Fund*.

Key words: Fiscal Unions, Recursive contracts, debt contracts, partnerships, limited enforcement, moral hazard, debt restructuring, debt overhang, sovereign funds

JEL classification: E43, E44, E47, E63, F34, F36

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*We thank participants in seminars and conferences where versions of this work have been presented for their comments; in particular, Marco Bassetto, James Costain, Alessandro Dovis, Aitor Erce, Andreja Lenarcic and Rody Manuelli. Most of the research leading to this paper has been conducted within the Horizon 2020 ADEMU project, "A Dynamic Economic and Monetary Union", funded by the European Union's Horizon 2020 Programme under grant agreement No. 649396; and Liu is grateful to the financial support from National Natural Science Foundation of China under grant No. 72173091. BSE is supported by the Severo Ochoa Programme for Centres of Excellence in R&D (CEX2019-000915-S).

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1 Introduction

"For all economies to be permanently better off inside the euro area, they also need to be able to share the impact of shocks through risk sharing within the EMU."

This 2015 quote from the *Five Presidents' Report* recognizes a widely accepted fact. Without a Federal budget or an institutional framework with fiscal automatic stabilizers for the Euro area, it is unlikely that the European Economic and Monetary Union (EMU) will efficiently exploit its capacity for risk sharing and economic stabilization.[1] This became evident during the Great Recession. Moreover, in response to the Covid-19 crisis 5 years later, the European Union has established, *temporarily,* an unemployment insurance mechanism (Support to Mitigate Unemployment Risks, SURE) and a new system of transfers and sovereign debts mostly financed with Eurobonds (Recovery and Resilience Facility, RRF). Since natural disasters, possibly associated with climate change, pandemics, and other shocks with asymmetric consequences are likely to be a feature of the decades to come, a permanent fiscal risk-sharing and stabilization framework is clearly needed, not only for the EMU but for other unions, countries or states with decentralized structures but not well-defined transfer systems (Spain, China, etc.). In the present paper, we provide the theory on how to design a constrained-efficient transfer system and illustrate its performance in the context of the Euro area financial crisis.[2]

To do this, we develop a dynamic model of a *Financial Stability Fund* (Fund) as a long-term partnership addressing three features that are usually seen as the most problematic for a risk-sharing institution to be sustainable when the partnership is a union of sovereign countries. First, sovereignty means that countries can always exercise their right to exit the institution (possibly defaulting on their obligations), but it also means that risk-sharing transfers should never become permanent transfers or go beyond the level of redistribution that is accepted by all partners. To take this into account, *Fund contracts* are subject to *limited enforcement constraints* (LE) which make the fund stable, in the sense that there are no defaults, and sustainable, i.e., there are no undesired losses. In particular, our specific design assumes that there are no expected losses at any point in time, i.e., the Fund does not provide any redistribution *ex ante* or *ex post.*[3]

Second, the Fund must take into account moral hazard problems, since governments may be able to reduce future social and economic risks by implementing policy reforms, but often fail to do so whenever these reforms have contemporaneous socio-political costs. Again,

[1]For example, using a 'semi-structural methodology', **?** shows that "without a transfer policy rule, the standard deviation of employment across states increased by about 1 percent in the Great Recession and it will increase by 1.5 percent in the long-run."

[2]Following our work, **?** propose to establish an *European Fiscal Fund* (EFF) in two phases: the first implementing, as much as possible, the *Financial Stability Fund* policies in the current EU and euro area framework, the second, which involves Treaty changes, establishing the EFF.

[3]For example, in a more centralized union (e.g., China) these limited enforcement constraints are less stringent.

sovereignty places constraints here, since the Fund may have limited capacity to fully monitor policy reform efforts. More importantly, our Fund design respects that national governments have 'ownership' of their policy reforms, while taking into account the potential excessive risk due to the presence of moral hazard. Thus, *Fund contracts* are based on country-specific risk assessments and subject to *moral hazard constraints* (MH). Given that Fund contracts are 'experience rated', countries have an incentive to reduce their risk profile before entering the Fund contract[4]. Moreover, given that these contracts incorporate moral hazard constraints, risk-sharing transfers are combined with 'performance-based' long-term rewards (and punishments), which provide incentives for governments to further pursue risk-reduction policy reforms within the contract. Nevertheless, policy reform efforts are not contractable and, accordingly, Fund contracts are not conditional on ex-ante reforms or austerity packages, which, not surprisingly, are usually perceived as a lender's imposition over the borrowing sovereign country, often becoming ineffective (**?**).

Third, risk sharing among *ex-ante* equal partners without debt liabilities is relatively easy to design and achieve but, unfortunately, this is not the case in existing unions, for example, the EMU. In fact, the Euro crisis has left a 'debt-overhang problem' which aggravates the euro area divide. Given this, proposals for a 'shock-absorbing facility' are systematically postponed to a later day of greater convergence (e.g., the *Five Presidents' Report*, 2015), which can result in a never-ending procrastination. Our Fund design allows for a greater level of heterogeneity regarding the countries' growth, risk and liability profiles, provided that the latter are sustainable. Moreover, we show that risky defaultable sovereign debts are more sustainable if they are transformed into safe Fund contracts. With such an operation, the Fund balance sheet expands with safe assets, allowing the Fund to issue 'safe bonds'. Thus, the Fund can also play an important role in resolving existing 'debt-overhang' problems, as well as in creating 'high quality liquid assets' for the union.

In sum, the *Financial Stability Fund* is a *constrained-efficient mechanism* which, by integrating the risk-sharing and crisis-resolution functions, becomes a powerful instrument to prevent and confront crises, and is therefore superior to the standard instrument used to smooth consumption: sovereign (defaultable) debt financing. As a by-product of its ability to transform existing risky liabilities into safe Fund contracts, the Fund can also become an important absorber of existing sovereign debts — at least partially — and an issuer of safe assets.

It should be noted that *Limited enforcement* (LE) and *moral hazard* (MH) constraints are *forward-looking* constraints (i.e., the future evolution of the contract is part of the current constraint). Given this, standard dynamic programming techniques cannot be applied to solve the Fund's contractual problem. We use *recursive contracts* (see **?**) to obtain and characterize the (constrained) efficient Fund contract. To our knowledge, this is the first paper using this approach to study optimal lending contracts with LE and MH constraints.

[4]In the same way that home-owners may pay for the installation of a proper alarm system before signing a home-insurance contract.

One of our main theoretical contributions is the characterization of the constrained allocation by showing how the LE and MH frictions interact to determine the risk-sharing properties of the Fund contract as well as the maximum sustainable level of risk sharing. We show how optimal long-term contracts (through state-contingent transfers) can provide sufficient transfers to make sure that borrowers are able to smooth consumption during crisis periods without relying on default. At the same time, the path of transfers are also bounded above to make sure that the Fund will never accumulate liabilities against the country. We also show that moral hazard considerations lead to transfers (the terms of the contract in general) that respond positively to declining government expenditures, which in turn signal high policy effort. Finally, we provide a version of the inverse Euler equation for our environment. As opposed to models that feature only moral hazard (or private information), our model does not feature immiseration, as limited enforcement constraints on the borrower's side prevent it.

A second theoretical contribution of the paper is a novel decentralization of the constrained efficient allocation with taxes on Arrow securities and endogenous borrowing constraints, under which we can obtain a version of the two welfare theorems. We show that endogenous borrowing constraints are in place to make sure that the competitive equilibrium is consistent with limited enforcement constraints, while asset taxes are required to align the private (country-level) incentives for exerting policy effort with the social incentives. In particular, our Pigouvian asset taxes are, in equilibrium, budget neutral and absorb all the asset value variations implied by the moral hazard constraint. Nevertheless, they may be difficult to implement in a competitive market, strengthening the role of Fund as the institution implementing the constrained efficient allocation.

A third contribution of the paper is a quantitative evaluation of how the Euro area 'stressed countries' would have performed under the Fund during the most recent financial crisis. Formally, the model of a Financial Stability Fund consists of a contract between a risk-averse, relatively small and impatient borrower (the sovereign country) and a risk-neutral lender (the Fund itself). To assess the efficiency of the Fund, we use as a benchmark an incomplete markets model where sovereign countries issue long-term defaultable debt (IMD) in order to smooth their consumption. In order to have a qualitative and quantitative comparison of the two economies, we 'decentralize' the Fund contract to generate asset holdings and prices that are comparable to those in the IMD economy. Both in the IMD economy with default and in the Fund economy, interest rates may differ from the risk-free rate. The *positive spreads* in the IMD economy reflect the risk of default. Interestingly, the Fund economy only generates *negative spreads*, reflecting the risk that the lender's limited enforcement (i.e., his limit for redistribution) constraint is binding. We set the Fund's 'limit for redistribution' to zero, meaning that the Fund never has expected losses. Therefore, the lender's limited enforcement constraint can also be viewed as a *Debt Sustainability Analysis* (DSA), since it sets the limit beyond which the contract imbeds permanent transfers. The negative spread reflects the risk of not passing the DSA, the signal that for the Fund is better to invest in the risk-free market.

Our quantitative results are based upon a calibration of the incomplete markets model using data from the Euro area countries that were most affected by the European sovereign debt crisis (Greece, Italy, Portugal, and Spain) for the period of 1980–2015. The calibrated economy provides a good fit regarding the key variables of interest. In particular, it generates the level of debt and the statistical properties of the spread (mean, volatility and correlation with output) that are in line with the data. We then solve for the constrained-efficient Fund allocation using the same parameters as in the incomplete markets economy to assess quantitatively how the euro area 'stressed countries' would have performed had they had a Fund contract. We compare the IMD and the Fund allocations in a number of ways. We contrast several long run moments of simulated data from both environments, we examine how the two economies respond to severe shocks that resemble the euro-crisis and we evaluate the welfare gains and debt absorbing capacity associated with the Fund. All these comparisons point in the same direction. The Fund is able to provide superior risk sharing (insurance) against shocks through multiple channels. First, it increases the borrowing capacity of the country significantly, smoothing the impact of shocks when they hit through borrowing. This also implies that the Fund can take over very large amount of debt from the borrowing countries without the risk of default episodes. Second, the Fund provides state-contingent payments, generating efficient counter-cyclical primary deficits. Third, while default is costly in the two economies, both because of direct output losses and because of temporary exclusion from the sovereign debt market, the design of the Fund eliminates default episodes. Fourth, in the absence of default, the borrower does not have to pay any penalties or high spreads on debt whenever borrowing is most desirable. Finally, the Fund contract provides incentives for higher risk-reduction effort in normal times, while it allows for lower effort than in the IMD economy in a crisis situation.

Quantitatively, we find that the welfare gains of the Fund are very significant: between 7 and 10 percent in consumption-equivalent terms, depending on the state of the economy.[5] The paper then provides a novel decomposition of these welfare gains. We show that the most important sources of welfare gains are the relaxation of the effective borrowing limits, which imply a higher borrowing capacity in the Fund, and the state contingency of payments. In the case of low initial productivity, these two elements constitute at least 95 percent of the total welfare gains, with somewhat higher weight on its higher borrowing capacity.

Literature review We are not the first to address how risks could be shared in a monetary union and how to deal with sovereign debt-overhang problems. For example, as an implicit criticism of different proposals to issue some form of joint-liability eurobonds, **?** emphasises the asymmetry issue: the optimal (one-period) risk-sharing contract with two symmetric countries

[5] It is worth to stress that, altough in this paper we assume the Fund absorbs all the sovereign debt, the superior welfare gains can still be achieved when relaxing this exclusivity assumption: following our work, **?** have shown that similar welfare gains exist if the Fund absorbs an (endogenously determined) minimal part of the debt, with competitive risk-neutral private lenders holding the rest, which becomes safe due to the Fund intervention.

is a joint liability debt contract serving as a risk-sharing mechanism, while the optimal contract between two countries with very different distress probabilities is a debt contract with a cap and no joint liability, where the cap depends on the extent of solidarity, which is given by the externality cost of debt default on the lender. With long-term relationships — as they are among sovereign countries that form a union — we show that better contracts can be implemented: the Fund contracts are *constrained-efficient* and they can be implemented as long-term bonds with state-contingent coupons and appropriate taxation of assets.

In terms of optimal long-term contracts, **?** and **?** study lending contracts in international contexts. Both of these papers consider only lack of commitment from the borrower's side. Similar to our paper, **?** also considers moral hazard, but with respect to consuming or investing the borrowed funds. Finally, in related and contemporaneous work, **?**, study dynamic sovereign lending contracts with moral hazard with respect to reform policy effort and one-sided limited enforcement. They provide an interesting characterization and decentralization of the constrained-efficient allocation in a stylized model (e.g., normal times are an absorbing state) and their mechanism heavily relies on complex *ex-post* default procedures. Closer in scope to our work, **?** decentralizes optimal contracts subject to two-sided limited enforcements and private information (adverse selection) through partial default and an active debt maturity management. In light of this, he proposes an alternative to the decentralization of **?**. He rationalizes sovereign default as a decision that is *ex-post* inefficient but *ex-ante* necessary to sustain the efficient interaction between the contracting parties. In periods of distress, the debt contract is implicitly made state-contingent by allowing for 'excusable' defaults with partial repayments **?**. Even though such events are rare, they imply losses for the lenders as debt remittance is only partial.

From the perspective of quantitative normative-positive analysis, it is interesting to know the *constrained efficiency* properties of contracts where *ex-ante* state-contingencies are replaced by *ex-post* active debt management, default episodes or debt renegotiations. This is the focus and contribution of the work of **?**, **?** and others. We could have considered other specifications of the incomplete markets with default (IMD) economy, as well as of state-contingent contracts, or arrangements, different from long-term Arrow securities, but our simplifying choices respond to the need of focusing on the design and characterization of the Fund. Two remarks are in order. First, our IMD calibration to the euro area four 'stressed' during the euro crisis, fits remarkably well to the observed level of self-insurance and cyclicality of these countries. Other calibrations with different models can obtain similar fits, but what determines the welfare gains from having a 'constrained-efficient Fund' is the difference between the time-series generated from the fitted IMD model and the economy with the Fund, given the same underlying stochastic process. Second, an IMD economy with *ex-post* state-contingent contracts would reduce, but not eliminate, the welfare gains of the Fund contract relative to our IMD economy, as long as these alternative 'constrained efficient contracts' entail costly defaults or renegotiations.

Finally, our model of the Fund as a partnership builds on the literature on dynamic optimal contracts with enforcement constraints (e.g., ?, ?, ?), but, as discussed earlier, we develop the theory further by incorporating moral hazard constraints. There is also a related literature on the decentralization of optimal contracts (e.g. ?, ?) and, since we introduce taxes on state-contingent bonds, to the new dynamic public finance literature (e.g., ?). Finally, our benchmark incomplete markets economy with long-term debt with default builds on the model of ?, who extends the sovereign default models of ? and ? to long-term debt.

The paper is organized as follows. Section 2 presents the economy with the Fund and with incomplete markets and defaultable long-term sovereign debt. Section 3 shows how to decentralize the Fund contract with state-contingent long-term bonds. Section 4 discusses the calibration. Section 5 quantitatively compares the IMD and Fund regimes, concluding with a welfare comparison and a counterfactual 'euro-crisis' simulation. Section 6 concludes. All proofs and more details of calibration are relegated to Appendix A and B respectively.

2 The Economy

We consider an infinite-horizon small open economy where the 'benevolent government' acts as a representative agent with preferences for current leisure, $\ell = 1 - n \in [0, 1]$, consumption, $c \geq 0$, and effort, $e \in [0, 1]$, valued by $U(c, n, e) \equiv u(c) + h(1 - n) - v(e)$. We make standard assumptions on preferences: u, h, v are differentiable; $u'(x) > 0, h'(x) > 0$ for $x \geq 0$, $v'(x) > 0$ for $x > 0$, and $v'(0) = 0$; $u''(x) < 0, h''(x) < 0$, and $v''(x) > 0$. The government discounts the future at the rate β, satisfying $\beta \leq 1/(1 + r)$, where r is the risk-free world interest rate. In general, we will assume the inequality to be strict.

The country has access to a decreasing-returns labour technology $y = \theta f(n)$, where $f'(n) > 0$, $f''(n) < 0$, and θ is a productivity shock $\theta \in \{\theta_m : m = 1, \dots, N_\theta\}$, $\theta_m < \theta_{m+1}$. The country also needs to cover its government expenditures, or liabilities, which are represented by g with $g \in \{g_k : k = 1, \dots, N_g\}$, $g_k > g_{k+1}$. To an extent, g is endogenous, since the current period effort of the government (representative agent) determines the distribution of expenditures next period, with costly higher effort resulting in a distribution of expenditures that first order stochastically dominates the distribution with lower effort. We assume that effort is not contractable, and we allow for technology and government shocks to be correlated. In sum, the exogenous state of the economy is given by $s = (\theta, g) \in S$ with the overall Markovian transition given by $\pi(s'|s, e)$.[6] As standard in models with private effort, we assume full support: $\pi(s'|s, e) > 0$ for all s', s and $e > 0$. This implies that (i) for interior effort, our model generates an ergodic set of S that includes all possible combinations of shocks with positive probability and (ii) our incentive problem is well-defined as there are no

[6]We order the elements of S as follows: $s(1) = (\theta_1, g_1), s(2) = (\theta_1, g_2), s(N_g) = (\theta_1, g_{N_g}), s(N_g + 1) = (\theta_2, g_1), \dots, s(\bar{N}) = (\theta_{N_\theta}, g_{N_g})$, where $\bar{N} = N_\theta \times N_g$; that is $s(i) = (\theta_m, g_k)$, where $i = N_g(m - 1) + k$ and $m \in \{1, \dots, N_\theta\}, k \in \{1, \dots, N_g\}$. In the theory part, we will assume that the $\{\theta_t\}$ and $\{g_t\}$ processes are independent and, therefore, $\pi(s' = s(i)|s, e) = \pi(\theta' = \theta_m|\theta)\pi(g' = g_k|g, e)$. We also implicitly assume that $\theta f(1) - g > 0$ for all s to guarantee feasiblity. In the quantitative part, we relax the independence assumption.

states of the world where infinite punishments can be used.

Most of our our analysis focuses on a set-up where the country can manage its private and public debt liabilities with the help of a Financial Stability Fund (Fund), which acts as a benevolent risk-neutral principal/planner who has access to the international capital markets at the risk-free rate. Note that, since the country is an open economy, $\theta f(n)-(c+g)$, does not need to be zero period by period, allowing for transfers between the country and the Fund. This framework is compared to an incomplete markets economy with default (IMD), that is, an economy where the government accesses directly the international capital markets by issuing non-contingent defaultable long-term debt. In order to make the economies quantitatively comparable Section 3.3 decentralises the Fund allocation as a competitive equilibrium.

2.1 The Economy with a *Financial Stability Fund* (Fund)

The Financial Stability Fund (Fund) is modeled as a long-term contract between a Fund (also called lender) and an individual partner (also called country or borrower) who is the government of the small open economy. The Fund contract chooses a state-contingent sequence of consumption, leisure and effort that maximises the life life-time utility of the borrower given some initial level of the borrower's debt. The optimal contract is self-enforcing through the presence of two limited-enforcement constraints. First, we assume that, if the country ever defaults on the Fund contract, it will not be able to sign a new contract with the Fund and will enter the markets for defaultable long-term debt as a defaulter. The Fund contract, however, makes sure that the country never finds it optimal to renege the contract. Second, the contract also prevents the Fund from ever incurring in undesired expected losses, i.e., undesired permanent transfers. In addition, the contract also has an incentive compatibility constraint, since effort to achieve a better distribution of government liabilities is non-contractable, i.e., it is private information, or a sovereign right of the country. Thus, the long-term contract must provide sufficient incentives for the country to implement a (constrained) efficient level of effort.

In sum, the Fund contract can provide risk-sharing and consumption smoothing with state-contingent transfers. However, these transfers are constrained by limited enforcement and moral hazard frictions. Note also that the Fund contract is based on a country-specific risk-assessment, as the allocation depends on all the underlying parameters describing preferences, technology and the shock process.

2.1.1 The Long Term Contract

In its extensive form, the *Fund contract* specifies that in state $s^t = (s_0, \ldots, s_t)$, the country consumes $c(s^t)$, uses labour $n(s^t)$ and exercises effort $e(s^t)$, resulting in a transfer to the Fund of $c_l(s^t) = \theta f(n(s^t)) - (c(s^t) + g)$, with $c_l(s^t) < 0$ implying that the country is effectively borrowing. With *two-sided limited enforcement* and *moral hazard*, an optimal Fund contract is a solution to the following problem:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\left[\mu_{b,0}\sum_{t=0}^{\infty}\beta^t U(c(s^t),n(s^t),e(s^t)) + \mu_{l,0}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_l(s^t)\middle| s_0\right]$$

$$\text{s.t.}\quad \mathbb{E}\left[\sum_{j=t}^{\infty}\beta^{j-t}U(c(s^j),n(s^j),e(s^j))\middle| s^t\right] \geq V^o(s_t), \tag{1}$$

$$v'(e(s^t)) = \beta \sum_{s^{t+1}|s^t} \partial_e \pi(s^{t+1}|s^t, e(s^t))V^{bf}(s^{t+1}), \tag{2}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty}\left(\frac{1}{1+r}\right)^{j-t} c_l(s^j)\middle| s^t\right] \geq Z, \tag{3}$$

$$\text{and}\quad c_l(s^t) = \theta(s^t)f(n(s^t)) - c(s^t) - g(s^t), \quad \forall s^t, t\geq 0, \tag{4}$$

where

$$V^{bf}(s^t) \equiv \mathbb{E}\left[\sum_{j=0}^{\infty}\beta^j U(c(s^{t+j}),n(s^{t+j}),e(s^{t+j}))\middle| s^t\right].$$

In the previous problem, $(\mu_{b,0}, \mu_{l,0})$ are the initial Pareto weights, which are key for our interpretation of the Fund contract as a lending contract. In particular, we show in Section 3.3 that the initial relative Pareto weight determines uniquely the level of debt that the Fund takes over when the country joins. Note also that the notation is implicit about the fact that expectations are conditional on the implemented effort sequence, as it affects the distribution of the shocks.

Constraints (1) and (3) are the *limited enforcement constraints* for the borrower and the lender, respectively, in state s^t. The outside value for the borrower is denoted by $V^o\left(s^t\right)$, and it will be defined formally in section 2.2, where we present the economy with incomplete markets and default (IMD). For now, we just need to assume that $V^o(s)$ is bounded below, increasing in θ and decreasing with a higher g,[7] and that there is a sufficient level of punishment for defaulters such that the Fund can provide some risk sharing beyond that is achievable in IMD markets. In principle, a diverse set of default scenarios can satisfy these requirements.

The finite outside option of the lender Z measures the extent of *ex post* redistribution the Fund is willing to tolerate.[8] That is, if $Z<0$ the Fund is allowed to make a permanent loss in terms of life-time expected net present value — i.e., in the international financial market, the Fund can find better investment opportunities and if it does not renege it is because it has committed to sustain $Z<0$. Clearly, the level of Z has an important impact on the amount of risk sharing in our environment and it can thus be interpreted as solidarity, as in **?**. In our benchmark calibration, we assume that $Z=0$ implying that the lender does not accept any permanent level of *ex-ante* (at the time of signing the contract) or *ex post* (at any later

[7]In particular, $V^o(s(1)) < V^o(s(\bar{N}))$ where $\bar{N}$ represents is the number of total possible combinations of θ and g.

[8]We can introduce state-dependence of this constraint without any conceptual difficulty.

period) redistribution. At the same time, the period by period transfers c_l can be positive or negative, hence the Fund can still generate risk sharing. In fact, we will show that, even with $Z = 0$, the Fund is superior to the IMD economy, since it can still provide significant risk-sharing gains and a higher debt capacity to the government.

Constraint (2) is the *moral hazard* (i.e., incentive compatibility) constraint with respect to the borrower's effort, which is not contractable and $V^{bf}(s^{t+1})$ represents the value of the Fund contract for the borrower in state s^{t+1}. The interpretation of this constraint is standard: the marginal cost of increasing effort has to be equal to the marginal benefit. The latter is measured as the change in life-time utility due to the change in the distribution of future shocks as a result of the increasing effort.[9] Note that (2) uses implicitly the *first-order condition approach*, that is, we replace the agent's full optimization problem with respect to effort by its necessary first-order conditions. Following **?**, we now introduce assumptions to guarantee that this condition is also sufficient. To do this, we denote the cumulative distribution function of s' with:

$$F_j(e, s) = \sum_{i=1}^{j} \pi(s' = s(i)|s, e).$$

Assumption 1 (*Independence, Differentiability, Monotonicity, and Convexity*)**.** The $\{\theta_t\}$ and $\{g_t\}$ processes are independent. $F_j(e, s)$ is differentiable in e implying that $\pi(s' = s(i)|s, e)$ is differentiable for every s, i and $e > 0$. For every s and $e > 0$, the ratio $\frac{\partial_e \pi(s'=s(i)|s,e)}{\pi(s'=s(i)|s,e)} = \frac{\partial_e \pi(g'=g_n|g,e)}{\pi(g'=g_n|g,e)}$ is increasing in n, where $i = N_g(m-1) + n$ for some productivity shock index m; and, $\partial_e^2 F_j^g(e, s) = \sum_{n=1}^{j} \partial_e^2 \pi(g' = g_n|g, e) \geq 0$.

These conditions generalize the assumptions in **?** so that we can apply his *first-order condition approach* in a simple static Pareto-optimization problem to our dynamic contracting problem with limited enforcement and moral hazard frictions.[10] In particular, the independence assumption is made to have a single shock variable, g, depending on effort, as in the standard moral hazard problem.

Finally, note that the formulation of the problem implicitly assumes interiority of effort as we impose the incentive compatibility constraint as equality. In our setting, this is guaranteed if full risk sharing is not the optimal allocation and appropriate Inada conditions are imposed on the cost $v(e)$ and the marginal benefit $\partial_e \pi(s'|s, e)$ of effort. Moreover, we will implicitly assume that the Fund can deliver risk sharing benefits in the strict sense. This implies that at least one of the two limited enforcement constraints are satisfied as a strict inequality. Given that the Fund can always replicate what the IMD markets can offer, this assumption can be easily satisfied in our setting because with $Z = 0$ as long as β is high enough, in which case

[9]Throughout the paper, we use ∂_x and ∂_x^2 to denote the first and second derivatives of a function with respect to variable x respectively.

[10]More precisely, the monotonicity condition is Rogerson's monotone likelihood-ratio condition (MLR) and $\partial_e^2 F_j(e, s) \geq 0$ is Rogerson's convexity of the distribution condition (CDF).

the risk sharing gains are strictly positive for any sequence of effort. Assumption 2 below provides formal conditions to generate interiority. This assumption is also necessary for the existence of the recursive formulation provided below, as it guarantees the boundedness of the associated Lagrange multipliers.

Assumption 2 (*Interiority.*)**.** There is an $\epsilon > 0$, such that, for all $s_0 \in S$ there is a program $\{\tilde{c}(s^t), \tilde{n}(s^t), \tilde{e}(s^t)\}_{t=0}^{\infty}$ satisfying constraints (1) and (3) when, on the right-hand side, $V^o(s_t)$ and $Z(s_t)$ are replaced by $V^o(s_t) + \epsilon$ and $Z(s_t) + \epsilon$, respectively and, similarly, when in (2) '$v'(e(s^t)) =$' is replaced by '$v'(e(s^t)) + \epsilon \leq$ '.

2.1.2 Recursive Formulation

It is known from **?** and **?** that we can rewrite the general fund contract problem as a saddle-point Lagrangian problem:[11]

$$
\begin{aligned}
\text{SP} \min_{\{\gamma_b(s^t),\gamma_l(s^t),\xi(s^t)\}} \max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\Bigg[\sum_{t=0}^{\infty} \beta^t \Big(& \mu_{b,t}(s^t) U(c(s^t), n(s^t), e(s^t)) - \xi(s^t) v'(e(s^t)) \\
& + \gamma_b(s^t) \big[U(c(s^t), n(s^t), e(s^t)) - V^o(s_t) \big] \Big) \\
+ \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \Big(\mu_{l,t+1}(s^t) [\theta(s^t) f(n(s^t)) & - c(s^t) - g(s^t)] - \gamma_l(s^t) Z \Big) \Bigg| s_0 \Bigg] \\
\text{s.t. } \mu_{b,t+1}(s^{t+1}) = \mu_{b,t}(s^t) + \gamma_b(s^t) + \xi(s^t) & \frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}, \\
\mu_{l,t+1}(s^t) = \mu_{l,t}(s^{t-1}) + \gamma_l(s^t), & \text{ with } \mu_{b,0}(s^0) \equiv \mu_{b,0}, \ \mu_{l,0}(s^{-1}) \equiv \mu_{l,0} \text{ given,}
\end{aligned}
$$

where $\beta^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \gamma_b(s^t)$, $\left(\frac{1}{1+r}\right)^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \gamma_l(s^t)$ and $\beta^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \xi(s^t)$, with $\boldsymbol{e}^{t-1} \equiv \{e(s^j)\}_{0 \leq j \leq t-1}$, are the Lagrange multipliers of the limited enforcement constraints (1), (3), and incentive compatibility constraint (2), respectively, in state s^t. The above formulation of the problem defines two new co-state variables $\mu_b(s^t)$ and $\mu_l(s^t)$, which represent the temporary Pareto weights of the borrower and the lender respectively. These variables are initialized at the original Pareto weights and they become time-variant because of the limited commitment and moral hazard frictions. In particular, a binding limited enforcement constraint of the borrower (lender) will imply a higher welfare weight of the borrower (lender) so that he does not leave the contract. In addition, the moral hazard friction (whenever $e > 0$ and $\xi > 0$, i.e., whenever the incentive compatibility constraint is binding) implies that the co-state variable of the borrower will increase or decrease depending on the sign of the likelihood ratio $\frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}$. In particular, a positive likelihood ratio, which occurs with a low government expenditure, provides a good signal about effort and hence the borrower will be rewarded with a higher temporary Pareto weight. Note that the monotonicity assumption guarantees that

[11] Following **?**, we only consider saddle-point solutions and their corresponding saddle-point multipliers. That is, given $\Phi(a, \lambda)$, (a^*, λ^*) solves SP $\min_\lambda \max_a \Phi(a, \lambda)$ if and only if $\Phi(a, \lambda^*) \leq \Phi(a^*, \lambda^*) \leq \Phi(a^*, \lambda)$, for any feasible action a and Lagrangian multiplier λ.

the the likelihood ratio is decreasing in g, that is, the the Pareto weight of the borrower is rewarded for low realizations of g next period. This leads the borrower to exert higher effort in the current period.

It turns out that, in the previous problem, only relative Pareto weights matter for the allocations, and this allows us to reduce the dimensionality of the co-state vector and write the problem recursively by using a convenient normalization. Let $\eta \equiv \beta(1+r) \leq 1$ and define the discounted relative pareto weight of the borrower as $x_t(s^t) \equiv [\beta(1+r)]^t \mu_{b,t}(s^t)/\mu_{l,t}(s^t)$. We normalize the multipliers as follows:

$$\nu_b(s^t) = \frac{\gamma_b(s^t)}{\mu_{b,t}(s^t)}, \quad \nu_l(s^t) = \frac{\gamma_l(s^t)}{\mu_{l,t}(s^{t-1})}, \quad \varrho(s^t) = \frac{\xi(s^t)}{\mu_{b,t}(s^t)},$$
$$\varphi(s^{t+1}|s^t, e(s^t)) = \varrho(s^t)\frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}.$$

Note that $\varphi_{t+1}(s^{t+1}|s^t, e(s^t))$ can be positive or negative depending on whether the derivative with respect to effort in the numerator is positive or negative. The law of motion of x can then be defined recursively as:

$$x_{t+1}(s^{t+1}) = \frac{1 + \nu_{b,t}(s^t) + \varphi_{t+1}(s^{t+1}|s^t, e(s_t))}{1 + \nu_{l,t}(s^t)}\eta x_t(s^t), \text{ with } x_0 = \mu_{b,0}/\mu_{l,0} \tag{5}$$

With this normalization, ν_b and ν_l become the multipliers of the limited enforcement constraints, corresponding to (1) and (3) and ϱ becomes the multiplier of the incentive compatibility constraint corresponding to (2). Moreover, the state vector for the problem (including the new co-state) becomes (x, s). The *Saddle-Point Functional Equation* (**SPFE**) — i.e., the saddle-point version of Bellman's equation — is given by:

$$FV(x,s) = \text{SP} \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Big\{ x[(1+\nu_b)U(c,n,e) - \nu_b V^o(s) - \varrho v'(e)] \tag{6}$$
$$+ [(1+\nu_l)(\theta(s)f(n) - c - g(s)) - \nu_l Z] + \frac{1+\nu_l}{1+r}\mathbb{E}[FV(x',s')\big|s,e] \Big\}$$
$$\text{s.t. } x'(s') = \frac{1+\nu_b+\varphi(s'|s,e)}{1+\nu_l}\eta x \quad \text{with} \quad \varphi(s'|s,e) = \varrho\frac{\partial_e \pi(s'|s,e)}{\pi(s'|s,e)}. \tag{7}$$

where the Fund's value value functions can be decomposed as follows:

$$FV(x,s) = xV^{bf}(x,s) + V^{lf}(x,s), \text{ with} \tag{8}$$
$$V^{bf}(x,s) = U(c(x,s), n(x,s), e(x,s)) + \beta\mathbb{E}\big[V^{bf}(x'_{xs}(s'), s')\big|s, e(x,s)\big], \text{ and} \tag{9}$$
$$V^{lf}(x,s) = c_l(x,s) + \frac{1}{1+r}\mathbb{E}\big[V^{lf}(x'_{xs}(s'), s')\big|s, e(x,s)\big], \text{ where} \tag{10}$$
$$c_l(x,s) = \theta(s)f(n(x,s)) - g(s) - c(x,s). \tag{11}$$

The derivation of this recursive SPFE follows the standard procedure of **?**. We study economies where the SPFE equation (6) has a solution for every (x, s). Without moral hazard constraints, which lead to an endogenous g process, a direct application of **?**, Theorem 3 would establish the existence of a unique solution to the SPFE (6) by contraction mapping, and no additional assumptions would be required. However, in the presence of moral hazard, the existence of a saddle-point solutions SPFE is more intricate and outside of the scope of the present paper[12]. Nevertheless, our assumptions guarantee that solutions, if they exist – as numerically we show to be the case – are well behaved and can be characterized, as we do.

In what follows, we assume the existence of a unique solution to the SPFE and we provide a preliminary charcaterization of the Fund allocation by looking at the optimality conditions. To simplify notation, we let the policy for the relative Pareto weight be given by $x'_{xs}(s') \equiv x'(x, s, s')$. The policy functions for consumption of the Fund contract must solve the first-order conditions of the SPFE. In particular, $c(x, s)$ and $n(x, s)$ must satisfy:

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x} \tag{12}$$

$$\frac{h'(1-n(x,s))}{u'(c(x,s))} = \theta f'(n(x,s)). \tag{13}$$

These conditions are standard, the borrower's consumption is determined by her endogenous relative Pareto weight and, given that preferences are separable, the labor supply is undistorted. The effort policy $e(x, s)$ is determined by the first order condition of the SPFE with respect to e, which can be conveniently expressed as:

$$\begin{aligned} v'(e(x,s)) = {} & \beta \sum_{s'|s} \partial_e \pi(x'_{xs}(s'), s') V^{bf}(x'_{xs}(s'), s') \\ & + \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(x'_{xs}(s'), s') V^{lf}(x'_{xs}(s'), s') \\ & - \frac{\varrho(x,s)}{1+\nu_b(x,s)} \left[v''(e(x,s)) + \beta \sum_{s'|s} \partial_e^2 \pi(x'_{xs}(s'), s') V^{bf}(x'_{xs}(s'), s') \right]. \end{aligned} \tag{14}$$

Equation (14) balances the marginal cost of effort with the benefits. The first line is the life-time utility benefit of effort to the borrower; the second line is the marginal benefit of effort to the lender, in terms of the borrower's marginal utility, given by (12); the third line accounts for the marginal relaxation/tightening effect of the moral hazard constraint (2) when there is a change in effort. With contractable effort, the Fund problem would not have the *incentive compatibility constraint* (2) and the effort decision would be given by the first two lines, with the second one accounting for the social value of effort. In contrast, with non-contractable effort, as we assume, constraint (2) is present and the first line is equal to

[12]**?** proves existence of a solution to a related dynamic contracting problem with moral hazard and limited commitment. The key difference compared to our approach is that his paper has lenders that solve an essentially static problem (they offer one period debt contracts); ours, instead, is a dynamic saddle-point problem, with a 'first-order approach' moral hazard constraint.

zero, namely:

$$v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s, e(x,s)) V^{bf}(x'_{xs}(s'), s'). \tag{15}$$

In this case, (14) reduces to

$$\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(x'_{xs}(s'), s') V^{lf}(x'_{xs}(s'), s')$$
$$= \chi(x,s) \left[v''(e(x,s)) - \beta \sum_{s'|s} \partial_e^2 \pi(x'_{xs}(s'), s') V^{bf}(x'_{xs}(s'), s') \right], \tag{16}$$

where $\chi(x,s) \equiv \frac{x\varrho(x,s)}{1+\nu_l(x,s)}$ can be interpreted as the marginal value of relaxing the ICE constraint in terms of the lender's valuation; that is, (16) accounts for the external effect of effort on the lender's value through its effect on the incentive compatibility constraint. Note that, although incentive compatibility implies that only the borrower's returns affect the effort decision directly, the benefits represented in (16) will affect incentives as they affect $\varrho(x,s)$ and hence the whole future path of allocations through (7).

Given the policy function $e(x,s)$, we denote by $\{s\}_{e(x,s)}$ the resulting Markov process of $\{\theta, g\}$ shocks. Furthermore, a recursive constrained-efficient Fund allocation also satisfies the following endogenous limited enforcement (constraint qualification) constraints:

$$\nu_b(x,s)\left[V^{bf}(x,s) - V^o(s)\right] = 0 \text{ with } \nu_b(x,s) = 0 \text{ if } V^{bf}(x,s) > V^o(s), \tag{17}$$

$$\nu_l(x,s)\left[V^{lf}(x,s) - Z\right] = 0 \text{ with } \nu_l(x,s) = 0 \text{ if } V^{lf}(x,s) > Z. \tag{18}$$

Definition 1 (Recursive Constrained Efficient Fund Contract)**.** Given an initial relative Pareto weight $x(s_0)$ and outside options $\{V^o(s), Z(s)\}$ for the borrower and lender, the policies for the allocations $\{c(x,s), c_l(x,s), n(x,s), e(x,s)\}$, multipliers $\{\nu_l(x,s), \nu_b(x,s), \varrho(x,s)\}$, value functions $\{V^{bf}(x,s), V^{lf}(x,s)\}$, relative Pareto weight $x'_{xs}(s')$, and the laws of motion for $\{\theta, g\}_{e(x,s)}$ are a recursive constrained efficient Fund contract if they satisfy conditions (7)–(13) and (15)–(18) for all (x,s).

2.1.3 Characterization of the Fund Contract

Given the existence of a unique solution to the SPFE, we can establish several important properties. The following ones follow directly from the first-order conditions (12) and (13), the resource constraint (11), and the constraint qualification conditions (17) and (18):

Lemma 1. *Given a unique solution of the SPFE (6), the value functions $V^{bf}(x,s)$ and $V^{lf}(x,s)$ are bounded and uniquely determined, strictly concave and strictly monotone — i.e., $V^{bf}(x,s)$ increasing while $V^{lf}(x,s)$ decreasing — in x, whenever neither of the limited enforcement constraints are binding, and constant, when one is binding.*

Proof. See Appendix A. □

In what follows, we provide an analytical characterization of the Fund allocations that relate our model to both the limited commitment and the private information/action literature. To do this, it will be convenient to denote by $\underline{x}(s)$ the maximum value of x for which the borrower's limited enforcement constraint is binding — i.e., $\underline{x}(s) \equiv \max\{x : V^{bf}(x,s) = V^{o}(s)\}$ – and by $\overline{x}$ the minimum value of x for which the lender's limited enforcement constraint is binding — i.e., $\overline{x}(s) \equiv \min\{x : V^{lf}(x,s) = Z\}$.

Like in other models with two-sided lack of commitment, it is easy to see that consumption allocations can be described through intervals of the relative pareto weight $[\underline{x}(s), \overline{x}(s)]$. If the initial relative Pareto weight is within this interval for a given s, consumption is determined directly by x. In contrast, if the initial relative pareto weight is outside this interval, consumption is determined by the binding limited enforcement constraint. The key difference between the standard two-sided limited commitment framework (see **?** for a textbook treatment) and our model is that the relative Pareto weight (and hence consumption) respond to the realization of the shocks between states even if none of the limited enforcement constraints bind due to the presence of moral hazard constraints (see (7)). The existence and uniqueness of these thresholds are the consequence of the finiteness of the outside options for the borrower and the Fund and the strict monotonicity of the value functions. The following two lemmas use these properties to characterize the policy functions and the ergodic set for our Fund economy.[13]

Lemma 2. *The Fund policy functions, which solve SPFE (6), satisfy the following properties:*

(i) If none of the limited enforcement constraints is binding, i.e. $\nu_b(x,s) = \nu_l(x,s) = 0$, *then,* $c(x,s)$ *is increasing in* x *and constant in* s, $n(x,s)$ *is decreasing in* x *and increasing on* θ, *but constant in* g, $V^{bf}(x,s)$ *is increasing in* x *and in* s, $V^{lf}(x,s)$ *decreasing in* x *and increasing in* s, *and* $x'(x,s,s')$ *is increasing in* x *and non-decreasing in* s'*;*

(ii) If at (x,s) *the borrower's limited enforcement constraint is binding — i.e.,* $x \le \underline{x}(s)$*, — then, for* $m = c, n, e, x', V^{bf}$ *and* V^{lf}, $m(x,s) = m(\underline{x}(s),s)$, *therefore the constraint is non-binding if* x *increases from* $\underline{x}(s)$ *(and policies follow (i)), but it is constant for* $x < \underline{x}(s)$*; with respect to* s, $\underline{x}(s)$ $V^{bf}(\underline{x}(s),s) = V^{o}(s)$, $c(\underline{x}(s),s)$ *and* $x'(\underline{x}(s),s,s')$ *are increasing, while* $n(\underline{x}(s),s)$ *is increasing in* g, *and* $x'(\underline{x}(s),s,s')$ *is non-decreasing in* s'*;*

(iii) If at (x,s) *the lender's limited enforcement constraint is binding — i.e.,* $x \ge \overline{x}(s)$ *then, for* $m = c, n, e, x', V^{bf}$ *and* V^{lf}, $m(x,s) = m(\overline{x}(s),s)$, *therefore the constraint is non-binding if* x *decreases from* $\overline{x}(s)$ *(and policies follow (i)), but it is constant for* $x > \overline{x}(s)$*; with respect to* s, $V^{bf}(\overline{x}(s),s)$, $c(\overline{x}(s),s)$ *and and* $x'(\overline{x}(s),s,s')$ *are increasing, while* $n(\overline{x}(s),s)$ *is increasing in* g, *and* $x'(\overline{x}(s),s,s')$ *is non-decreasing in* s'.

Proof. See Appendix A. □

[13] In the following Lemma, and related statements, $s \ge \tilde{s}$ if and only if $(\theta, -g) \ge (\tilde{\theta}, -\tilde{g})$, i.e., given n, available surplus is at least as high in s than in $\tilde{s}$. Note that the ordering of g in s is in the opposite of its numerical magnitude: higher g is worse for the borrower.

This lemma extends the results of the standard two-sided limited commitment model to endogenous labor supply and effort. In the region when neither limited enforcement constraint is binding, the allocation is efficient: consumption of the borrower is increasing with her relative Pareto weight, while labor supply and effort decrease ('wealth effect'); given x, consumption is constant and labor supply is increasing in θ and, since the borrower is more impatient than the lender, $x' < x$ and, therefore, consumption decreases, as long as x' does not bind. In fact, by (7) and (12), $x'(x, s)$ and $c(x, s)$ always co-move and, given our Assumption 1, $x'(x, s, s')$ is non-decreasing in s'. An increase in the exogenous state s increases the borrower's outside option and, therefore, $\underline{x}(s)$, but also increases $\overline{x}(s)$ since the lender's outside option, Z, does not depend on s and, therefore the borrower absorbs the increase of surplus. How effort reacts to x and s depends on how these variables change the marginal benefit in the incentive compatibility constraint – i.e. the RHS of (15) – which in general is not determined. Although, if the 'wealth effect' dominates, increases in (x, s) should weaken the IC constraint and, therefore, lower $e(x, s)$ in (15). For further details,

An immediate consequence of Lemma 2, given that $\eta < 1$, is the following result.

Lemma 3. *A recursive constrained-efficient Fund contract defines a long-run steady-state allocation determined by the exogenous ergodic set of* $\{s_t\}$ *and the endogenous ergodic set of* $\{x(s_t)\}$*, with support* $[\underline{x}, \overline{x}] = [\underline{x}(s(1)), \overline{x}(s(\bar{N}))]$*,* $V^{bf}(\underline{x}, s(1)) = V^o(s(1))$ *and* $V^{bf}(\bar{x}, s(\bar{N})) = V^o(s(\bar{N}))$*.*

This result follows from the fact that given any initial exogenous state s_0, $x(s_0)$ is arbitrary, and it could be that $x(s_0) \notin [\underline{x}, \overline{x}]$. In this case, the limited enforcement constraints, together with the fact that $\eta < 1$ imply that x will enter $[\underline{x}, \overline{x}]$ in finite time as long as $\eta < 1$. Once in the ergodic set, given that $\eta < 1$, both the minimum and maximum limited enforcement constraints of the borrower can be achieved with probability one, and the allocation policies on (x, s), together with the Markov matrix (given by $\pi(s'|s, e(x, s))$ and the law of motion $x'_{xs}(s')$ — determine the long-run distribution of allocations (and multipliers).

All the properties described above are also reflected in Figure 1, displaying the policy functions for the relative Pareto weight of the borrower consumption, labor, and effort.[14] All the policies in the figure are displayed for the worst ($\underline{g} = g_1$) and best ($\overline{g} = g_3$) government shocks and for the worst ($\underline{\theta} = \theta_1$) and best ($\overline{\theta} = \theta_{27}$) values of the technology shock. Moreover, since $x'_{xs}(s')$ depends on the future shocks s', the figure plots the future pareto weight for intermediate realizations of the two shocks tomorrow.

Three observations are worth noting. First, we see how efficiency prevails without binding limited enforcement constraints, $(\nu_b(x, s) = \nu_l(x, s) = 0)$; in particular, since preferences are separable and 'log' for consumption, labor is decreasing in x (due to 'wealth effects') and consumption is not disrupted by the moral hazard constraint and, in fact, consumption is equal to the relative Pareto weight, $c = x$ and, given borrower's relative impatience, $x'_{xs}(s') = \eta x < x$, so that future relative Pareto weights (and consumptions) monotonically decrease

[14]The specific stochastic processes and functional forms are described in Section 4.

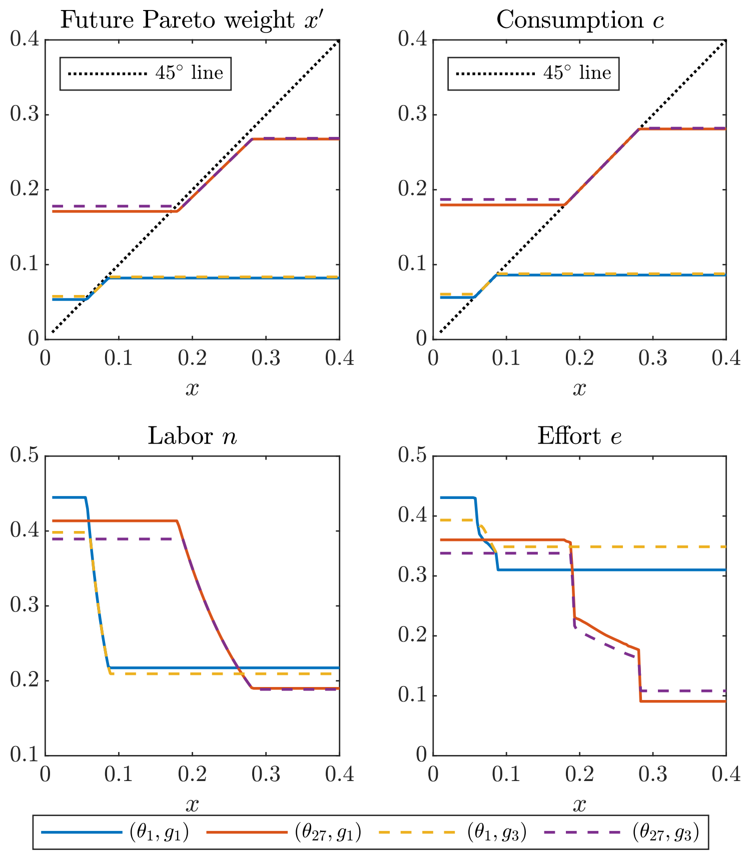

Figure 1: Fund policies with extreme shocks

Notes: θ_1 and θ_{27} refer to the worst and best productivity shock states respctively, while g_1 and g_3 refer to the worst and best government consumption shock states respectively.

over time until the borrower's limited enforcement constraint binds. Similarly, the lender's limited enforcement constraints deter x from being too high, defining the horizontal lines to the right of the 'decay line' of slop η. Note that the upper-left panel of Figure 1, displaying the $x'(x, s)$ policy is *the engine* determining the Fund contract policies. This brings us to the second observation: how efficient policies are disrupted by limited enforcement constraints, making them insensitive for too low or too high values of x, given s and, as a result, reversing some effects – for example, labor being higher at lower productivity levels (lower-left panel of Figure 1).

In general, the figure shows that government expenditure shocks g play a smaller role, with the exception of effort, compared to productivity shocks θ. This is the third relevant observation: how effort behaves and how limited commitment and dynamic moral hazard frictions interact. Regarding the former, we see that the 'wealth effect' dominates: the country exerts more effort in bad productivity states and in bad expenditure shock states, in which case the country enhances the probability of moving to a good expenditure shock state to relieve the pressure on consumption. Regarding the latter, we can better characterize (also analytically) how the interaction of constraints determines the dynamics of consumption in our model.

First, using the optimality condition with respect to consumption in (12) and the definition of $\varphi(s'|s,e)$, we can express the inverse of the marginal utility of consumption next period as:

$$\frac{1}{u'(c(x'_{xs}(s'),s'))}\frac{1+\nu_l(x'_{xs}(s'),s')}{1+\nu_b(x'_{xs}(s'),s')} = \eta\left[\frac{1}{u'(c(x,s))} + \chi(x,s)\frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}\right]. \tag{19}$$

Note that this condition relates consumption in two consecutive periods. If neither limited enforcement is binding next period and there is no moral hazard ($\chi(x,s) = 0$) consumption follows a deterministic path determined by η. As we have discussed above, In our environment of relatively impatient borrowers, this implies declining consumption over time. Moral hazard introduces state-contingency in this intertemporal pattern. In particular, consumption is adjusted upwards (downwards) in states that provide a positive (negative) signal about the borrower's effort through the likelihood ratios. Under our monotonicity assumption, this implies that for high (low) g realizations consumption will decrease (increase), ceteris paribus. Finally, if this path reaches a violation of the borrower's (Fund's) limited enforcement constraints, consumption is adjusted upwards as ν_b is positive (downwards as ν_l is positive). Note also that the presence of moral hazard introduces a wedge between the marginal rates of substitution of the borrower and the lender even if the enforcement constraints are not binding. In particular, if $V^{bf}(x'_{xs}(s'),s') > V^o(s')$ and $V^{lf}(x'_{xs}(s'),s') > Z$ equation (19) implies:

$$\beta\frac{u'(c(x'_{xs}(s'),s'))}{u'(c(x,s))}\left[1+\chi(x,s)u'(c(x,s))\frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e\,(x,s))}\right] = \frac{1}{1+r} \tag{20}$$

The following result characterizes the inverse Euler condition in our setting.

Lemma 4. *In a recursive constrained-efficient Fund contract the inverse Euler equation takes the following form:*

$$\sum_{s'|s}\pi(s'|s,e)\left[\frac{1}{u'(c(x'_{xs}(s'),s'))}\frac{1+\nu_l(x'_{xs}(s'),s')}{1+\nu_b(x'_{xs}(s'),s')}\right] = \eta\frac{1}{u'(c(x,s))} \tag{21}$$

Proof. See Appendix A. □

In (constrained or unconstrained) dynamic social planning/mechanism design problems, the inverse Euler equation characterizes the intertemporal allocation of consumption. In an unconstrained efficient allocation, this is equivalent to the traditional individual Euler equation that is a result of optimal individual intertemporal consumption choice. Versions of this equation have been derived both for dynamic moral hazard models (see **?**) and for dynamic adverse selection problems (see e.g. **?**). Our version of this equation, (21) embeds the inverse Euler equations of other problems if limited enforcement constraints are never binding, in which case (21) implies:

$$\mathbb{E}\left[\frac{1}{u'(c(x'_{xs}(s'), s'))}\bigg| s\right] \leq \frac{1}{u'(c(x, s))}, \tag{22}$$

with strict inequality if $\eta < 1$. In this latter case, it follows that he inverse of the marginal utility process is a 'positive supermartingale'. Therefore, by the *supermartingale theorem*, it converges almost surely to $-\infty$ without a borrower's limited enforcement constraint, which is the well-known *immiseration* result (see **?** and **?**). Alternatively, if $\eta = 1$, and there is only borrower's one-sided limited commitment (i.e., $\nu_l = 0$), (22) is a 'left bounded positive submartingale' which, without moral hazard, would lead to consumption increase and converge to the level of consumption given by $\underline{x}(s(N))$. In general, limited enforcement constraints of the borrower prevent the immiseration in this environment and put a lower bound on the supermartingale. In sum, in our formulation with $\eta < 1$, two sided-limited commitment and moral-hazard, the (inverse) marginal utility process is characterized by the binding *limited enforcement constraints* recurrently truncating the positive supermartingales processes that are perturbed every period due to moral hazard constraints.

2.2 The Economy with *Incomplete Markets and Default* (IMD)

We now describe the economy with incomplete markets and sovereign debt financing with possible default. This is our second benchmark economy, which plays three roles in our analysis. First, we use this economy as the status quo and we therefore calibrate it to euro area 'stressed countries' — in other words, the *risk assessment* of these countries is done with the IMD model economy. Second, as we have discussed above, the outside option of the borrower in the Fund economy is equivalent to the endogenous outside option of the borrower in the IMD economy. Third, we compare this benchmark economy with the economy with a Fund, to assess the value of introducing this fund in the euro area. The incomplete market model with default is a quantitative version of the seminal model by **?** with endogenous labor supply, policy effort, long-term bonds, and an asymmetric default penalty, to achieve a more complete description of the business cycle dynamics of a small open economy with sovereign debt.[15]

With sovereign debt financing, the borrower can issue or purchase *long-term* bonds, which promise to pay constant cash flows across different states. We model long-term bonds in the same way as **?**. A unit of long-term bond is parameterized by (δ, κ), where δ is the probability of continuing to pay out the coupon in the current period, and κ is the coupon rate. Alternatively, $1 - \delta$ is the probability of maturing in the current period, and this event is independent over time. The coupon rate κ provides a flexible way to capture the coupon payment, where $\delta\kappa$ equals to the expected coupon payment on each unit of outstanding debt.

[15] One can consider alternative benchmark economies where, defaults and/or the bond long-term structure play a larger role as contingencies, as in **?** or **?**. Nevertheless, to focus on the Fund mechanism it helps to have a simpler benchmark economy.

Note that δ directly captures the maturity of the bond, namely, if $\delta = 0$ and $\kappa = 0$, the bond becomes the standard one-period debt — as in **?** and **?** — and, in general, the average maturity of the bond equals to $1/(1-\delta)$, which is increasing in δ. By a purchase of one bond we mean, more precisely, the purchase of one unit of a portfolio of a continuum of bonds of infinitesimal size and the same (δ, κ), but with independent realizations within the portfolio. Thus, one unit of bond (δ, κ) repays $(1-\delta) + \delta\kappa$ in any given period (as long as the borrower does not decide to default).

Since the setup of the IMD model is standard, we give only a brief description in what follows. Let b be the size of the long-term bond portfolio held by the borrower at the beginning of a period,[16] and (s, b), $s = (\theta, g)$, be the state. Let $V_n^b(b, s)$ denote the value function of the borrower when the borrower chooses *not* to default. Then it satisfies:

$$V_n^b(b,s) = \max_{c,n,e,b'} U(c,n,e) + \beta \mathbb{E}\big[V^b(b',s')\big|s,e\big] \qquad (23)$$
$$\text{s.t. } c + g + q(s,b')(b' - \delta b) \le \theta f(n) + (1 - \delta + \delta\kappa)b,$$

where $V^b(b', s')$ denotes the continuation value. When the borrower chooses to default, the value function $V^o(s)$ satisfies

$$V^o(s) = \max_{n,e} \big\{u(\theta^p(\theta)f(n) - g) + h(1-n) - v(e)\big\} + \beta\mathbb{E}\big[(1-\lambda)V^o(s') + \lambda V_n^b(0,s')\big|s,e\big], \quad (24)$$

where $\theta^p(\theta)$ denotes the productivity net of a penalty, and λ is the probability to come back to the market and be able to borrow again, starting with 0 debt. As is standard in the literature, his outside option is temporary autarky. Moreover, $V^o(s)$ also represents the outside option that the borrower faces in the Fund contract when she contemplates whether to default or not.[17] Finally, the default choice is given by:

$$D(s,b) = 1 \text{ if } V^o(s) > V_n^b(b,s) \text{ and } 0 \text{ otherwise},$$

where $D(s, b) = 1$ denotes default. It follows that borrower's value function, *prior to* the default decision, is

$$V^b(b,s) = \max\big\{V_n^b(b,s), V^o(s)\big\}, \qquad (25)$$

When the borrower chooses not to default, the optimality condition with respect to effort takes the following form:

$$v'(e) = \beta \sum_{s'|s} \partial_e \pi(s',s) V^b(b',s'). \qquad (26)$$

This equation has a similar form as the incentive compatibility constraint (2) and the same

[16] We assume that $b \in [b_{\min}, b_{\max}]$, with $-\infty < b_{\min} < 0 \le b_{\max} < \infty$., where we will choose $b_{\min}$ and $b_{\max}$ so that in equilibrium the bounds are not binding.

[17] This implies, in particular, that once the borrower defaults with the Fund, she will be in autarky for at least one period, and will re-enter the incomplete credit market with probability λ at zero debt.

interpretation. Moreover, this condition implies that the optimal effort decision only depends on b through b', hence we can write the policy function as $e(s, b')$. This simplifies considerably the pricing equation of the bond and consequently our computations.

Denoting the expected default rate by $d(s, b') = \mathbb{E}[D(s', b')|s, e(s, b')]$, the equilibrium bond pricing function $q(s, b')$ satisfies the following recursive equation:

$$q(s,b') = \frac{(1-\delta)+\delta\kappa}{1+r}(1-d(s,b')) + \delta\frac{\mathbb{E}[(1-D(s',b'))q(s',b''(s',b')|s,e(s,b')]}{1+r} \tag{27}$$

Note that, for a one-period bond ($\delta = 0$), this would reduce to the more familiar expression $q(s, b') = \frac{1-d(s,b')}{1+r}$. Note also that the price of a *riskless* long-term bond (δ, κ) is $q = \frac{(1-\delta)+\delta\kappa}{r+1-\delta}$. Furthermore, the implied interest rate on a risky bond is given by

$$r^i(s,b') = \frac{(1-\delta)+\delta\kappa}{q(s,b')} - (1-\delta) \tag{28}$$

resulting in a *positive spread* $r^i(s, b') - r \geq 0$, strictly positive if $d(s, b') > 0$ for some b'.

The optimal policies when there is no default $(c(s, b), n(s, b), b'(s, b), e(s, b'(s, b))$ and those when there is default $(n^a(s), e^a(s))$ are standard dynamic programming solutions to (23)–(25), whereas the bond price $q(s, b')$ and implied interest rate $r^i(s, b')$ are a solution to (27) and (28) respectively. Finally, in order to keep track of debt flows and in order to compare with a counterpart for c_l in the Fund contract, it will be useful to define the primary surplus of the borrower, which is also the transfer to the lender, as:

$$c_l^i(s,b) = \theta f(n(s,b)) - (c(s,b)+g) = q(s,b')(b'-\delta b) - (1-\delta+\delta\kappa)b. \tag{29}$$

In essence, if the country consumes more than it produces, $c_l^i(s, b) < 0$, we say that the country is running a deficit, whereas the country runs a surplus if the opposite. In this sense, we will call $c_l^i(s, b)$ primary surplus (or primary deficit if negative). Here, it is important to note that, in our economy, taxes (and transfers) are implicitly defined by $\theta f(n) - c$. This implies that (29) defines both the primary surplus of the government and the net exports. The two key assumptions behind this equivalence are that only the government has access to any intertemporal borrowing/saving technology and we do not have physical capital accumulation in our model.

3 Decentralization of the Fund Contract

In what follows, we show how to decentralize the optimal Fund contract as a competitive equilibrium with endogenous borrowing constraints and taxes on assets. This will allow us to compare the Fund contract more directly with the debt contract of the economy with sovereign debt. To do this, we build on the work of **?** and **?** on limited commitment models, but we consider long-term *state-contingent bonds (assets or securities)* to make it more comparable with the incomplete market model. Our decentralization is also related to the new dynamic public finance literature where they show that taxes on capital income or capital holdings are required to provide for efficient incentive provision (see e.g. **?**). To our knowledge, our paper is the first one that provides a decentralization to a constrained-efficient allocation that has both limited commitment and dynamic moral hazard frictions. Our decentralization will also highlight why it is plausible to believe that private markets would have a difficulty to provide the allocation of the Fund.

3.1 Asset Structure

At the beginning of a period, in state s, the borrower holds a portfolio a of securities (δ, κ), where a fraction $1-\delta$ of the portfolio matures in the current period and a fraction δ pays a coupon κ. The borrower can trade in S securities $a'(s')$ with a unit price of $q(s'|s)$; and $a'(s')$ pays *corresponding units of asset* next period only if state s' is realized. The borrower is subject to state contingent taxes $\tau^a(s')$ on the 'Arrow security' holdings and it receives a lump sum transfer $\overline{\tau}(s)$ (that make these taxes budget neutral in equilibrium). We will discuss the role of these taxes in the equilibrium in Section 3.3. As in the IMD economy, the borrower chooses the amount of net debt issuance $q(s'|s)(a'(s')(1+\tau^a(s'))-\delta a)$. Therefore, the borrower's budget constraint is:[18]

$$c + \sum_{s'|s} q(s'|s)(a'(s')(1+\tau^a(s')) - \delta a) \leq \theta(s) f(n) - g(s) + (1-\delta+\delta\kappa)a + \overline{\tau}(s).$$

To make the model as comparable as possible to the IMD economy, we note that the state contingent portfolio can be decomposed into (i) a common 'bond' $\bar{a}'$ that is carried to the next period, is independent of the next period state and is traded at the implicit bond price $q(s) = \sum_{s'|s} q(s'|s)$, and (ii) an insurance portfolio of S assets $\hat{a}(s')$, with $\hat{a}(s') = a'(s') - \bar{a}'$, $\bar{a}' = \sum_{s'|s} q(s'|s)a'(s')/q(s)$ and hence $\sum_{s'|s} q(s'|s)\hat{a}(s') = 0$. The budget constraint can then be rewritten as:

$$c + q(s)(\bar{a}' - \delta a) + \sum_{s'|s} q(s'|s)\hat{a}(s') + \sum_{s'|s} q(s'|s)a'(s')\tau^a(s') \leq \theta(s) f(n) - g(s) + (1-\delta+\delta\kappa)a + \bar{\tau}(s). \quad (30)$$

As in other 'decentralization problems' alternative forms of asset market structures could potentially deliver the same allocation (e.g., **?**). However, our main purpose here is to have clear comparison between the two regimes and this asset structure works well for that purpose, since $(a, \bar{a}')$ can be 'identified' with (b, b') in the IMD economy, while $\hat{a}(s')$ corresponds to the additional insurance component provided by the Arrow securities. In addition, we can use the bond price of this equilibrium $q(s)$ to compute spreads in this economy, which can be

[18] From now on, to simplify the notation, we simply write $q(s'|s)$ instead of $q(s'|a, s)$.

compared with the spreads generated by the IMD economy.

3.2 The Recursive Competitive Equilibrium (RCE)

With the above financial structure we can characterize the equilibrium in the economy with the Fund as a *recursive competitive equilibrium* (in strict sense, a partial equilibrium since the world interest rate is given as the opportunity cost of the lender). In this formulation, the borrower has access to long-term state-contingent assets and solves the following dynamic programming problem:

$$W^b(a,s) = \max_{\{c,n,e,a'(s')\}} U(c,n,e) + \beta \mathbb{E}\left[W^b(a'(s'),s')\big|s,e\right] \quad \text{s.t.} \tag{31}$$

$$c + \sum_{s'|s} q(s'|s)(a'(s')(1+\tau^a(s')) - \delta a) \leq \theta(s)f(n) - g(s) + (1-\delta+\delta\kappa)a(s) + \overline{\tau}(s), \tag{32}$$

$$a'(s') \geq \mathcal{A}_b(s'), \tag{33}$$

where $\mathcal{A}_b(s')$ is an endogenous borrowing constraint that will be defined below. The policies that solve this problem are denoted by $c(a,s)$, $n(a,s)$, $e(a,s)$, and $a'_{as}(s') \equiv a'(a,s,s')$.

The first-order conditions of (31), with respect to the choice of consumption, labour and effort are given by:

$$u'(c(a,s)) = \lambda(a,s), \tag{34}$$

$$\frac{h'(1-n(a,s))}{u'(c(a,s))} = \theta(s)f'(n(a,s)), \tag{35}$$

$$v'(e(a,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s,e) W^b(a'_{as}(s'),s'), \tag{36}$$

where $\lambda(a,s)$ is the Lagrange multiplier of the intertemporal budget constraint of the borrower. Let $\gamma_b(a,s)$ be Lagrange multiplier on the borrowing constraint,[19] and let

$$A(s') = (1-\delta+\delta k) + \delta q(s'), \quad \text{with } q(s') = \sum_{s''|s'} q(s''|s').$$

Then the first order condition with respect to asset holdings is given by: [20]

$$q(s'|s) = \beta\pi(s'|s,e(a,s))\frac{u'(c(a'_{as}(s'),s'))}{u'(c(a,s))}\frac{1}{1+\tau^a(s')}A(s') + \frac{\gamma_b(a'_{as}(s'),s')}{u'(c(a,s))(1+\tau^a(s'))}, \tag{37}$$

where $\gamma_b(a'_{as}(s'),s') \geq 0$, with $\gamma_b(a'_{as}(s'),s') = 0$ if $a'_{as}(s') > \mathcal{A}_b(s')$.

[19] This $\gamma_b(a,s)$ shall be distinguished from the multipliers to borrower's limited enforcement constraints used in the recursive formulation of the Fund in Section 2.1.2. The same applies below for $\gamma_l(a,s)$ used for lender's borrowing constraint.

[20] Equilibrium prices in RCE also depend on a, a state variable in addition to s. To simplify notation, we suppress a in $q(\cdot)$.

The lender (i.e., the Fund), who has linear preferences for — possibly, negative — consumption solves the following problem:

$$W^l(a_l, s) = \max_{\{c_l, a_l'(s')\}} c_l + \frac{1}{1+r}\mathbb{E}\big[W^l(a_l'(s'), s')\big|s, e\big] \tag{38}$$

$$\text{s.t. } c_l + \sum_{s'|s} q(s'|s)(a_l'(s') - \delta a_l) = (1 - \delta + \delta\kappa)a_l, \tag{39}$$

$$a_l'(s') \geq \mathcal{A}_l(s'), \tag{40}$$

where $\mathcal{A}_l(s')$ is an endogenous borrowing constraint that will be defined below. The policies that solve this problem are denoted by $c_l(a, s)$ and $a'_{l,as}(s') \equiv a_l'(a, s, s')$. [21] Let $\gamma_l(a, s)$ be Lagrange multiplier on the borrowing constraint. The optimality condition with respect to asset holdings implies:

$$q(s'|s) = \frac{1}{1+r}\pi(s'|s, e(a, s))A(s') + \gamma_l(a'_{l,as}(s'), s'), \tag{41}$$

with $\gamma_l(a'_{l,as}(s'), s') \geq 0$ and $\gamma_l(a'_{l,as}(s'), s') = 0$ if $a'_{l,as}(s') > \mathcal{A}_l(s')$.

In the competitive equilibrium, the asset market and goods market clearing conditions are given by:

$$a'_{as}(s') + a'_{l,as}(s') = 0, \tag{42}$$

$$c(a, s) + c_l(a, s) = \theta(s)f(n(a, s)) - g(s), \tag{43}$$

with the initial asset holdings $a(s_0)$ and $a_l(s_0) = -a(s_0)$ given.

The transfers are determined such that the government's budget constraint is cleared period by period:

$$\overline{\tau}(s) = \sum_{s'|s} q(s'|s)a'_{as}(s')\tau^a(s') \tag{44}$$

Definition 2 (Recursive Competitive Equilibrium)**.** Given initial asset holdings $\{a(s_0), a_l(s_0)\}$, borrowing limits $\{\mathcal{A}_b(s'), \mathcal{A}_l(s')\}$, and taxes and transfers $\{\tau^a(s'), \overline{\tau}(s)\}$, a *recursive competitive equilibrium* consists of policies functions for the allocations

$$\{c(a, s), n(a, s), e(a, s), a'_{as}(s'), c_l(a, s), a'_{l,as}(s')\},$$

prices $q(s'|s)$, value functions $\{W^b(a, s), W^l(a, s)\}$, and laws of motion for $\{\theta, g\}_{e(a,s)}$ such that: (i) Given the taxes, transfers, outside options and asset prices $q(s'|s)$, the policies for the allocations $\{c(a, s), n(a, s), e(a, s), a'_{as}(s')\}$, together with the value function $W^b(a, s)$, solve the borrower's problem (31) given $\mathcal{A}_b(s')$, and the allocations $\{c_l(a, s), a'_{l,as}(s')\}$, together with the value function $W^l(a, s)$, solve the lender's problem (38) given $\mathcal{A}_l(s')$; and (ii) the

[21] To ease the notation, we have used the fact of $a_l = a$ as required by market clearing condition. This allows us to write the policy function of the lender as a function of a instead of a_l.

market clearing conditions and government's budget constraint in (42)–(44) are satisfied.

3.3 Implementation of the Fund Contract as a RCE

We now show how the recursive constrained-efficient Fund contract can be decentralized as a recursive competitive equilibrium with long-term state contingent assets, state contingent taxes on the assets, and borrowing limits that are not too tight: a version of the Second Welfare Theorem. This allows us to obtain asset prices and holdings supporting the Fund contract, which we can compare to the debt prices and holdings of the incomplete markets economy. Moreover, we show that, with the right taxes, the recursive competitive equilibrium with borrowing limits that are not too tight is also a recursive constrained-efficient Fund allocation: a version of the First Welfare Theorem.

Before we state the welfare theorems, it is useful to see that we can define the corresponding Arrow security price in the competitive equilibrium using (37) and (41) as follows:

$$q(s'|s) = \pi(s'|s, e(a,s))A(s') \max\left\{\beta \frac{u'(c(a'_{as}(s'), s'))}{u'(c(a,s))}\frac{1}{1+\tau^a(s')}, \frac{1}{1+r}\right\}, \tag{45}$$

where $A(s) = (1-\delta+\delta\kappa) + \delta q(s)$ and $q(s) = \sum_{s'|s} q(s'|s)$. Using the Arrow security prices, we can then define the intertemporal discount factor as:

$$Q(s'|s) = \frac{q(s'|s)}{A(s')}. \tag{46}$$

We are now ready to state the two welfare theorems, whose proofs are relegated to Appendix A.

Proposition 1 (Second Welfare Theorem). *Given initial conditions $(s_0,\, x(s_0))$, outside options $\{V^o(s), Z(s)\}$, and a constrained efficient recursive Fund contract with policy functions for the allocions $\{c(x,s), c_l(x,s), n(x,s), e(x,s)\}$ and multipliers $\{\nu_l(x,s), \nu_b(x,s), \varrho(x,s)\}$, value functions $\{V^{bf}(x,s), V^{lf}(x,s)\}$, and laws of motion for $\{\theta, g\}_{e(x,s)}$ and relative Pareto weight $x'_{xs}(s')$, there are: uniquely defined value functions $\{W^b(a,s), W^l(a,s)\}$ allocation $\{c(a,s), c_l(a,s), n(a,s), e(a,s)\}$ and asset policies $\{a'_{as}(s'), a'_{l,as}(s')\}$ with an initial allocation of asset holdings $(a(s_0), a_l(s_0)$, asset prices $q(s'|s)$ and asset taxes $\{\tau^a(s'), \overline{\tau}(s)\}$, borrowing limits $\{\mathcal{A}_b(s'), \mathcal{A}_l(s')\}$, and a law of motion for $\{\theta, g\}_{e(a,s)}$, which constitute a recursive competitive equilibrium.*

Proposition 1 shows that the *recursive Fund contract* with initial Pareto weights (μ_{b0}, μ_{l0}) can be 'decentralized' as a *recursive competitive equilibrium* with 'Arrow security' taxes and endogenous borrowing constraints with specific initial asset holdings $a_0(s_0)$ and $a^l_0(s_0) = -a_0(s_0)$. There are four key steps in the proof of Proposition 1, which are worth mentioning. First, one can get (34) from (12), i.e.,

$$\frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x} = u'(c(x,s)) = u'(c(a,s)) = \lambda(a,s),$$

and, in doing so, a one-to-one map between multipliers and states — (x, s) and (a, s)— is established. Second, using Lemma 4 one can obtain $q(s'|s)$ in (45) for given Arrow securities taxes $\tau^a(s')$. Third, one can define these taxes by letting:

$$\frac{1}{1+\tau^a(s')} = 1 + \chi(x,s)u'(c(x,s))\frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}, \tag{47}$$

where $\chi(x, s)$ is given by equation (16). Finally, the endogenous borrowing constraints are determined at the level that just keeps the borrower (or the lender) indifferent between honoring its obligations or taking its outside options (similar to **?**).

Note that the right-hand side of (47) can also be written as $1 + \frac{\varphi(s'|s,e)}{1+\nu_b(x,s)}$, and the asset price equation (45) as:

$$\begin{aligned} q(s'|s) &= \frac{1}{1+r}\pi(s'|s,e(x,s))A(s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')}\frac{1}{1+\frac{\varphi(s'|s,e(x,s))}{1+\nu_b(x,s)}}\frac{1}{1+\tau^a(s')}, 1\right\} \\ &= \frac{1}{1+r}\pi(s'|s,e(x,s))A(s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')}, 1\right\}, \end{aligned}$$

where the last equality follows from the definition of taxes (47). It follows that there will be a *negative spread* if and only if the Fund's limited enforcement constraint is binding; i.e. $\nu_l(x', s') > 0$. In other words, asset taxes (transfers) absorb all the price variability needed to enforce the incentive compatibility constraint, and asset prices are decoupled from moral hazard considerations. Let us briefly discuss the role of asset taxes in this decentralization. It is clear that taxes need to satisfy (47). Since effort is not contractable in our economies, the Fund imposes the incentive compatibility constraint, which creates a wedge between the intertemporal rate of substitution of the borrower and the lender in the Fund allocation (see Lemma 4). In the equilibrium, the taxes on Arrow securities directly account for this wedge. Note that these taxes also guarantee that similar to the Fund allocation, the inverse Euler equation (21) holds in equilibrium as well.[22] Given these taxes, it is not necessary to impose incentive compatibility as a constraint in the borrower's problem in equilibrium when we decentralize the Fund allocation. This is in contrast to **?**, who constrain arbitrarily the contract space of competitive equilibria with the incentive compatibility constraint to decentralize constrained-efficient allocations with moral hazard.[23] We consider this decentralization as a relevant theoretical contribution of the present paper.

The presence of these taxes in the competitive equilibrium that decentralizes the Fund allocation also highlights why it is not straightforward to deliver this allocation through private international capital markets for sovereigns. First, typically, private international lending contracts do not have legal power to impose state-contingent taxes in the domestic

[22]Some form of wealth taxes are used often in the dynamic public finance literature (see e.g., **?**, **??**) to align equilibrium incentives with the inverse Euler equation. Those models exhibit private information instead of private actions and hence taxes take a different form.

[23]See also **?**.

countries. Second, although these taxes are ex ante optimal taking into account the lender and the borrower's preferences, it is not obvious that the borrower will have an incentive to impose them unilaterally given the conditions available on the financial markets.

The 'decentralization' above allows us to compare prices and asset allocations in the economy with the Fund and in the economy with incomplete markets and default (IMD). To do this, we let $Q^*(s) = \sum_{s'|s} Q^*(s'|s)$, where $Q^*(s'|s)$ is defined as in (46) using the equilibrium prices $q(s', s)$. The implicit interest rate in the decentralized economy can be obtained from the price of the long-term bond:

$$r^*(s) = \frac{1}{Q^*(s)} - 1,$$

which results in a possibly *negative spread*, $r^*(s) - r \leq 0$, since $Q^*(s) \geq \frac{1}{1+r}$ due to (46) and (45).

To understand the negative spread, consider first the case with *no moral hazard*, implying that $\chi(x, s) = \tau^a(s'|x, s) = 0$ for all s'. Looking at the expression for $q(s'|s)$, it is clear that the *negative spread* in this case reflects the fact that the lender's intertemporal limited enforcement constraint is binding for some state tomorrow, that is, $Q^*(s) > \frac{1}{1+r}$ only if $\nu_l(x'(s'), s') > 0$ for some s'. In that state, the borrower's Pareto weight, $x'(s')$, is typically relatively high given the state s', and given this, the borrower's liabilities are in risk to become permanent transfers, i.e., the Fund is in danger of making permanent losses. The negative spread then discourages the Fund from lending, since the lender is better off lending (saving) at the riskless interest rate r in the international market — i.e., the negative spread indirectly imposes a constraint on the amount of insurance the borrower can get. Consider now the case with *moral hazard*. In this case, whenever no limited enforcement constraint is binding the asset taxes make sure that the intertemporal rate of substitution is equalized across states (see (47) and (19)) and hence equal to $\frac{1}{1+r}$. This implies that exactly the same argument holds as above. In sum, *the negative spread*, $r^*(s) - r < 0$, reflects the *wedge* that aligns the market price with the lender's unwillingness to lend in some states of the future.

We now show that a recursive competitive equilibrium allocation with borrowing limits that are not too tight also implements a recursive constraint efficient Fund allocation as long as taxes can create the right wedge between the borrower's and the lender's marginal rates of substitution in order to internalize the full social cost and benefit of effort present in the Fund.

Proposition 2 (First Welfare Theorem). *Given initial asset holdings $\{a(s_0), a_l(s_0)\}$, taxes and transfers $\{\tau^a(s'), \overline{\tau}(s)\}$, and a recursive competitive equilibrium with policy functions for the allocations $\{c(a, s), n(a, s), e(a, s), a'_{as}(s'), c_l(a, s), a'_{l,as}(s')\}$, laws of motion for $\{\theta, g\}_{e(a,s)}$, prices $q(s'|s)$, value functions $\{W^b(a, s), W^l(a, s)\}$, and borrowing limits $\{\mathcal{A}_b(s'), \mathcal{A}_l(s')\}$ that are not too tight with respect to the outside options $\{V^o(s_t), Z(s)\}$, then there exists a recursive constrained-efficient Fund contract with an initial condition $x(s_0)$, laws of motion $\{\theta, g\}_{e(x,s)}$*

and $x'_{xs}(s')$, and multipliers $\{\nu_l(x,s), \nu_b(x,s), \varrho(x,s)\}$, implementing the same allocation with policies $\{c(x,s), c_l(x,s), n(x,s), e(x,s)\}$, as long as taxes satisfy equation (47).

To prove Proposition 2, we need to revert the steps, i.e., we need to go from the asset prices and optimal policies of the recursive competitive equilibrium to the optimal policies and the structure of the Fund contract to show that, provided that a recursive competitive equilibrium exists, there is a corresponding *constrained-efficient Fund contract*. This result extends the one in **?** to an economy where the borrower decides the level of effort by adding taxes on Arrow securities that create a wedge between the marginal valuations of the borrower and the lender to account for the full social value of effort in the Fund. In other words, in the presence of *moral hazard*, the Arrow security taxes make the borrower to endogenize the social effect of effort by changing his marginal valuation.

In order to decentralize the Fund as an equilibrium, it is useful to think about it as an intermediary transforming uncontingent defaultable debt into a state-contingent non-defaultable contract. As a result, the Fund has a safe asset in its balance sheet and, therefore, can issue safe non-contingent bonds in the international capital markets to finance its loans to the debtor country. In fact, in our quantitative exercise, we will take a representative economy with a given level of liabilities under incomplete markets (IMD) and study how its economic performance is affected if the Fund takes over its debt.

4 Calibration

4.1 Functional Forms and Parameter Values

We calibrate the model parameters so that the IMD economy with defaultable debt is representative of the four 'stressed countries' in the European debt crisis, i.e., Portugal, Italy, Greece, and Spain (henceforth GIPS), over the period 1980–2015. We target key data moments by taking the average across the GIPS countries. The model period is assumed to be one year. The utility of the borrower is additively separable in consumption, leisure and effort. In particular, we assume that $u(c) = \log(c)$, $h(1-n) = \gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma}$ and $v(e) = \omega e^2$ so that:

$$U(c,n,e) = \log(c) + \gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma} - \omega e^2$$

The preference parameters (σ, γ) are set to $\sigma = 0.34$ and $\gamma = 1.734$. These are used to match the average fraction of working hours, together with the volatility of labor relative to GDP, of the GIPS countries. The effort cost parameter is set to $\omega = 0.0087$, and this choice is explained below with the specification the government expenditure shock. The risk free interest rate is set to $r = 2.48\%$ to match the average real short-term interest rates of the Euro area.

The parameters of the long-term bond (δ, κ) are set to $\delta = 0.814$ and $\kappa = 0.083$ to match the average maturity and the average coupon rate (coupon payment to debt ratio) of long-term debt for the GIPS countries. After a country defaults in the IMD economy, it faces exclusion for a random number of periods, and the probability that it comes back to the market with sovereign debt upon default is set to be $\lambda = 0.264$. Moreover, if a country defaults in the IMD economy, it is subject to an asymmetric default penalty of the form (?):

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.189$. The latter two parameters (λ, ψ), together with the discount factor $\beta = 0.929$ are chosen to match jointly the spread level, the spread volatility and the average debt to GDP ratio. Note that this implies a different discount factor for the lender of $\frac{1}{1+r} = 0.9758$, as well as a growth rate for the relative Pareto weight of the borrower of $\eta = \beta(1+r) = 0.9684$ in the optimal contract. The fact that the borrower is less patient than the lender implies that the borrower will tend to get indebted in both economies. As it is well known, in the absence of any frictions (limited commitment or moral hazard) consumption of the borrower would converge towards zero in the long run.

In the Fund economy, the limited enforcement constraint of the lender is set to $Z = 0$, implying that there will be no expected permanent transfers between the borrower and the lender at any time or state. In other words, the Fund is not build on an assumption of solidarity, which would require permanent transfers.

We assume that $f(n) = n^{\alpha}$ with the labor share of the borrower set to $\alpha = 0.566$ to match the average labor share across the GIPS countries. Table 1 summarizes the parameter values, and Appendix B contains additional information regarding data sources.

4.2 Shock Processes

4.2.1 Productivity Shock

We start by constructing the model consistent measure of productivity. The original productivity θ^o_{it} of country i in year t equals to real output divided by total working hours to the power α. We further detrend $\{\log \theta^o_{it}\}$ with a country specific linear trend, and then normalize the detrended series to the same mean and volatility across the GIPS countries. The final productivity series $\{\log \theta_{it}\}$ is homogeneous across i, and it can be viewed as repeated samples from the same data generating process, which is representative for the GIPS. Finally, we estimate the following panel Markov regime switching AR(1) model:

$$\log \theta_{it} = (1 - \rho(\varsigma_{it}))\mu(\varsigma_{it}) + \rho(\varsigma_{it}) \log \theta_{it-1} + \sigma(\varsigma_{it})\varepsilon_{it},$$

where $\varsigma_{it} = 1, 2, 3$ denotes the regime of country i at time t, $\mu(\varsigma_{it})$, $\rho(\varsigma_{it})$, and $\sigma(\varsigma_{it})$ are the regime-specific parameters, and $\varepsilon_{it} \overset{\text{iid}}{\sim} N(0, 1)$. Regime ς_{it} follows a Markov chain with a 3×3 transition matrix P. Table 2 displays the estimated parameters of the process. As shown

Table 1: Parameter values

Parameter	Value	Definition	Target moment
A. Direct measures from data			
α	0.566	labor share	average labor share
r	0.0248	risk-free rate	Euro area short-term risk-free rate
δ	0.814	bond maturity	average bond maturity
κ	0.083	bond coupon rate	average bond coupon rate
B. Based on model solution			
β	0.929	discount factor	average b/y
λ	0.264	return probability	average and volatility of spread
ψ_0	0.189	productivity penalty	
σ	0.34	labor elasticity	average n and $\sigma(n)/\sigma(y)$
γ	1.734	labor utility weight	
ϕ	0.975		
ϖ	0.005	g distribution	average, 1 and 99 percentile
w	0.72		of g/y; $\rho(g,y)$, $\sigma(g)/\sigma(y)$,
g_1	0.0385		
g_2	0.0315	g grid points	and $\sigma(ps/y)/\sigma(y)$
g_3	0.0285		
ω	0.0087	effort disutility weight	$\mathbb{E}\zeta(e) = 0.5$
C. By assumption			
Z	0	Fund's outside value	

Notes: Apendix B contains details on data sources; parameters in panel B are calibrated jointly, with groups indicating main sources of identification; and ps denotes primary surplus.

in the table, regime 1 has the lowest conditional mean and the highest conditional volatility. Thus, we can interpret regime 1 as a 'crisis' regime,[24] regime 3 as normal times, while regime 2 as an intermediate more temporary regime that typically precedes a crisis. A Markov regime switching specification allows us to capture potentially nonlinear dynamics observed in the data, such as the sudden drops of productivity around the financial crisis and the euro debt crisis in a convenient manner.[25] Finally, we discretize the process into a 27-state Markov chain, with 9 grid points in each regime.

4.2.2 Government Consumption Shock

We first explain the parameterization of π^g assuming g to be independent of θ, and then extend baseline specification so that g and θ are correlated in the full quantitative model.

[24] The smoothed regime probabilities shown in the Appendix B confirm that regime 1 concentrates on the European debt crisis periods.

[25] **?** use a similar regime switching model to calibrate heterogeneous productivity dynamics across countries. Focusing on the euro debt crisis, **?** also choose to capture the rich dynamics in the data by a regime switching structure for the shock process.

Table 2: Parameters of the regime switching productivity process

	$\mu(\varsigma)$	$\rho(\varsigma)$	$\sigma(\varsigma)$	P	$\varsigma' = 1$	$\varsigma' = 2$	$\varsigma' = 3$	invariant dist.
$\varsigma = 1$	-0.1893	0.8680	0.0132	$\varsigma = 1$	0.8931	0.0000	0.1069	0.3289
$\varsigma = 2$	0.0118	0.8264	0.0048	$\varsigma = 2$	0.1862	0.8138	0.0000	0.1568
$\varsigma = 3$	0.1060	0.9021	0.0129	$\varsigma = 3$	0.0116	0.0567	0.9317	0.5143

Notes: ς denotes the current regime of productivity shock, and ς' denotes that of the next period.

Independent g and θ as a preliminary step In order to parametrize π^g, we adopt the renowned *spanning condition* of **?** by assuming

$$\pi^g(g'|g, e) = \zeta(e)\pi^l(g'|g) + (1 - \zeta(e))\pi^h(g'|g), \tag{48}$$

where π^l and π^h are two distributions independent of e while $\pi^h(\cdot|g)$ first-order stochastically dominates $\pi^l(\cdot|g)$ for all g, and the weighting function $\zeta(e) = (1 - e)^2 \in (0, 1)$ satisfies $\zeta'(e) < 0$ and $\zeta''(e) < 0$. It is straightforward to verify that the *monotonicity* and *convexity* assumptions on π^g are satisfied by (48).

For a parsimonious parameterization, we assume g to take three values: $g_1 > g_2 > g_2 > 0$. Furthermore, we choose a 2-parameter specification of π^l and π^h,

$$\pi^h = \begin{bmatrix} 2\phi - 1 & \frac{4}{3}(1-\phi) & \frac{2}{3}(1-\phi) \\ 0 & 2(\phi + 2\varpi) - 1 & 2(1 - \phi - 2\varpi) \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 4\varpi & 1 - 4\varpi & 0 \\ 2\varpi & 2(1 - \phi - \varpi) & 2\phi - 1 \end{bmatrix}, \tag{49}$$

and calibrate the parameter values so that when $\mathbb{E}\zeta(e) = 0.5$ holds in the ergodic mean of the IMD model, the *average* transition matrix $\bar{\pi}^g(g'|g) = \bar{\zeta}\pi^l(g'|g) + (1 - \bar{\zeta})\pi^h(g'|g)$ equals to:[26]

$$\bar{\pi}^g = \begin{bmatrix} \phi & \frac{2}{3}(1-\phi) & \frac{1}{3}(1-\phi) \\ 2\varpi & \phi & 1 - \phi - 2\varpi \\ \varpi & 1 - \phi - \varpi & \phi \end{bmatrix}. \tag{50}$$

The relevant parameters of the transition matrix are set to $\phi = 0.965$ and $\varpi = 0.005$, while the state space of g is set to $\{0.0385, 0.0315, 0.0285\}$. In addition, we calibrate effort disutility parameter $\omega = 0.0087$ so that $\bar{\zeta} = \mathbb{E}\zeta(e) = 0.5$. We discuss the target moments below after introducing correlation between g and θ for the full quantitative model.

Correlated g and θ for the full model In order to improve the model fitting of the correlation between g and GDP, in together with the volatility of g and primary surplus relative to GDP respectively, as observed in the data, we assume that g and θ may be correlated in the full quantitative model. To this end, we exploit the Markov regime transition structure in

[26] Note that this specification of the transition matrix is motivated by the one-period-crash Markov chain of **?**. The numerical values of π^l, π^h, and $\bar{\pi}^g$ are displayed in Appendix B.

our calibration of θ: by conditioning the distribution of g' on ς, the regime to which θ belongs, in addition to g and e, it follows that g' and θ' are correlated.[27] Moreover, we introduce one more parameter $w \in [0,1]$ to control for the influence of ς on g': with $w = 0$, g' is independent of ς; while with $w = 1$, g' depends only on ς but no longer on g.

To sum up, as displayed in Table 1, we use 6 parameters related to the disribution of g and the moral hazard setup to match 6 moments in the data: the level and the lower 1 and upper 99 percentiles of g/y, the correlation between g and y, and the relative volatilities of g and the primary surplus with respect to y.

Lastly, to improve the convergence properties of the IMD economy, we follow the practice of **?** by adding a small iid shock to the g shock described as above. In particular, we assume that it is uniformly distributed over $[-\bar{m}, \bar{m}] = [-0.001, 0.001]$ and discretize it into 5 equally spaced grid points over the range with equal probability for each point.

4.3 The IMD Model Fit

Table 3 provides an exhaustive account of our benchmark calibration. To compute the data moments, we first compute the corresponding moments for each country, and then take average across the GIPS countries, resulting in a set of moments representative of the common features of the GIPS countries. Furthermore, for the second moments, we HP filter data series with a filtering parameter 6.25 to extract the business cycle frequency fluctuations for annual data (**?**). To compute the model moments, we execute 50,000 short run simulations of the IMD model with 300 periods each, and we discard the first 100 periods. Similar to the data moments, we HP filter the simulated data to compute the second moments.[28]

The IMD economy matches most moments remarkably well, with the exception of the average primary surplus to GDP. In particular, the model is able to produce a significant amount of debt together with a realistic level, volatility and cross-correlation of spreads, but it generates a positive average primary surplus to GDP. Note that, in any stationary model without growth, whenever there is debt in the long run, we need to have primary surplus which allows the country to pay the interest rate on its debt. This is not true in the data, as the countries in the sample were able to run deficits and increase their debt, possibly expecting growth, given that there is (moderate) growth during the sample period. What is more important than the level for our purpose, however, is that we match well the relative volatility of the primary surplus over GDP, the positive correlation of government and technology shocks, and the positive correlation of primary surplus with GDP, even though this last moment is not targeted. Note that this moderate but positive correlation enhances consumption insurance: resources come in whenever the country's output is low.

As the tables reflect, the model also matches well the moments of consumption and labor,

[27] Appendix B contains the details on the transition probability specification.

[28] Note that there is default in the IMD economy, in which case debt and the primary surplus are zero, by construction, and the spread is not defined. Therefore, all the moments involving the debt to GDP ratio, primary surplus over GDP and the spreads are conditional on borrowing (i.e., not in default).

Table 3: IMD Model Fit and Comparison with Fund

Target Moments				Non-target Moments			
Variables	Data	IMD	Fund	Variables	Data	IMD	Fund
			A. 1^{st} Moments				
b'/y (%)	78.33	78.57	191.00	ps/y (%)	-1.00	1.14	4.70
spread (%)	4.15	4.17	-0.003				
g/y %	21.68	21.74	20.97				
1% of g/y	13.38	15.22	14.44				
99% of g/y	32.80	32.14	32.62				
n (%)	36.37	36.56	37.82				
e	n.a.	0.29	0.34				
			B. 2^{nd} Moments				
$\sigma(n)/\sigma(y)$	1.00	0.91	0.70	$\sigma(c)/\sigma(y)$	1.51	1.39	0.36
$\sigma(g)/\sigma(y)$	1.02	1.03	0.70	$\rho(c, y)$	0.63	0.64	0.62
$\sigma(ps/y)/\sigma(y)$	1.00	0.97	0.86	$\rho(n, y)$	0.70	0.10	0.94
σ(spread)	1.67	1.74	0.00	$\rho(\text{spread}, y)$	-0.38	-0.06	-0.48
$\rho(g, y)$	0.38	0.38	0.47	$\rho(ps/y, y)$	0.18	0.23	0.93

Notes: all data moments are the averages of country specific moments over GIPS countries; second moments are calculated after removing trends by HP-filter, both in the data and IMD/Fund model solutions; for the Fund solution, debt/output ratio is defined as $\bar{a}'/y$ (cf. (30)); and ps denotes primary surplus.

although the correlation of labor and GDP is lower than in the data. However, this is not the focus of our inquiry and — except for the fact that welfare comparisons are easier with separable preferences — our main results do not depend on our specific choice of preferences.

5 Quantitative Analysis

This section investigates quantitatively how the Fund improves on the IMD economy using the calibrated parameters described above in the two economies. We do this in three steps. First, we assess how differently the Fund operates in normal times, by comparing the long run properties of the two economies using both a few key statistics and representative long-run simulations in Section 5.1. Then, in Section 5.2, we compute and discuss the welfare implications of these improvements and evaluate the debt absorption capacity of the Fund. We conclude our quantitative analysis in Section 5.3 by contrasting the paths of the two economies under a crisis event that approximates well the onset of the Euro debt crisis for the GIPS countries.

5.1 Comparing the Allocations in 'Normal Times'

We base our description of how the Fund operates during normal times by comparing long-run statistics of the two economies, displayed in Table 3, and by presenting representative simulation paths of both economies on Figures 2 and 3, subject to the same sequence of shocks in the long run stationary distribution. The long-run simulations are initialized at the the ergodic mean of the two economies.

In Figure 2, the upper left panel shows the history of shocks for 100 periods, while the output, consumption and labor allocations in the IMD and Fund regimes are shown in the other panels. In addition, Figure 3 displays the levels of effort, surplus over GDP, debt over GDP and spreads in the two economies. In order to make the two economies comparable, we plot simulations in which they face exactly the same sequence of productivity and government expenditure shocks. The grey periods in the figures correspond to periods of default in the IMD economy. To obtain comparable variables (e.g., debt holdings or spreads) in the two economies, we rely heavily on Section 3.

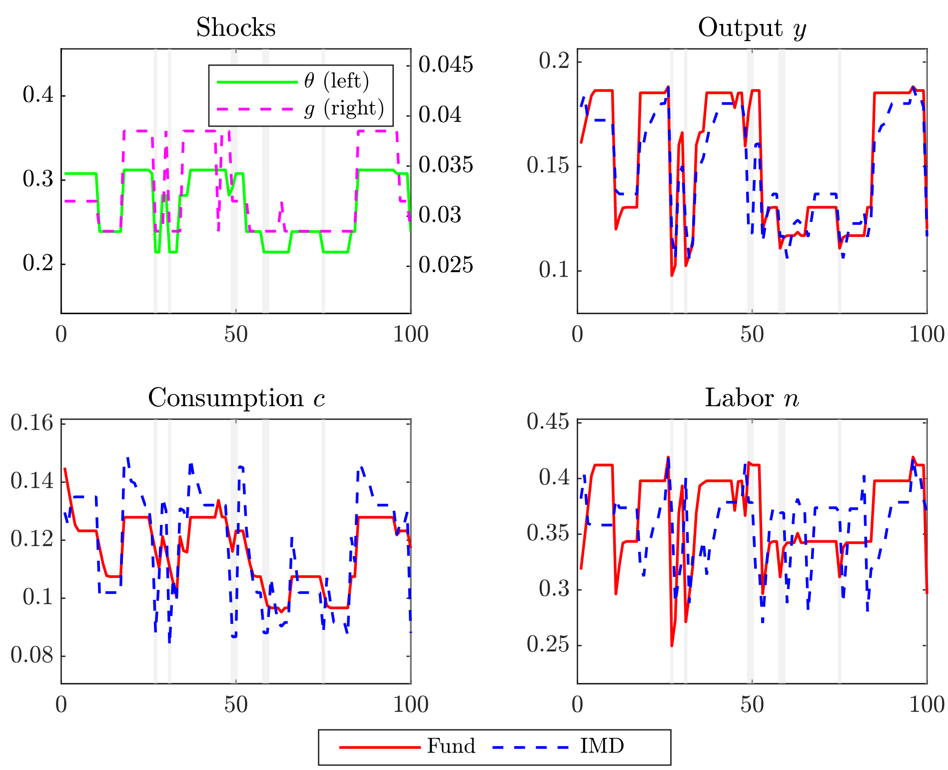

Figure 2: Business cycle paths of real variables

Notes: horizontal axis is for time period; and grey bars indicate default periods in the IMD economy.

The differences between the two economies are striking in many dimensions. The first stark difference we would like to emphasise is that the Fund is able to *absorb much higher level of debt*. The long-run average debt over GDP is 191 percent under the Fund compared with 78.6 percent in the IMD economy. Note that our calibration implies that the borrower is effectively more impatient than the lender (the markets) and hence accumulating debt is desirable. In the IMD economy, a high level of debt cannot be sustained as it increases the probability of future default and consequently the cost of borrowing (the spread). In fact, Figure 2 reflects that defaults are primarily associated with drops in productivity with relatively large levels of initial debt. Moreover, the frequency of default and the long-term nature of debt implies that spreads are relatively high in the IMD economy even in normal times, and they spike just before a default episode, making further borrowing prohibitively costly. Even though the same limited commitment friction (with the same exact outside option, $V^o(s)$) is imposed in the Fund, the Fund allocation provides a much higher utility through improved risk sharing and by avoiding costly default episodes, implying that a much higher debt is sustainable with the Fund. By eliminating default episodes, even with these much higher levels of debt, the borrower faces no positive spread. As we explained in Section 3.3, binding future limited enforcement constraints of the Fund may lead to negative spreads along the equilibrium path. However this does not happen along the particular simulation path displayed in the Figure.

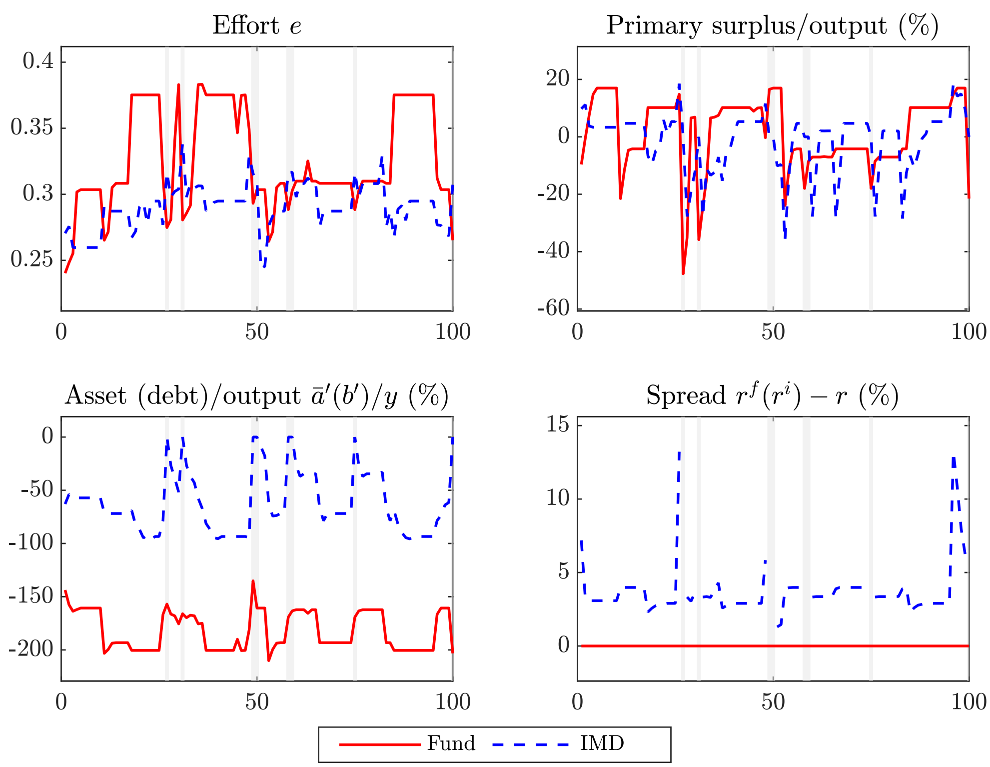

Figure 3: Business cycle paths of financial variables

Notes: horizontal axis is for time period; $\bar{a}'$ refers the 'bond' component of asset positions in the Fund economy (cf. (30)); and grey bars indicate default periods in the IMD economy.

The second key difference between the Fund and IMD allocations is the amount of *consumption insurance*. Table 3 shows that the relative volatility of consumption drops from 139 percent in the IMD economy to 36 percent in the Fund. The smoother path of consumption is also clearly reflected on Figure 2. How does the Fund deliver a smoother consumption path? The key statistic to understand this is the co-movement between output and the primary surplus. Full consumption insurance would imply a very strong co-movement of the primary surplus and output, as constant consumption can be achieved through surpluses (capital outflows or savings) during good times and through deficits (capital inflow or borrowing) during bad times. Under separable preferences, this co-movement is not perfect solely due to the presence of government expenditure shocks. In the IMD economy, the presence of spreads induces an implicity tight borrowing constraint, leading to a mildly procyclical (0.23) primary surplus. At the same time, the Fund allows for much more state-contingency and the procyclicality of the surplus rises to 0.94. Indeed, when we compare the path of θ on Figure 2 with that of the primary surplus on Figure 3, we can visualize the strong co-movement of the two variables. This is particularly important when the economy is hit by negative shocks, triggering default in the IMD economy. Instead, in most of these cases the borrower enjoys a large primary deficit under the Fund that allows her to keep consumption much smoother. Under incomplete markets, this deficit would imply an immediate increase of outstanding debt. In contrast, Figure 3 shows that the borrower is (partially) insured against this extreme realizations under the Fund and the outstanding debt is actually reduced. Here, consumption smoothing is delivered by the state contingent part of the debt ($\hat{a}(s')$, see (30)), which makes the primary surplus more procyclical.

Finally, the Fund also leads to a *more efficient allocation of labour and effort*. First, as Table 3 reflects, labor supply in the Fund is strongly correlated with output. This is an indication of improved efficiency, as in the unconstrained optimal allocation of this economy, labor supply is solely determined by productivity under our separable utility function specification. Another interesting observation is that the Fund provides on average considerably better incentives for exerting effort, especially in normal times. Table 3 indicates that the long-term mean of effort is 17 percent higher under the fund than under the IMD economy. This is because the Fund provides long run incentives directly, connecting future realizations of the government expenditure shock to future lifetime utility through the law of motion of x (see (5)). Section 3 shows that these rewards and punishments can be translated to changing terms (cost) of borrowing. In contrast, defaultable debt markets can mostly provide short term incentives through the immediate utility drop and spread rise associated with high realizations of the g shock.

5.2 Welfare Implications

In what follows, we quantify the actual welfare implications of the Fund regime compared to the IMD economy as well as the increased capacity to absorb debt by the Fund. The results are displayed in Table 4. We start the discussion by looking at the debt absorption. The second and third columns of the table display the maximum end of period debt to output ratio in percentage terms that the country has for different values of the shocks.[29] These latter

[29] For this exercise, we set the value of the idiosyncratic component of government expenditure to its zero mean.

measures are intended to capture the absorbing debt capacity of the borrower. Note that debt capacity is not straightforward to measure in the IMD economy, as there are no explicit debt limits. However, given the impatience of the borrower, the actual debt choices reflect the debt capacity in this case. Hence, we choose the highest equilibrium debt/output ratio for a given state s across all feasible levels of current debt b (or a for the Fund). In the case of complete markets, the borrower has a whole portfolio of debt (and assets) for each future state. However, in section 3.3, we have shown that from any portfolio of Arrow securities, we can construct a bond component a' and an insurance portfolio, with the bond component being the comparable measure to the debt choice in the incomplete markets economy. Given this, we follow the same logic and we present the maximum of debt/output ratio using the bond component across all values of current debt for a given state.[30]

Table 4: Welfare gains at zero debt and debt capacity

$s = (\theta, g)$	Welfare Gain %	IMD max $\frac{-b'(s,\cdot)}{y(s,\cdot)}$ %	Fund max $\frac{-\bar{a}'(s,\cdot)}{y(s,\cdot)}$ %
(θ_l, g_h)	9.31	1.87	98.89
(θ_m, g_h)	7.73	88.21	169.09
(θ_h, g_h)	7.00	183.75	292.82
(θ_l, g_l)	10.28	1.60	104.08
(θ_m, g_l)	8.36	90.89	160.42
(θ_h, g_l)	7.27	182.52	273.46
Average	8.48		

Notes: the average welfare gains in the last row equals to $\sum_s \Pr(s)W(s)$, where $\Pr(s)$ denotes the ergodic distribution of shock s, and $W(s)$ denotes the welfare gain for shock s with zero debt; $\bar{a}'$ refers to the 'bond' component of asset holdings in the Fund economy (cf. (30)); and the maximum (end of period) debt capacity is taken over the state space for the current debt, i.e., b for IMD and a for Fund.

The difference between the debt capacities in the two economies are striking. As we see, the Fund is able to absorb much higher debt-to-GDP ratios in all states, while the capacity to absorb debt in the IMD economy is substantially smaller, particularly in bad states. This is because, due to the relatively high persistence of the shocks, a low realization of the shock today implies a high spread on any significant amount of debt, as the country will have only a small chance to pay it back through a better realization of the shocks. Moreover, due to the asymmetric default penalty specification, there is no output penalty for low shock realizations, and default is not particularly costly in this case. Another interesting feature of the IMD economy is that the borrowing limits are relatively loose in normal times (for medium productivity levels). This is due to the fact that, in this case, the countries suffer an output loss upon default and the value of staying in the financial markets is higher in relative terms.

[30] Note that, for the Fund economy, one can actually compute the maximum borrowing capacity as the bond component of the portfolio that allows for the maximum amount of borrowing across all possible realizations of the future shocks: $\frac{\sum_{s'|s} q(s'|s)\mathcal{A}_b(s')}{\sum_{s'|s} q(s'|s)}$, where $\mathcal{A}_b(s')$ is the state contingent borrowing constraint of the country. However, for comparison with the incomplete markets economy, we choose the alternative measure described in the text, which has values that are necessarily tighter than the maximum borrowing limit.

Nevertheless, the Fund will be able to support much more borrowing as default becomes less attractive compared to the utility the Fund can deliver. Below, we provide a measure the welfare impact of this increased debt capacity.

These results also indicate that the Fund can 'take over' very large amounts of debt from potential member countries at the verge of defaulting. For example, if a borrower country has an outstanding debt of around 80 percent of its GDP and it is hit by a crisis (θ_l, g_h), then under the IMD scenario, it will default on its debt, suffering all the consequences of default. In contrast, the Fund will be able to absorb all this debt, enrolling the country in the long-term debt program and able to provide strictly higher utility than under autarky (as the accumulated debt is strictly below the debt absorbing capacity of the Fund), and hence it would not violate the limited enforcement constraint of either parties.

Now, we turn our attention towards the welfare gains. The first column of Table 4 displays the welfare gains of the Fund in (annual) consumption equivalent terms when countries have zero initial debt for different values of the shocks (θ, g) (in the next section, we will measure welfare gains also for an already indebted borrower). The Table reflects that the welfare gains are very substantial under the Fund: the consumption-equivalent steady-state average welfare gain is around 8.5 percent and, even more relevant, the gain is of 10.3 percent in the worst state. As discussed earlier, two of the features of the Fund that lead to welfare gains are the fact that it provides more risk sharing through state contingent assets and the fact that it allows for a much higher debt capacity. Both of these features are particularly important with bad shocks, leading to substantially higher welfare gains. In other words, the welfare gains of the Fund are the highest when the country is in trouble. The gains are still substantial when the country is hit by good shocks. Note that this is partially due to the fact that agents are forward-looking and gain benefits from the future insurance against bad shocks, and partially because at the higher shock levels they still have much higher debt capacity and still benefit from the state contingency of the Fund contract.

Next, we go deeper to inspect how important are these different features of the Fund for the welfare gains. To do this, we propose a novel decomposition of welfare gains that implements a series of counterfactual exercises to evaluate the main channels of welfare improvements. The first important difference between the IMD and Fund economies is that default occurs in equilibrium in the IMD economy but not in the Fund economy. Given this, we first simulate a counterfactual IMD economy where we keep the asset prices, asset holdings and default decisions at the same level, except that *(i) no output penalty* is imposed and *(ii)* no penalty is imposed, in the sense that there is *no market exclusion* after default. By comparing the lifetime values obtained in the first counterfactual economy with the value functions of the IMD economy, and then comparing the values of the first and second economies, we obtain the isolated effect of the output penalty and exclusion, respectively. To evaluate the effect of *(iii)* a *higher debt capacity*, we solve for counterfactual economies with looser constant exogenous debt limits in which default is not allowed. In particular, the debt limits are set at the endogenous borrowing constraints $\mathcal{A}_b(s)$ associated with a given value of the state vector s under the Fund economy. Comparing the value of this counterfactual exercise to case (ii) provides us with the measure of welfare gains due to an increased debt capacity in the Fund (note that all the direct costs of default were already taken care by the previous case). Finally, note that the previous three counterfactuals do not account for the fact that Fund is able to provide *(iv) state-contingent payments* as opposed to the IMD economy (apart from the costly default episodes). This is captured by the (residual) difference between the welfare in the Fund economy and counterfactual (iii).[31] The results of the counterfactuals are displayed in Table 5 below for a selection of initial states.

Table 5: Welfare decomposition at 0 debt/asset for selected shock states

$s = (\theta, g)$	(i) No θ penalty %	(ii) Immediate return to market %	(iii) Greater debt capacity %	(iv) State-contingent insurance %
(θ_l, g_h)	6.58	1.67	63.65	28.10
(θ_l, g_l)	5.31	1.38	51.92	41.39

Notes: see the main text for the explanation on how to decompose the welfare gains into the four components.

The table reflects that, for all values of the shocks, the higher debt capacity and insurance through the state contingent assets provided by the Fund are the two most important factors contributing to the welfare gains. In particular, these two factors account for more than 90% of the welfare gains in both cases presented. We also see that the contribution of not having a penalty upon default is relatively small. The reason is that matching debt levels and spreads simultaneously in the IMD economy requires an asymmetric default penalty that does not impose penalties for low productivity levels, hence the output penalty kicks in only whenever during the default period when productivity increases. We also see that the fact that a country is excluded from the financial markets upon default is less important. This is due to the fact that returning to the market with low shocks implies very tight borrowing limits under the IMD economy and hence limited potential welfare gains of market return.

The key result is that both the increase in debt capacity and the state contingency of payments are quantitatively significant in explaining the welfare gains with increased greater debt capacity being the more important component. Whenever, low productivity is combined with high government expenditure state-contingency tends to be (relatively) more important because the desire for borrowing is highest in this case and even the relaxed borrowing constraint limits consumption smoothing extensively. To summarize, the Fund leads to substantial welfare gains that arise primarily from the fact that it provides insurance through the state contingent assets as well as a higher debt capacity.

[31] One caveat is that step (iii) of this decomposition can be computed credibly only for the lowest levels of the shocks because a constant debt limit set at the level of the endogenous borrowing constraint associated with medium shocks under the fund would be not sustainable even under the Fund

5.3 Comparing the Allocations in Crisis Times

In this section, we investigate how the Fund responds in a crisis situation. In particular, we compute a counterfactual simulation that compares how the representative economy would have done under the IMD and Fund regimes when hit by a crisis that resembles some of the aspects of and the Euro debt crisis following the 2008 Financial crisis. To do this, we initialize the economy at a state with low spreads of around 0.8 and a level of debt of around 70% of GDP, which are consistent with the average levels of debt to GDP and spreads in the pre crisis times during 2005–2007. Subsequently, we hit the economy with a negative productivity shock and a bad (high) government shock at period 1 and we compute, under each regime, the average path for 50,000 independent simulations with the same pre crisis initial asset holdings and spreads but different shock realizations from the (partially endogenous) Markov structure of our economy after period zero.

Table 6: Statistics around the onset of European debt crisis

	Periods	Avg. b'/y %	Avg. spread %
Before crisis:	2005–2007	78.31	0.78
Crisis eruption:	2009–2010	99.14	4.04

Notes: all moments are the averages over the GIPS countries.

Table 6 displays the pre and post crisis average levels for the debt to GDP, spreads and GDP, while Figures 4 and 5 display the simulated average impulse response paths after the crisis for the economy under the IMD and Fund regimes. As the table and figures reflect, the model is able to match relatively well the post crisis increase in debt, the considerable increase in spreads after the crisis, and the decrease in GDP.

The smooth path of all the key variables in the figures reflects the fact that we depict the average path for many independent economies. It is important to note that there are many default episodes in the IMD economy after period 1, generating the positive spreads in Figure 5. For the real variables (shocks, output, consumption, labour, effort, and primary surplus), we take an average over all economies in every period, while for debt over GDP and the spread we only average over for those who are not in default, as these variables are not defined for those who are in default.

Looking at the pictures, the differences between the IMD and the Fund are even more striking than in the long run simulations. The paths for consumption and labor clearly indicate that the Fund is able to stabilize the crisis considerably more in the short run: consumption is higher for the first 10 periods and, due to efficiency considerations, the Fund allows for a reduction in labor supply in the short run. At the same time, for the IMD economy, labor supply needs to increase exactly when productivity is low to limit the consumption drop. In turn, the lower labor supply implies that output drops more under the Fund.

Inspecting Figure 5, we see how consumption smoothing is achieved in the Fund. First of all, under the Fund, the borrower is able to deal with a crisis by running a large deficit

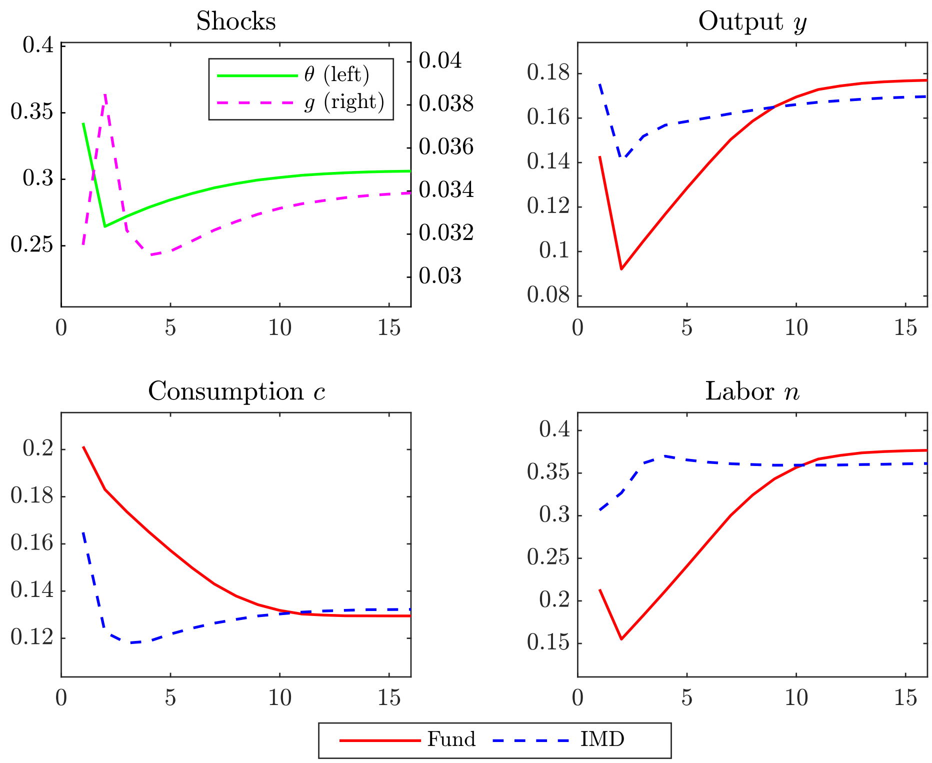

Figure 4: Counterfactural simulation of real variables

Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 50,000 IMD/Fund model economies are simulated with stochastic θ and g shocks that are independent in the cross section, and the final results are the average of 50,000 simulations.

during the first periods of the crisis. These deficits are financed by a large reduction of the debt that is in contrast with the sharp rise of debt under the IMD economy during the same period. This debt reduction is due to the state contingent nature of the Fund contract: the country is (partially) insured against severe negative shocks. Also note the dramatic rise of the spread upon the shock in the IMD economy. This shock brings our economy from the normal stage to the intermediate stage of our Markov switching process. That is, the level of productivity is lower but not at the crisis level yet, but the probability of a crisis becomes high. This implies no default in the current period but a high probability of it in future periods, hence a high spread in the IMD economy. This behaviour is confirmed by Figure 6, displaying the proportion of countries defaulting over time. Again, the large positive spreads are eliminated under the Fund economy and replaced by small negative spreads because, in some future states the contingent assets (insurnace) requested by the borrower surpass the limits implied by her future primary surplus, risking a permanent loss (transfer) for the Fund. As a result, the Fund is better off by restraining the borrower.

It is important to note that one has to interpret the paths of debt and the spread under the IMD economy with caution, since we only depict these two variables for the selected set

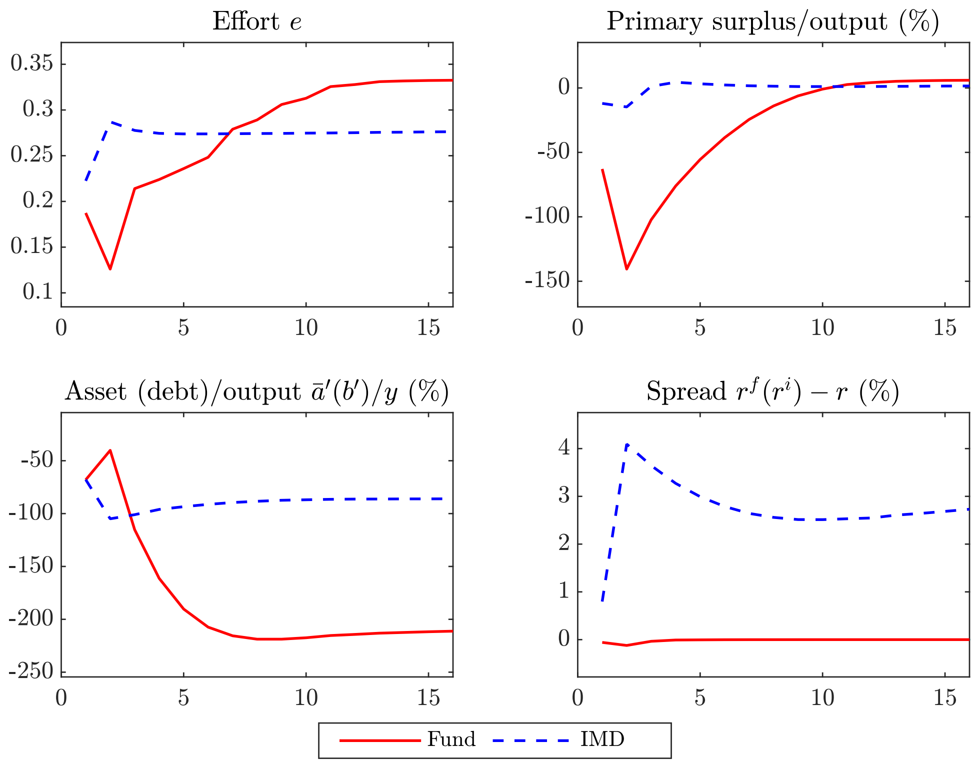

Figure 5: Counterfactural simulation of financial variables

Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 50,000 IMD/Fund model economies are simulated with stochastic θ and g shocks that are independent in the cross section; and the final results are the average of 50,000 simulations, but for the last two panels the averages are taken over cross-section units that are not in default at time t.

of countries that are not in default. In particular, the fact that debt is increasing under the IMD economy is affecting that the sample of countries depicted for this variable is selected as they experienced (higher) realizations of the shocks not triggering default.

In the long run, average consumption in the Fund is slightly lower, while average labor supply is slightly higher. This is is due to the fact that the country accumulates (on average) a much higher stock of debt under the Fund. At first sight, this may imply that the Fund cannot improve welfare compared to the IMD economy, because of lower long-term average consumption and leisure. Note, however, that this is offset by the two other features of the Fund contract. First, as we have seen above, it offers a much smoother consumption. Second, given the fact that the borrower is more impatient than the lender, this front-loading of consumption is increasing *ex-ante* welfare. In this counterfactual experiment, the welfare gains achieved by the Fund are around 10.59 percent in consumption equivalent terms. Note that this implies, that welfare gains are significantly higher when the Fund is taking over a significant amount of debt (the welfare gains with these initial shocks and zero initial debt would have been be 8.57%).

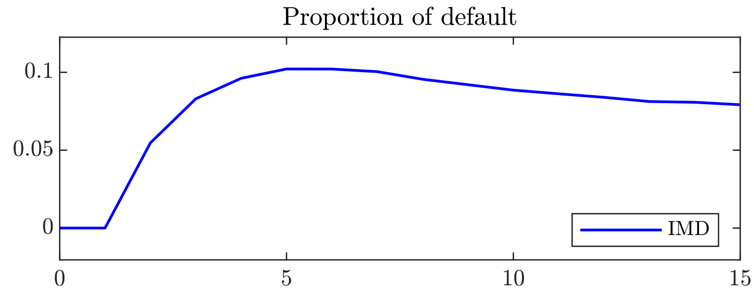

Figure 6: Counterfactural simulation of the default wave

Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 5,000 IMD model economies are simulated with stochastic θ and g shocks that are independent in the cross section; and the final results are the proportion of cross-section units in default at time t.

Finally, the response of effort shows an interesting pattern. While we have seen that the Fund provides better incentives for exerting effort in normal times, the IMD economy imposes more discipline than the Fund in bad times. As we see, effort increases in the IMD economy right after the bad shock, while it decreases in the Fund. In the long run, however, effort is higher in the Fund. In the IMD economy, incentives are provided through prices and through the fact that when a country is effectively borrowing constrained higher effort increases the probability of a budget relief (a lower government expenditure). These channels are obviously stronger in crisis times (and under temporary autarky). However, the effort under the Fund indicates that this type of 'austerity' is not part of the efficient allocation.

This counterfactual has demonstrated several important aspects. First, faced with a crisis of similar magnitude to the 2008 financial crisis, a borrower with a similar level of debt as the distressed countries in the EU in the pre-crisis times would have had access to a much larger debt capacity under the Fund, in the sense that the Fund could have absorbed all the preexisting debt and more. Second, the state contingency of the Fund allocation would have lead to considerably more risk sharing through counter-cyclical primary deficits during the crisis times. Third, costly default episodes (both in terms of lack of consumption smoothing and in terms of costly effort) could have been avoided in the Fund. Given this, we find that, even with very limited redistribution, the Fund could have improved efficiency in the EU area considerably and could have led to substantial welfare gains both in normal times and even more in crisis times.

6 Conclusion

By developing and computing a model of a *Financial Stability Fund* as a *constrained-efficient mechanism* we have contributed to the existing literature on risk sharing and sovereign debt, and to the current policy debate on stabilization, risk-sharing, debt and crisis-resolution mechanisms for the European Economic Monetary Union (EMU) after the pandemic. In particular, we have quantitatively shown that a Fund with well designed contracts, integrates and internalizes these different aspects, resulting in substantial welfare gains, even if we have calibrated the model for euro area 'stressed countries', and we have set a 'tight constraint' on risk-sharing transfers: the fund should always have non-negative expected profits from its Fund contracts. We have also shown that accounting for *moral hazard* the Fund provides better incentives to reduce endogenous risks in normal times, without imposing excessive effort in crisis times.

We have made an 'exclusivity assumption' in having the Fund absorbing all the sovereign debt of a participant country but, as already noted, the follow-up work of **?** relaxes this assumption (considering only LE frictions): absorbing only a fraction, the Fund stabilizes *all* the country's sovereign debt. We leave it for other future work to study, and quantify, how the Fund can be more effectively simplified — in the sense of making it less contingent, or relying on a simpler financial structure — as has been proposed (e.g., a 'rainy day' Fund to absorb 'large economic shocks'). The advantage of our framework is that it allows for a characterizations and quantitive evaluation of the tradeoff between *simplicity* and *efficiency*, providing a guide for further 'contractual engineering' work which should help its implementation. Similarly, it also provides a guide, for further work, on how more effectively determine *ex-ante* and *ex-post* conditionality: the Fund *ex-ante* conditionality is based on a range of different risk-assessed contracts, while its *ex-post* conditionality is not a renegotiation process, but the realization of a *ex-ante* agreed state-contingencies.

We started working in this project in the aftermath of the euro-crisis, when the emphasis was on risk-sharing, solving the debt-overhang problem and producing safe-assets (i.e., safe Eurobonds). As noted, our Fund as a *constrained-efficient mechanism* has the virtue to encompass all these different aspects and to provide a benchmark for other fiscal frameworks. In fact, the EU and EA fiscal framework has changed in response to the Covid-19 crisis. Now circa 30% of the euro-area sovereign debt being held by the Eurosystem and elements of the new European Stability Mechanism (ESM) programmes and of *Next Generation EU* are more inline with the proposed theory. In particular, SURE and RRF debts are designed to avoid transfers (but allow for pandemic solidarity grants[32]); they display more flexible *ex-ante* conditionality and, importantly, programmes are financed with Eurobonds. However, our theory also emphasizes the need for permanent — unique, or well coordinated — institutional arrangements, the use of state-contingent (not simply debt) contracts and the key role of country-specific risk assessments. In sum, our theory suggests that there is ample room for future contractual and institutional enhancement in the European Union (**?**) and, possibly, in other unions too.

[32]Formally, in our model this would mean that there are 'pandemic states' $\tilde{s}$ for which $Z(\tilde{s}) < 0$.

Appendix

A Proofs

A.1 Characterization of the Fund Solution

Proof of Lemma 1. Note that if we define $\widetilde{FV} : \mathbb{R}^2_+ \times S \to \mathbb{R}$ by $\widetilde{FV}((x,y),s) \equiv yFV(x,s)$, then $\widetilde{FV}$ — the non-normalized version of SPFE — is monotone of degree one in (x,y) (**?**). Given, as we assume, the solution to the SPFE is unique, the decomposition $\tilde{FV}((x,1),s) = xV^{bf}(x,s) + V^{lf}(x,s)$ is unique and $\widetilde{FV}$ is differentiable (**?**); furthermore, $\widetilde{FV}((x,1),s)$ is monotonically increasing and concave in x. FV shares the properties of $\widetilde{FV}$, at $(x,1)$; in particular, $\partial_x FV(x,s) = \partial_x \widetilde{FV}((x,1),s) = V^{bf}(x,s)$. Given our strict concavity/convexity assumptions on u, f and v, whenever neither limited enforcement constraint is binding: V^{bf} is strictly concave in x, as well as $V^{lf}(x,s) = FV(x,s) - xV^{bf}$, and both functions are strictly monotone, increasing and decreasing, respectively. □

Before we prove Lemma 2 we prove the following supporting lemma.

Lemma 5. *Given the monotonicity assumption (MLR), the law of motion $x'_{xs}(s'(i))$ is non-decreasing in i and it is constant if $\chi(x,s) = \varrho(x,s) = 0$.*

Proof. It follows from Assumption 1 (i.e., the monotone likelihood-ratio condition) — i.e., $\frac{\partial_e \pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}$ is nondecreasing in i for every e. Recall (7):

$$x'_{xs}(s'(i)) = \frac{1 + \nu^b + \varphi(s'(i)|s,e)}{1 + \nu_l}\eta x \text{ and } \varphi(s'(i)|s,e) = \varrho \frac{\partial \pi(s'(i)|s,e)/\partial e}{\pi(s'(i)|s,e)}. \qquad \square$$

Lemma 5 implies that $V^{bf}(x'_{xs}(s'(i)), s'(i))$ is nondecreasing in $s'(i)$.

Proof of Lemma 2. Given (x,s) the eight equations (8), (11)–(13) and (15)–(18) determine the three allocations policy values, the three corresponding multipliers and the two values, $V^{bf}(x,s)$ and $V^{lf}(x,s)$. Then, given s', equation (7) determines $x'_{xs}(s')$, and the recursive resolution follows. More precisely:

In *(i)*, the differentiability and strict concavity and convexity assumptions of the functional forms guarantee the local uniqueness of the allocation policy values, as well as of the two values the value functions must satisfy; in particular, equations (15) and (16) contribute to determine $e(x,s)$ and $\chi(x,s)$; and, therefore, $\varrho(x,s)$. Equation (12) — now, $u'(c(x,s)) = 1/x$ — shows that $\partial_x c(x,s) > 0$ and $\partial_s c(x,s) = 0$, (13) shows that $\partial_x n(x,s) < 0$ and, since $\partial_s c(x,s) = 0$ it is constant in g and increasing in θ. There is perfect risk-sharing $V^{bf}(x,s)$ and $V^{lf}(x,s)$ increase with positive exogenous shocks, s (provided the expected value of future surpluses is higher), and with their relative shares of the surplus, x and $1/x$, respectively; furthermore, by (7) $x'_{xs}(s') \equiv x'(x,s,s')$ follows the the same pattern of $c(x,s)$ and is increasing with x and, by Lemma 5, non-decreasing in s'.

In *(ii)* $c(\underline{x}(s), s)$ is determined by $\partial_{c+}u(c(\underline{x}(s), s)$.[33] All other policies and values are constant for all $x \leq \underline{x}(s)$ and equal their policy at $\underline{x}(s), s)$ — except for $FV(x, s)$ which increases at the rate x. In this case there is no perfect risk-sharing and an increase in s increases $V^o(s)$ (and $\nu^b(\underline{x}(s), s) > 0$) and, therefore, $\underline{x}(s)$, $c(\underline{x}(s), s)$, $V^{bf}(\underline{x}(s), s)$ and $x'(\underline{x}(s), s, s')$ increase; while $n(\underline{x}(s), s)$ decreases in g, given θ, since consumption decreases.

In *(iii)* $c(\overline{x}(s), s)$ is determined by $\partial_{c-}u(c(\overline{x}(s), s)$. As in *(ii)* there is no perfect risk-sharing and policies and values are constant for all $x \geq \overline{x}(s)$. However, if there is an increase in s, since $V^{lf}(\overline{x}, s) = Z$, $\overline{x}(s)$ does not change, the borrower absorbs the increase of surplus; i.e. $c(\overline{x}(s), s)$, $V^{bf}(\overline{x}(s), s)$ and $x'(\overline{x}(s), s, s')$ and $n(\overline{x}(s), s)$ follows the same pattern than in *(ii)*. □

Proof of Lemma 4. Using the optimality condition with respect to consumption in (12), we can express the inverse of the marginal utility of consumption next period as:

$$\frac{1}{u'(c(x'_{xs}(s'), s'))} = \frac{1 + \nu_b(x'_{xs}(s'), s')}{1 + \nu_l(x'_{xs}(s'), s')} x'_{xs}(s') = \frac{1 + \nu_b(x'_{xs}(s'), s')}{1 + \nu_l(x'_{xs}(s'), s')} \frac{1 + \nu_b(x, s) + \varphi(s'|s, e)}{1 + \nu_l(x, s)} \eta x.$$

Moreover, using the definition of $\varphi(s'|s, e)$, this equation can be rewriten as

$$\frac{1}{u'(c(x'_{xs}(s'), s'))} \frac{1 + \nu_l(x'_{xs}(s'), s')}{1 + \nu_b(x'_{xs}(s'), s')} = \eta \left[\frac{1}{u'(c(x, s))} + \chi(x, s) \frac{\partial_e \pi(s'|s, e(x, s))}{\pi(s'|s, e(x, s))} \right], \tag{A.1}$$

which is equivalent to (19) in the main text. Multiplying both sides of equation (A.1) by $\pi(s'|s, e)$ and adding up over all s' we obtain

$$\sum_{s'|s} \pi(s'|s, e) \left[\frac{1}{u'(c(x'_{xs}(s'), s'))} \frac{1 + \nu_l(x'_{xs}(s'), s')}{1 + \nu_b(x'_{xs}(s'), s')} \right]$$
$$= \eta \frac{1}{u'(c(x, s))} + \eta \chi(x, s) \sum_{s'|s} \partial_e \pi(s'|s, e(x, s))$$

Moreover, as the transition probabilities add to 1, it follows that $\sum_{s'|s} \partial_e \pi(s'|s, e(x, s)) = 0$. In sum, the inverse Euler equation in our framework with limited enforcement constraints takes the following form:

$$\sum_{s'|s} \pi(s'|s, e) \left[\frac{1}{u'(c(x'_{xs}(s'), s'))} \frac{1 + \nu_l(x'_{xs}(s'), s')}{1 + \nu_b(x'_{xs}(s'), s')} \right] = \eta \frac{1}{u'(c(x, s))}.$$

Note that, if enforcement constraints are not binding, it follows that:

$$\mathbb{E} \left[\frac{1}{u'(c(x'_{xs}(s'), s'))} \middle| s \right] \leq \frac{1}{u'(c(x, s))}, \tag{A.2}$$

[33] Given s, $\partial_{c+}u(c(\underline{x}(s), s)$ denotes the right-hand side derivative of $u(c(x, s))$ at $c(\underline{x}(s), s)$ and $\partial_{c-}u(c(\overline{x}(s), s)$ denotes the left-hand side derivative of $u(c(x, s))$ at $c(\overline{x}(s), s)$; i.e., by (12), $\partial_{c+}u(c(\underline{x}(s), s) = \lim_{t \searrow 0}(\underline{x}(s) + t)^{-1}$ and $\partial_{c-}u(c(\overline{x}(s), s) = \lim_{t \searrow 0}(\overline{x}(s) - t)^{-1}$.

with strict inequality if $\eta < 1$. □

A.2 Proofs of Proposition 1 and 2

Proof of Proposition 1. To prove the proposition, we show that we can construct asset prices, asset holdings, policies, multipliers, borrowing limits and value functions such that all the conditions characterizing the recursive competitive equilibrium are satisfied by the recursive Fund allocations. The proof is based on **?**, but it is set in a recursive competitive framework, takes advantage of the Fund characterization, in Lemma 1 and follow-up results, and extends **?** by accounting for the moral hazard friction that requires the introduction of Arrow security taxes.

We use the recursive Fund allocation policies and multpliers to define asset prices and taxes as functions of (x, s). First, state contingent taxes take the following form:

$$\frac{1}{1+\tau^a(s', x, s)} = 1 + \chi(x, s)u'(c(x, s))\frac{\partial_e \pi(s'|s, e(x, s))}{\pi(s'|s, e(x, s)}.$$

Second, we construct the RCE asset prices $\{q, Q\}$ in (45) and (46), namely:

$$q(s'|s) = \pi(s'|s, e(x, s))A(x', s') \max\left\{\beta\frac{u'(c(x'_{xs}(s'), s'))}{u'(c(x, s))}\frac{1}{1+\tau^a(s', x, s)}, \frac{1}{1+r}\right\},$$

where $A(x, s) = (1 - \delta + \delta k) + \delta q(s)$, with $q(s) = \sum_{s'|s} q(s'|s)$, and

$$Q(s'|s) = \frac{q(s'|s)}{A(x'_{xs}(s'), s')}, \quad Q(s''|s) = \frac{q(s''|s')}{A\big(x''_{x's'}(s''), s''\big)}Q(s', x'_{xs}(s')|s), \quad \ldots,$$

recursively for any state and any time in the future, with $Q(s_0, x(s_0)|s_0) = \frac{1}{A(x(s_0), s_0)}$.

Third, we can use the budget constraints of the RCE, (32) and (39), to obtain the initial allocation of asset holdings $(a(s_0), a_l(s_0))$, which are the transfers of the Second Welfare Theorem. But, at this point, it helps note the fact that in the RCE taxes are budget neutral. Therefore, as in (44), we can define:

$$\overline{\tau}(x, s) \equiv \sum_{s'|s} q(s'|s)a'(x, s, s')\tau^a(s', x, s).$$

This implies that taxes do not appear in the present value budgets since, by the above definitions, intertemporal taxes and transfers cancel out. Furthermore, by Lemma 2, the constrained-efficient allocation process is stationary and the value functions are bounded. Therefore, the transversality conditions are satisfied. In particular,

$$a(s_0) = \sum_{t=0}^{\infty}\sum_{s_t|s_0} Q(s_t|s_0)[c(x_t, s_t) - \theta_t f(n(x_t, s_t)) - g_t], \tag{A.3}$$

$$a_l(s_0) = \sum_{t=0}^{\infty} \sum_{s_t|s_0} Q(s_t|s_0) c_l(x_t, s_t). \quad \text{(A.4)}$$

Now we have achieved the key result for the proof: a one-to-one identification between the initial condition $x(s_0) = \mu_{b,0}/\mu_{l,0}$ and $(a(s_0), a_l(s_0))$, where $a_l(s_0) = -a(s_0)$. The same logic extends to a one-to-one mapping between (x, s) and (a, s) for *any* period, once they are taken as the initial state. The one-to-one mapping also implies that for borrower's policies: $m(a, s) = m(x, s)$, for $m = c, n, e$. We can proceed with our construction of asset holdings: $\bar{a}'(a(s_0), s_0)$ is given by

$$q(s_0)(\bar{a}'(a(s_0), s_0) - \delta a(s_0)) = \theta_0 f(n(a(s_0), s_0)) - g_0 - c(a(s_0), s_0) + (1 - \delta + \delta\kappa)a(s_0),$$

and then we can obtain — still using $q(s'|s_0)$ — the risk-sharing assets $\hat{a}'(a(s_0), s_0, s')$ and the period zero asset portfolio $a'(a(s_0), s_0, s')$. For the construction of the assets' low of motion $a'_{as}(s') \equiv a'(a(s), s, s')$, it suffices to note the one-to-one mappings between (i) a and x, and (ii) a' and x' in any future state, and the law of motion for $x'_{xs}(s') \equiv x'(x, s, s')$. Regarding the lender's policies, there is also one-to-one correspondence between states (x, s) and (a_l, s), resulting in $c_l(a_l, s) = c_l(x, s)$ and, starting from $a_l(s_0)$ in (A.4), a similar recursive process, using the lender's budget constraint, defines the lender's asset policy $a'_l(a_l, s)$ and, by period-by-period resource feasibility implies that $a'_l(a_l, s') = -a'(a, s, s')$.

Fourth, given the characterization of the Fund's optimal policies (see Lemma 2), we can define the borrowing and lending limits for every s:

$$\mathcal{A}_b(s) = a(\underline{x}(s), s) \text{ and } \mathcal{A}_l(s) = a(\overline{x}(s), s)$$

Note that these limits are history-independent and hence they are functions of only the exogenous state s. Note also that these borrowing constraints imply that $a'(a, s) \geq \mathcal{A}_b(s)$ and $a'_l(a_l, s) \geq \mathcal{A}_l(s)$ for all s; i.e., the constructed asset holdings satisfy the competitive equilibrium borrowing constraints (33) and (40).[34] In sum, the policy functions, as functions of (a, s) and (a_l, s), satisfy all the constraints of borrowers and lenders' competitive equilibrium problems.

Fifth, we show that the defined policy functions also satisfy the corresponding optimality conditions. Using the consumption first order condition in the Fund, **(**12**),** we can set

$$\lambda(a, s) = \frac{1 + \nu_l(x, s)}{1 + \nu_b(x, s)} \frac{1}{x}.$$

This guarantees that the consumption policy $c(a, s)$ is optimal in the competitive equilibrium as it satisfies (34). Since $c(x, s)$ and $n(x, s)$ satisfy the Fund labor optimality condition in (13), $c(a, s)$ and $n(a, s)$ satisfy the equilibrium labor optimality condition in (35). For

[34] In particular, $\mathcal{A}_b(s)$ and $\mathcal{A}_l(s)$ satisfy **?**'s condition of being 'not too tight.'

the risk-neutral lender $c_l(a_l, s) = c_l(x, s)$ is an optimal consumption policy, as long as the corresponding asset-portfolio is optimal. To see whether asset policies are optimal competitive policies we need to show that asset policies bind exactly when the limited enforcement constraints bind in the Fund.

Note that if $a'(a(s), s, s') > \mathcal{A}_b(s')$, then

$$\begin{aligned} q(s'|s) &\equiv \pi(s'|s, e(x,s))A(a', s')\beta \frac{u'(c(a'_{as}(s'), s'))}{u'(c(a,s))} \frac{1}{1+\tau^a(s', a, s)} \\ &\geq \pi(s'|s, e(x,s))A(a', s')\frac{1}{1+r}. \end{aligned}$$

Given the taxes we have constructed, it must be that $\nu_l(x', s') \geq \nu_b(x', s') = 0$. Similarly, if $a'_l(a_l, s') > \mathcal{A}_l(s')$, then

$$\begin{aligned} q(s'|s) &\equiv \pi(s'|s, e(x,s))A(a', s')\frac{1}{1+r} \\ &\geq \pi(s'|s, e(x,s))A(a', s')\beta \frac{u'(c(a'_{as}(s'), s'))}{u'(c(a,s))} \frac{1}{1+\tau^a(s', a, s)}, \end{aligned}$$

it must be that $\nu_b(x', s') \geq \nu_l(x', s') = 0$. Therefore the asset portfolio polices satisfy the equilibrium optimality conditions with respect to assets prices in (37) and (41).

Finally, provided that the effort policy $e(a, s)$ is also an optimal policy, the identification of value functions, $W^i(a, s) = V^i(x, s)$ for $i = b, l$, is consistent with their definitions: (9) and (10) become (31) and (38). But, by construction, $e(a, s) = e(x, s)$. If $W^b(a, s) = V^b(x, s)$ and $e(x, s)$ satisfies the IC constraint in (15), then $e(a, s)$ satisfies the equilibrium optimality condition for effort (36) as well. □

Proof of Proposition 2. We show that given a RCE we can design a Fund contract satisfying equations (7)–(13) and (15)–(18). First, as in the proof of Proposition 1, we obtain a one-to-one mapping between the states (a, s) and (x, s), establishing this correspondence with the initial state. The key equation for this mapping is (12); in particular, establishing the identitities:

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)} \frac{1}{x} = \lambda(a,s) = u'(c(a,s)). \tag{A.5}$$

Given the RCE initial asset holdings $(a(s_0), a_l(s_0))$, (A.5) determines $x(s_0)$ if the limited enforcement constraints are not binding. However, in a RCE the limited enforcement constraints at s_0 are simply the *individual rationality* constraints, which must be satisfied. Therefore, given a state s_0, we can associate to this state: $x(s_0) = 1/u'(c(a(s_0), s_0))$, since $a(s_0) \geq \mathcal{A}_b(s_0)$. This allows us to identify $m(x(s_0), s_0) = m(a(s_0), s_0)$, $m = c, n, e$ and $c_l(x(s_0), s_0) = c_l(a(s_0), s_0)$. More generally, we can identify (a, s) policies with (x, s) policies.

To see this, note that using the definition $\chi(x,s) \equiv \frac{x\varrho(x,s)}{1+\nu_l(x,s)}$ equation (7) can be written as

$$x'(s') = \frac{1}{u'(c(x,s))}\eta + \chi(x,s)\eta \frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}.$$

Let $\chi(a,s)$ be the solution to

$$\frac{1}{1+r}\sum_{s'|s} \partial_e \pi(s'|s,e(a,s)) V^{lf}(a'_{l,as}(s'),s') = \chi(a,s)\left[v''(e(a,s)) - \beta \sum_{s'|s} \partial_e^2 \pi(s'|s,e(a,s)) V^{bf}(a'_{as}(s'),s')\right],$$

then identifying $m(x,s) = m(a,s)$, $m = c, n, e$ and χ, not only equation (16) is satisfied but also the IC constraint (15) and the law of motion (7), which defines a one-to-one mapping between $a'_{as}(s') \equiv a'(a(s),s,s')$ and $x'_{xs}(s')$.

Regarding the limited enforcement multipliers and the constraint qualification constraints (17) and (18) we let $\nu_b(x,s) = 0$ if $a(s) > \mathcal{A}_b(s)$, while if $a(s) = \mathcal{A}_b(s)$, $\nu_b(x,s) \geq 0$ is determined by (A.5); similarly, we let $\nu_l(x,s) = 0$ if $a_l(s) > \mathcal{A}_l(s)$, while if $a_l(s) = \mathcal{A}_l(s)$, $\nu_l(x,s) \geq 0$ is determined by (A.5). Then, with the above identification of value functions, (17) and (18) are satisfied and, given that the borrower and lender's intertemporal budget constraints are satified — therefore, the resource feasibility constraints are also satisfied — equations (8)–(13) are also satisfied. In sum, there is a a *constrained-efficient Fund contract* which implements the RCE. □

B More Details on Calibration

B.1 Data Sources and Model Consistent Measures

The main data sources and relevant definitions of data variables are listed in Table B.1.

To map the data to the model, we construct model consistent data measures as below.

Labor input For the aggregate labor input n_{it}, we use two series from AMECO, the aggregate working hours H_{it} and the total employment E_{it} of each country over the period 1980–2015. We calculate the normalized labor input as $n_{it} = H_{it}/(E_{it} \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for most of the data moment computations, we use H_{it} directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

Fiscal position and private consumption We hold the premise of fitting the *observed* fiscal behavior across the GIPS countries, so that we use directlyd the *data measures* of government consumption and primary surplus to calibrate the model. However, the cost of

Table B.1: Data Sources and Definitions

Series	Time	Sources[a]	Unit
Output	1980–2015	AMECO (OVGD)	1 billion 2010 constant euro
Government consump.	1980–2015	AMECO (OCTG)	1 billion 2010 constant euro
Total working hours	1980–2015	AMECO (NLHT)[b]	1 million hours
Employment	1980–2015	AMECO (NETD)	1000 persons
Government debt	1980–2015	AMECO EDP[c]	end-of-year percentage of GDP
Debt service	1980–2015	AMECO (UYIGE)[d]	end-of-year percentage of GDP
Primary surplus	1980–2015	AMECO (UBLGIE)[e]	end-of-year percentage of GDP
Bond yields	1980–2015	AMECO (ILN)[f]	percentage, nominal
Debt maturity	1990–2010	OECD, EuroStat, ESM[g]	years
Labor share	1980–2015	AMECO[h]	percentage

[a] Strings in parentheses indicate AMECO labels of data series.
[b] PWT 8.1 values for Greece in 1980–1982.
[c] General government consolidated gross debt; ESA 2010 and former definition, linked series.
[d] AMECO for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[e] AMECO linked series for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[f] A few missing values for Greece and Portugal replaced by Eurostat long-term government bond yields.
[g] Average across different data sources; sporadic time coverage over countries, see text below; ESM data are obtained from private correspondance.
[h] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

such a strategy is on the model consistent measure of private consumption. Note that in the model, primary surplus equals to $y-g-c$, therefore private consumption equals to y mius the sum of g and primary surplus. This is the model consistent measure of private consumption we use in our calibration. Nevertheless, due to small magnitudes in primary surplus relative to GDP, the model consistent measure of private consumption tracks closely the dynamics of the alternative data measure of consumption,[35] and the correlation between the two measure is well beyond 0.97.

Government debt, spread, and maturity Since one of the major purposes of this paper is to provide a quantitaive assessment of the Euro Area 'stressed' countries, we choose to capture the overall debt burden of those countries by calibrating the general government consolidated gross debt. Indeed, **?** argue that matching the overall public debt allows a quantitative sovereign default model to better fit crisis dynamics.

We use the nominal long-term bond yields in AMECO to measure the nominal borrowing costs of the Euro Area 'stressed' countries. For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtaied from EuroStat with label `irt_st_q`) for 1999–2015, and the annulized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1980–1998, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany

[35]Indeed, the alternative measure is private absorption defined as the sum of private consumption and investment as measured in the data, since there is no capital in our model.

from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting the 'stressed' countries' credit risk, we split the sample into to two parts. After the introduction of Euro, we can directly use the spread between the 'stressed' countries' long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in euro and thus subject to the same inflation expectation. The question is much trickier for the period before Euro. Motivated by **?**, we use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into each stressed country's local currency, hence deriving a synthetic local currency risk free rate, and then take the difference between the local nominal long-term bond yield with the synthetic risk free rate. Since the synthetic risk free rate is denominated in the local currency as well, it is subject to the same inflation expectations as the long-term bond yield, and consequently, the difference is equivalent to the real spread.

The information on the maturity structure of the government debt for the GIPS countries is not comprehensive. The overall time coverage is unequal across countries: 1998–2010 and 2014–2015 for Ireland, 1998–2015 for Greece, 1991–2015 for Spain, 1990–2015 for Italy, and 1995–2015 for Portugal.

B.2 More Details on Productivity Shock Estimation

We implement the panel Markov regime switching AR(1) estimation of the productivity process following the expectation maximization approach outlined in **?**. To overcome the local maximum problem, we randomize the initialization by 50,000 times. Apart from the parameter estimation results reported in the main text, Figure B.1 shows the smoothed probability for each regime across the GIPS countires. Evidently, regime 3 concentrates around the global financial crisis and the European debt crisis. As a last remark, we discretize the regime switching AR(1) process with 9 grid points for each regime using the method detailed in **?**.

B.3 The Numerical Values of the Transition Matrix for g

The parameter values imply the following transition matrices:

$$\bar{\pi}^g = \begin{bmatrix} 0.9650 & 0.0233 & 0.0117 \\ 0.0300 & 0.9650 & 0.0050 \\ 0.0150 & 0.0200 & 0.9650 \end{bmatrix}, \ \pi^h = \begin{bmatrix} 0.93 & 0.0466 & 0.0234 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}, \ \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 0.06 & 0.94 & 0 \\ 0.03 & 0.04 & 0.93 \end{bmatrix}.$$

Note that the *average* distribution $\bar{\pi}^g$ of g constraints the possible effect of effort. Even in this 'extreme' case, the effect of effort is limited. For example by moving effort from 0 to 1 the borrower can increase the chance of reducing government expenditure from 0 to only 7% if the current expenditure is very high.

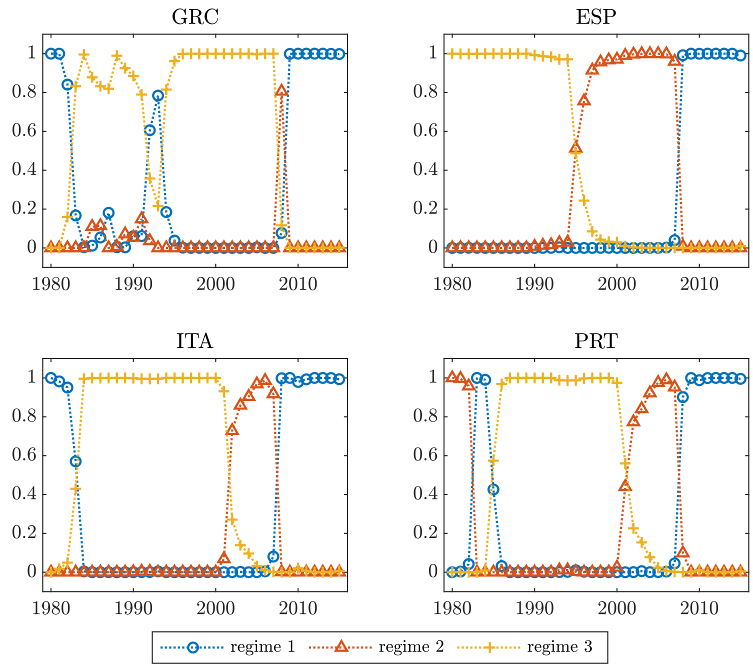

Figure B.1: Smoothed probability for each regime

B.4 Transition Probabilities for Correlated g and θ

Based on the convenient fact that the number of regimes for θ and the number of values g can take both equal to 3, we extend the baseline conditional distribution (48) for $\pi^g(g'|g,e)$ to $\tilde{\pi}^g(g'|\varsigma,g,e)$ as follows:

$$\begin{aligned}
&\tilde{\pi}^g(g' = g_j|\varsigma = i, g = g_k, e) \\
&= w[\zeta(e)\pi^l(j|4-i) + (1-\zeta(e))\pi^h(j|4-i)] \\
&\quad + (1-w)[\zeta(e)\pi^l(j|k) + (1-\zeta(e))\pi^h(j|k)], \quad i,j = 1,2,3,
\end{aligned} \tag{B.1}$$

where i denotes the regime of θ, j denotes the value of future g', and k denotes the value of current g. The additional parameter $w \in [0,1]$ controls for the influence on the distribution of g' coming from regime ς of θ: if $w = 1$, then g' only depends on ς but not on g; in contrast, if $w = 0$, then g' does not depend on ς, and the transition probability is identical to the baseline specification in (48). Moreover, the index $4-i$ in the second line suggests that when the current regime for θ is high, i.e., i is larger, then not only future θ' is high since ς is persistent, but g' also tends to be higher by the persitency inherent to the structure of π^h and π^l in (49). This feature induces positive correlation between g and θ for any w. Given $\tilde{\pi}^g$ so defined, it is straightforward to construct the overall transition matrix $\pi(s'|s,e)$ accordingly.