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Pairwise justifiable changes in collective choices*

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Abstract

Consider the following principle regarding the performance of collective choice functions. "If a rule selects alternative x in situation 1, and alternative y in situation 2, there must be an alternative z, and some member of society whose appreciation of z relative to x has increased when going from situation 1 to situation 2." This principle requires a minimal justification for the fall of x in the consideration of society: someone must have decreased its appreciation relative to some other possible alternative. On appropriately restricted domains, pairwise justifiability, along with anonymity and neutrality, characterizes Condorcet consistent rules, thus providing a foundation for the choice of the alternatives that win by majority over all others in pairwise comparisons, when they exist. We also study the consequences of imposing this requirement of pairwise justifiability on a large class of collective choice correspondences that includes social choice and social welfare functions as particular cases. When preference profiles are unrestricted, pairwise justifiability implies dictatorship, and both Arrow's and the Gibbard-Satterthwaite's theorems become corollaries of our general result.

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1 Introduction

We introduce a novel principle called pairwise justifiability, regarding the performance of collective choice rules. In the simple case where society's decisions are singletons and individual preferences are strict, the principle says the following:

"If a rule selects alternative x in situation 1, and alternative y in situation 2, when both were available in either case, there must be an alternative z, and some individual whose appreciation of z relative to x has increased when going from situation 1 to situation 2; likewise, there must be some alternative w and some individual whose appreciation of w relative to y has decreased."

Pairwise justifiability demands a minimal reason for the fall of an alternative x or the rise of an alternative y in the consideration of society as a response to any change in the situation that it faces: that is, as a response to changes in the set of feasible alternatives or to changes in someone's preferences. In fact, we have stated the requirement at full length for clarity, but we only need to impose it in one of its two directions, since each one carries the other. Extending the condition to cases where individual preferences admit indifferences requires to introduce some nuances in the case where changes in the social decision may result from the use of tie-breaking procedures among alternatives that are considered indifferent by some agent. We leave a discussion of this and other technical requirements for further sections.

We study the consequences of imposing this novel principle and its extensions to a general class of collective choice rules, that are defined on collections of situations, consisting of preference profiles included in a set $\mathcal{D}$ and of subsets of alternatives, called agendas, belonging to a collection $\mathcal{B}$. A collective choice rule selects, for each profile in $\mathcal{D}$, one or several alternatives in each of the agendas in $\mathcal{B}$: if it may select more than one, we call it a collective choice correspondence; if only a singleton can be selected, we call it a collective choice function.

Pairwise justifiability is not only intuitively attractive: it has bite when trying to discern between different collective choice rules. This is exemplified, in different directions, by the main results we obtain in this paper. When applied to anonymous and neutral rules, it is an implication of Condorcet consistency, and equivalent to this respected and classical requirement in many situations of interest.

Yet, as it happens with most combinations of attractive normative criteria, pairwise justifiability may not be satisfied by any non-dictatorial collective choice rule defined on the universal domain. Indeed, we establish that only dictatorial collective choice functions can satisfy pairwise justifiability, and also reach the same conclusion for collective choice correspondences satisfying an additional condition that we call weak decisiveness. These general results have Arrow's and the Gibbard-Satterthwaite's theorems as corollaries. This proves that these classical results have a common root, even in their most general version.

Collective choice rules as defined can be used to analyze many interesting real-life situations in which specific subsets of alternatives are faced by society, and others are not. Consider, for example the case where the citizens in a jurisdiction must decide what subset of projects to undertake simultaneously out of a list of them whose joint cost exceeds the community's available resources. Then, the set of feasible agendas consists of those collections of projects whose total cost is within the community's constraint.

In addition to economic constraints, sometimes society's decisions must accommodate to institutional ones. For example, the composition of parliamentary committees typically must satisfy a principle of proportionality, but the constraints imposed by this principle may vary across committees depending on their jurisdictions. In that case, the feasible agendas would consist of those that respect the admissible proportionality bounds, typically more than one and less than the total number of those conceivable. As a final example, consider the case where, in order to create a diverse environment for students, school districts implement controlled school choice programs providing parental choice, while maintaining the racial, ethnic or socioeconomic balance at schools. Controlled public school admission policies put hard or soft bounds on the groups of students who can be admitted to a school, and these bounds may vary according to the admission criteria. Each set of admission criteria identifies what subsets of agendas (in that case, what partition of students into groups) constitute feasible agendas.

A more technical but very important question, also derived from the fact that we work with rules defined on a variety of situations, refers to the comparison between pairwise justifiability and other conditions that social choice theorists have deemed important for the study of collective decision-making methods. These conditions are usually defined for specific subcases of our encompassing definition of collective choice rules, and it is not always obvious how to compare them with ours, due to the use of different frameworks. But we also proceed to establish a number of comparisons, enough to prove that pairwise justifiability is independent or weaker than other well known normative criteria.

Our main results address important issues that have been debated for a long time. Social choice theory has seen the advocates of scoring rules oppose those who support Condorcet consistent rules and the majority principle. While several axiomatizations have been offered in the literature for scoring rules[1], very few axiomatic characterizations of the Condorcet principle have been proposed. Campbell and Kelly (2015), and more recently Yonta Mekuko et al. (2021) are two notably exceptions.

The relation between Arrow's and the Gibbard-Satterthwaite's theorems has also been a matter of interest for a long time in social choice theory. Well before the work of Gibbard (1973) and Satterthwaite (1975). Vickrey (1960) had already conjectured that there was a strong connection between

[1]Fishburn (1973), Smith (1973), Young (1974), Richelson (1978), Nitzan and Rubinstein (1981), Myerson (1995) to name a few.

strategy-proofness and Arrow's condition of Independence of Irrelevant Alternatives.[2] Gibbard used Arrow's theorem as an intermediate step of his proof. Satterthwaite explicitly analyzed the mutual implications between the conditions involved in these two theorems. The latter presented further evidence of the parallelism between strategy-proofness and Arrow's conditions, as well as in their respective proofs, and so did Pattanaik (1978), Muller and Satterthwaite (1977) and later Reny (2001), among other authors.

But it was only in 2004 that Kfir Eliaz (Eliaz, 2004) proved that these two major results could be obtained as corollaries of a single theorem predicated on rules that contain Arrowian social welfare functions and social choice functions as particular cases. He defined such a class of rules, that he called social aggregators, and proved that when these satisfy a condition termed preference reversal, which is implied by Arrow's conditions and by strategy-proofness in strict preference domains, they must be dictatorial.[3] Our results differ from his in several respects and extends it substantially: it is expressed in terms of a different formulation of the aggregation problem, it uses the requirement of pairwise justifiability, which is a weaker condition than his, and, very importantly, it covers the general case where Arrow's theorem applies in a framework where both individual and collective preferences admit indifferences. Such an extension is much more than technical: it is necessary for a full coverage of our target theorems in their most general expression and differentiates our condition of pairwise justifiability from others in the literature. Although many equivalences arise between properties of functions defined on strict preference domains, pairwise justifiability is weakest when indifferences are not excluded, and allows for a direct use in the large framework where both preference profiles and agendas are relevant

[2]Vickrey stated that "social welfare functions that satisfy the nonperversity and the independence postulates, and are limited to rankings as arguments are (...) immune to strategy. It can be plausibly conjectured that the converse is also true".

[3]An unpublished paper by Barberà (2001) states a similar result but takes a different strategy of proof.

variables.[4]

The paper proceeds as follows. In Section 2 we provide notation and definitions. Section 3 investigates the connections between the requirement of Condorcet consistency and that of pairwise justifiability under different domains. In Section 4 we state a general dictatorship result for collective choice correspondences satisfying pairwise justifiability and weak decisiveness, and also a characterization result in the same spirit for collective choice functions defined on strict individual preferences. In Section 5 we show that Arrow's and the Gibbard-Satterthwaite theorem are corollaries of our main dictatorship result. Section 6 discusses the connections between pairwise justifiability and other conditions proposed in the social choice literature. Section 7 concludes with some final remarks. Although some proofs are outlined in the text, they are collected in their formal and complete form in the Appendix, which is organized by sections, and also contains several examples.

2 Notation and definitions

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$. Let A be a set of *alternatives* with $\#A \geq 3$. We denote subsets of alternatives as B, B',... and we call them *agendas*. We denote by $\mathcal{A}$ the set of all nonempty subsets of A and by $\mathcal{B} \subseteq \mathcal{A}$ a *collection of subsets of alternatives*, or equivalently, *a collection of agendas.*

Let $\mathcal{R}$ be the set of all *preferences* on A (that is, all complete, reflexive, and transitive binary relations on A). Elements of $\mathcal{R}$ are denoted by R_i, R_j,... The top of a preference $R_i \in \mathcal{R}$ in $B \in \mathcal{B}$, denoted by $t(R_i, B)$, is the set of alternatives $x \in B$ such that xR_iy for all $y \in B$. As usual, P_i and I_i denote the strict and indifference preference relation induced by R_i, respectively.

[4]More recently, Man and Takayama (2013) also proved interesting results in the spirit of Eliaz (2004) and Barberà (2001), but again using additional properties and assuming that preferences are strict in some cases, which blurs the comparison between the two theorems to be unified.

Let $\mathcal{R}^n$ be the set of all possible *preference profiles,* also called *the universal domain,* and $\mathcal{D} \subseteq \mathcal{R}^n$ be a subset of preference profiles. Elements of $\mathcal{R}^n$ are denoted by $R = (R_1, R_2, ..., R_n)$. When we have a partition of N into different sets $S_1, S_2, ..., S_k$, we write a preference profile as $R = (R_{S_1}, R_{S_2}, ..., R_{S_k})$.

Let $\mathcal{P}^n \subseteq \mathcal{R}^n$ denote the subset of all preference profiles where agents' preferences on A are *strict* (that is, also antisymmetric), also called *the strict universal domain.*

A *situation* is a pair $(R, B) \in \mathcal{D} \times \mathcal{B}$.[5]

A *collective choice correspondence* on $\mathcal{D} \times \mathcal{B}$ is a mapping $C : \mathcal{D} \times \mathcal{B} \to \mathcal{A}$ that assigns a non-empty subset of alternatives $C(R, B) \in 2^B \backslash \{\varnothing\}$ for each situation $(R, B) \in \mathcal{D} \times \mathcal{B}$.

A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ has *full range* if for each $B \in \mathcal{B}$ and $x \in B$ there exists $R \in \mathcal{D}$ such that $x \in C(R, B)$.

Definition 1 *A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ is **anonymous** on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if, for any preference profile $R = (R_1, R_2, ..., R_n) \in \mathcal{D}'$, any $B \in \mathcal{B}$, and any permutation ρ of N such that $(R_{\rho(1)}, R_{\rho(2)}, ..., R_{\rho(n)}) \in \mathcal{D}'$, then $C(R, B) = C\big((R_{\rho(1)}, R_{\rho(2)}, ..., R_{\rho(n)}), B\big)$.*

Definition 2 *A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ is **neutral** on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if, for any preference profile $R = (R_1, R_2, ..., R_n) \in \mathcal{D}'$, any $B \in \mathcal{B}$, and any permutation μ of B such that $(\mu(R_1), \mu(R_2), ..., \mu(R_n)) \in \mathcal{D}'$, then $\mu(C(R, B)) = C\left((\mu(R_1), \mu(R_2), ..., \mu(R_n)), B\right)$.*

We now formalize the principle whose introduction and analysis is the object of our work, that of *pairwise justifiability.* We first state it in a general form, that applies to collective choice correspondences. Immediately after that, we discuss the consequences of applying it in two special cases that will

[5] A situation is a pair formed by a preference profile and an agenda in Le Breton and Weymark (2011).

be the main object of our attention in what follows, and we discuss why we attach importance to this principle in each of its forms.

Definition 3 *A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ satisfies* ***pairwise justifiability*** *on $\mathcal{D}' \times \mathcal{B}$, $\mathcal{D}' \subseteq \mathcal{D}$ if, for any two situations (R, B), $(R', B') \in \mathcal{D}' \times \mathcal{B}$ such that $x \in C(R, B)$, $x \notin C(R', B')$, and there exists $y \in C(R', B')$ such that $x, y \in B \cap B'$, then either (1) there is some agent $i \in N$ and some alternative $z \in A \backslash \{x\}$ such that xP_iz and zR'_ix, or (2) there is some agent $i \in N$ such that xI_iy and $R_i \neq R'_i$.*

A special type of collective choice correspondences that we are especially interested in are those that, in fact, select one and only one alternative for each situation: we call them collective choice functions. Clearly, they result from restricting the range of the correspondence to singletons.

We may also concentrate on correspondences that are defined on different restricted domains, but at this point, we limit our attention to a simple one, that is rather natural in many contexts, especially when the set of alternatives is finite. The constraint is that correspondences be defined on domains that only include strict individual preferences.

Our informal statements regarding pairwise justifiability in the introduction refer to the special but important case of collective choice functions defined on domains of strict preferences. There, the requisite that every fall in the social appreciation of an alternative needs to be justified by its fall in the favor of at least one agent is implied without any reservation. This is the basis of our proposal to retain pairwise justifiability as an attractive requirement for collective choice functions and correspondences.

Why, then, add as a special proviso, the exceptional case where this requirement may be ignored if some agent who changes preferences from one situation to another was indifferent between the respective outcomes in the two situations (or with parts of these outcomes when considering correspondences)? We accept this exception because individual indifferences leave room for social choices that are indifferent for some agents and yet result from the use of tie breaking rules that may be quite elaborate and hard to track down by an external observer. This implicit acceptance that tie breaking rules may obscure otherwise transparent conditions is recognized in many parts of the literature. For example, Arrow's notion of a dictator allows for other agents to influence the outcome when the dictator is indifferent. Therefore, even if our general condition involves a subtle departure from its neatest and naïve form when indifferences are not present, we feel that it continues to capture its essence.

Finally, let us also introduce another special case of collective choice correspondences that result from imposing a restriction on their range when their domain is restricted. We refer to the case where, when defined on the subdomain of strict individual preferences, the correspondence is required to select singletons, and only allowed to become multivalued when some agents' preferences admit indifferences. We say that these correspondences satisfy weak decisiveness. Admittedly, weak decisiveness is a demanding requirement, but it will be useful to formulate some of our results. Formally:

Definition 4 *A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$, $\mathcal{D} \subseteq \mathcal{R}^n$ satisfies **weak decisiveness** if for any situation (R, B) such that $R \in \mathcal{D} \cap \mathcal{P}^n$ then $\#C(R, B) = 1$.*

Observe that a collective choice function trivially satisfies weak decisiveness on $\mathcal{D} \times \mathcal{B}$, for any $\mathcal{D} \subseteq \mathcal{R}^n$ but not any collective choice correspondence satisfying weak decisiveness is a collective choice function. The latter is because when individual indifferences exist, more than one alternative can be chosen. Moreover, any collective choice correspondence satisfying weak decisiveness is a collective choice function when defined on any domain $\mathcal{D} \subseteq \mathcal{P}^n$.

3 Pairwise justifiability and Condorcet consistency

In this section we show that pairwise justifiability is strongly related with the classical and appealing requirement of Condorcet consistency. We begin by formally defining this property in the context of collective choice functions and briefly elaborate about its intrinsic interest. Then we prove that our proposed principle of justifiability, when applied to anonymous and neutral collective choice correspondences, is in general weaker than Condorcet consistency but becomes equivalent to it in many well identified cases. That indicates that our notion of justifiability is rather fundamental, as it provides new additional arguments in favor of that classical principle. It also shows that, in spite of the negative character of the results that we will exhibit in the next section, pairwise justifiability may be used as a useful guide to discriminate between alternative collective choice rules defined on restricted domains.

Let $(R, B) \in \mathcal{D} \times \mathcal{B}$ be a situation. We say that an alternative $y \in B$ *defeats* alternative $z \in B$ by *majority* at $R \in \mathcal{D}$ if the number of agents who strictly prefers y over z is greater than the number of those who strictly prefer z over y. We say that an alternative $y \in B$ is the (unique) *strong Condorcet winner at* (R, B) if y defeats any other alternative in B by majority at R.

Definition 5 *A collective choice function C on $\mathcal{D} \times \mathcal{B}$ is **Condorcet consistent** on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}$ if for each situation $(R, B) \in \mathcal{D}' \times \mathcal{B}$ we have that $C(R, B)$ selects the strong Condorcet winner at (R, B) when it exists.*

For centuries now, this requirement, which demands that if one alternative is a strict majority winner over all others, it should be selected, has attracted much attention. This is understandable, because a first and foremost question in the theory of voting has been how to extend the notion of majority to the case where society faces more than two alternatives, especially as part of a criticism to the use of plurality voting, which can grossly deviate from any reasonable idea of respect to majorities. While many social choice theorists find Condorcet's principle very attractive and compelling, others defend the use of scoring methods, which are notoriously distanced from this view.[6] But since respect of the Condorcet principle remains high in the list of favorite extensions of majority, we consider very significant to exhibit its strong connection with pairwise justifiability, thus, proving the strength and the interest of our new proposed principle.

Since we consider collective choice rules, the consistency criterion has bite for all those situations for which, given the preference profile R, a strong Condorcet winner exists for the agenda B. Hence our definition:

Definition 6 *Given a set $\mathcal{B}$ of agendas and a set $\mathcal{D}$ of admissible profiles, the **Condorcet domain** $\mathcal{D}_{C\mathcal{B}}$ **of** $\mathcal{B}$ **in** $\mathcal{D}$ is the subset of preference profiles such that for all situations in $\mathcal{D}_{C\mathcal{B}}\times\mathcal{B}$, there exists a strong Condorcet winner.*

Notice that the notion of a Condorcet domain is conditional to the reference sets $\mathcal{B}$ and $\mathcal{D}$. Enlarging $\mathcal{D}$ or adding agendas to any given $\mathcal{B}$ may possibly narrow down the relevant Condorcet domain.[7]

Specifically, if $\mathcal{D}$ is the universal domain the Condorcet domain $\mathcal{D}_{C\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{D}$ is the set of all preferences profiles for which a strong Condorcet winner exists for all agendas in $\mathcal{B}$. But we can also define the Condorcet domain for other cases of interest: for example, those where $\mathcal{D}$ only contains strict preferences or where $\mathcal{D}$ is the set of all single-peaked preferences profiles.

Our Theorem 1 offers a characterization of Condorcet consistent rules defined on the largest set of preference profiles where a strong Condorcet

[6] See Moulin (1988), Chapter 9, for an illuminating discussion of the tension between scoring methods and Condorcet consistent rules.

[7] For example, consider the case with three agents, four alternatives $A = \{x, y, z, t\}$, $\mathcal{B} = \{A\}$, and $\mathcal{D}$ the universal domain. Then, any profile of preferences where each agent has x as the top alternative in $\mathcal{B}$ belongs to the Condorcet domain of $\mathcal{B}$ in $\mathcal{D}$. However, not all of these profiles can guarantee the existence of a strong Condorcet winner for $\mathcal{B} = \{A, B\}$ with $B = \{y, z, t\}$ in $\mathcal{D}$: consider the profile R such that $xP_1yP_1zP_1t$, $xP_2zP_2tP_2y$, $xP_3tP_3yP_3z$.

winner exists given the collection of agendas, and shows that the Condorcet principle is intimately linked with our property of pairwise justifiability.

Theorem 1 *Let $\mathcal{B}$ be a collection of agendas such that for all $B, B' \in \mathcal{B}$, $\langle B \cup B' \rangle \in \mathcal{B}$ and $\mathcal{D} = \mathcal{R}^n$. A collective choice function C on $\mathcal{D} \times \mathcal{B}$ is anonymous, neutral, and satisfies pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ if and only if C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$.*

The proof of Theorem 1 is in the Appendix. The same result holds for the strict universal domain and for any collection $\mathcal{B}$.[8]

Several observations are in order. First, notice that the result can be extended, in one direction, and show that any Condorcet consistent collective choice function is anonymous, neutral, and satisfies pairwise justifiability on any subdomain of a Condorcet domain, as stated in the following proposition proved in the Appendix.

Proposition 1 *Let $\mathcal{B}$ be a collection of agendas, $\mathcal{D} \subseteq \mathcal{R}^n$ a subset of preference profiles, and C be a collective choice function on $\mathcal{D} \times \mathcal{B}$. If C is Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$, then C satisfies pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$ for any $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$.*

Second, it is also worth noting that Condorcet consistency is not always a consequence of pairwise justifiability. In particular, this may not be true for all subdomains of the Condorcet domain on $\mathcal{B}$.[9] Still, the result of Theorem 1 applies to the two most popular cases in which the existence of Condorcet winners is guaranteed: it directly results from the proof of the theorem, both in case of intermediate preferences and in the case of single-peaked profiles.[10]

[8]Its proof follows the same logic as that of Theorem 1 but it is much simpler. It is available upon request.

[9]Let $N = \{1, 2\}$, $\mathcal{B} = \{B\}$ where $B = \{x, y, z\}$, and R, R' be two admissible profiles such that yP_1xP_1z, yP_2xP_2z and $yP'_1zP'_1x$, $yP'_2zP'_2x$. Define C such that $C(R, B) = x$ and $C(R', B) = z$. Note that C satisfies pairwise justifiability but it does not choose the strong Condorcet winner at any feasible situation.

[10]See Grandmont (1978) for intermediateness and Penn, Patty, and Gailmard (2011) for the definition of extended single-peakedness.

Third, observe that, although Condorcet consistency is usually predicated for situations involving a single agenda, we are also able to cover the consequences of pairwise justifiability on the choice of strong Condorcet winners by collective choice rules defined over situations that include multiple agendas. In the case of a single agenda and strict preferences, Campbell and Kelly (2015) show that the Condorcet rule is the unique rule satisfying anonymity, neutrality, and strategy-proofness on a Condorcet domain, under specific relationships between the number of agents and alternatives.[11] Our extension to multiple agendas is also interesting. For example, it may be used in the analysis of the many cases in political economy where preference profiles lie in single-crossing domains and society may confront agendas of different size. In these cases, for any agenda, the alternative preferred by the median voter is a strong Condorcet winner.

4 Dictatorship results

In the preceding section we showed that, when defined on properly restricted domains of preferences, pairwise justifiability is very closely related to the possibility of respecting the desirable objective of Condorcet consistency.

By contrast, in the present section we explore the consequences of imposing our condition of pairwise justifiability on collective choice rules defined on the universal domains of profiles formed by either strict or weak preference profile. We offer three similar results in this vein, showing that, as it is known to happen in other contexts and under different conditions, ours is also too demanding and precipitates dictatorship.[12]

[11] Their result and ours are independent given that the Condorcet domain they are considering does not have a Cartesian product structure and strategy-proofness and pairwise justifiability are independent properties in such domains (examples are available upon request).

[12] Examples are abundant. To mention one, consider for instance the notion of self-selectivity introduced by Koray (2000) which requires that a social choice function should choose itself from among other rival such functions when it is employed by the society to

Theorem 2 *If $A \in \mathcal{B}$, any full range collective choice correspondence C on $\mathcal{R}^n \times \mathcal{B}$ satisfying weak decisiveness and pairwise justifiability on $\mathcal{R}^n \times \mathcal{B}$ is dictatorial.*

Corollary 1 *If $A \in \mathcal{B}$, any full range collective choice function C on $\mathcal{R}^n \times \mathcal{B}$ satisfying pairwise justifiability on $\mathcal{R}^n \times \mathcal{B}$ is dictatorial.*

Theorem 3 *If $A \in \mathcal{B}$, a full range collective choice function C on $\mathcal{P}^n \times \mathcal{B}$ satisfies pairwise justifiability on $\mathcal{P}^n \times \mathcal{B}$ if and only if it is dictatorial.*

Notice that Theorem 2 applies to collective choice correspondences. It is the result, out of the three above, that refers to the larger class of collective choice rules. Observe that the statement in Theorem 2 is robust in the sense that when ruling out only one of the properties imposed, we can define rules satisfying the other properties.[13] Corollary 1 obtains from Theorem 2, because it only refers to collective choice functions, and this directly implies that weak decisiveness must be satisfied. However, we record it as an interesting result because it has content of its own and can be used directly in some applications. Theorem 3 is at the basis of the proof for the preceding results, and it is also of independent interest, since environments where individuals have strict preferences are often considered in the literature. Moreover, it is a full characterization result. Notice that both Theorem 2 and its corollary, when applied to the strict domain, imply one of the directions of Theorem 3, but not the other, since not all dictatorial rules on that domain are pairwise justifiable when individual indifferences are admitted to be part of preference profiles.

The detailed proofs of these results are in the Appendix. We provide here an outline of how we proceed. We start by proving Theorem 3, thus concentrating on the case where individual preferences are strict and the collective

make this choice as well. Koray (2000) shows that a unanimous and neutral social choice function is universally self-selective if and only if it is dictatorial.

[13]It is not difficult to think of collective choice rules violating either full range or pairwise justifiability. An example of a violation of weak decisiveness is available upon request.

choice rule is a function. Later on, we extend the necessity part of that result to the case contemplated in Theorem 2, by relaxing the requirements on the image of our rules and allowing for it to be a correspondence, and moreover allowing for indifferences in the preferences of individuals, while imposing the auxiliary condition of weak decisiveness. Two lemmas are used in the transition between one result to the other. As mentioned above, the proof of Corollary 1 is a straightforward consequence of Theorem 2.

Let us be more specific about the strategy of the proofs, starting by that of Theorem 3.

Checking the "if" part is straightforward. The "only if" part consists of several steps. In the first step we fix an agenda with at least three alternatives and show that our property implies the existence of a dictator on such fixed agenda. The proof of this step contains the novel definition of when an agent is determinant, which is different to the more usual and weaker notion of being pivotal. In the second step we compare the outcomes of the rule for varying agendas. The argument involves two cases, depending on whether or not both agendas have two elements each or at least one of them contains three or more alternatives. But the common starting point for both cases is that, since $A \in \mathcal{B}$, for any pair of agendas there will exist a third one with at least three alternatives, containing the two initial ones. Applying the previous step to that inclusive agenda precipitates the result that one and the same dictator prevails at all admissible profiles and for all relevant agendas. This is the dictator.

Now, to extend the dictatorship result in Theorem 3 to collective choice correspondences satisfying weak decisiveness, in order to obtain Theorem 2, we use the following Lemmas, which are proven in the Appendix.

Lemma 1 *If C on $\mathcal{R}^n \times \{B\}$ is a full range collective choice correspondence satisfying pairwise justifiability and weak decisiveness on $\mathcal{R}^n \times \{B\}$ then C on $\mathcal{P}^n \times \{B\}$ has full range.*

Lemma 2 *Let C be a collective choice correspondence satisfying pairwise justifiability on $\mathcal{R}^n \times \{B\}$. If there is a dictator i for the restriction of C to $\mathcal{P}^n \times \{B\}$, then C is dictatorial and i is the dictator on $\mathcal{R}^n \times \{B\}$.*

5 Applications: Gibbard-Satterhtwaite's and Arrow's impossibility results

As announced and motivated in the introduction, we now apply our main theorem to prove that Arrow's and the Gibbard-Sattherthwaite theorems are both corollaries of our results in the preceding section.

5.1 Gibbard-Satterhtwaite's theorem

In this subsection we present a first application using the results in Theorem 3 and Corollary 1. The latter says that pairwise justifiability triggers dictatorship for any collective choice function defined on the universal domain and on any collection of agendas $\mathcal{B}$ that contains the set A of alternatives. We show that the Gibbard-Satterthwaite theorem can be obtained as a corollary of our results.[14]

Social choice functions, in fact, can be viewed as specific collective choice functions as follows:

A **social choice function** $f : \mathcal{D} \to B$ where $B \subseteq A$ is a collective choice function C on $\mathcal{D} \times \mathcal{B}$ for $\mathcal{B} = \{B\}$.

Note that properties on f can be trivially translated as properties on C, and viceversa. Thus, we use f and C indistinctly.

Observe that when $\mathcal{B} = \{A\}$, Theorem 3 and Corollary 1, applied to $\mathcal{D} = \mathcal{P}^n$ and $\mathcal{D} = \mathcal{R}^n$, respectively, provide the same impossibility result

[14] If we assumed that only agendas with two alternatives are feasible, by May (1952) we already know that the majority rule is pairwise justifiable and the result of Theorem 3 does not hold.

as in Gibbard-Satterthwaite using pairwise justifiability instead of strategy-proofness, where the latter is defined as usual.

Definition 7 *Let $B \in \mathcal{B}$ and $\mathcal{D} \in \{\mathcal{P}^n, \mathcal{R}^n\}$. A social choice function $f : \mathcal{D} \to B$ is **strategy-proof** on $\mathcal{D}$ if for any agent $i \in N$, any preference profile $R \in \mathcal{D}$, and any agent i's preference R'_i, $f(R) R_i f(R'_i, R_{N \setminus \{i\}})$.*

Proposition 2 below clarifies why Gibbard-Satterthwaite's results can be obtained as a corollary.

Proposition 2 *Let $B \in \mathcal{B}$ and $\mathcal{D} \in \{\mathcal{P}^n, \mathcal{R}^n\}$. A social choice function $f : \mathcal{D} \to B$ is strategy-proof on $\mathcal{D}$ if and only if it satisfies pairwise justifiability on $\mathcal{D} \times \{B\}$.*

The proof is included in the Appendix. When individuals are allowed to be indifferent among alternatives, some dictatorial social choice functions may not satisfy strategy-proofness. Likewise, it is also possible to construct a dictatorial social choice function that violates pairwise justifiability when the dictator is indifferent between two alternatives.

5.2 Arrow's theorem

This subsection is devoted to discuss in detail how we prove that Arrow's result derives from ours. The proof that Arrow's theorem is also a corollary of our results is more challenging than the proof that the Gibbard-Satterthwaite's theorem is a corollary of our results.

First, we define social welfare functions in terms of Arrow's language and their relationship with collective choice correspondences.

A **social welfare function** F on $\mathcal{D}$ is a mapping from $\mathcal{D}$ to $\mathcal{R}$. For any $R \in \mathcal{D}$, $F(R) \in \mathcal{R}$ denotes the binary relation that F assigns to R.

Definition 8 *A social welfare function F on $\mathcal{D}$ is **dictatorial** if for any $R \in \mathcal{D}$ and any $x, y \in A$, there exists $i \in N$, the dictator, such that if $x R_i y$ then $x F(R) y$.*

Definition 9 *A social welfare function F on $\mathcal{D}$ satisfies the **weak Pareto** condition if for any $R \in \mathcal{D}$ and any $x, y \in A$, if xP_iy for any $i \in N$, then $xF(R)y$ but not $yF(R)x$.*

Definition 10 *A social welfare function F on $\mathcal{D}$ satisfies **independence of irrelevant alternatives** if for any $R, R' \in \mathcal{D}$ and any $x, y \in A$, if [for any $i \in N$, $xR_iy \Longleftrightarrow xR'_iy$] then [$xF(R)y \Longleftrightarrow xF(R')y$].*

To connect social welfare functions with collective choice correspondences we need the following definition.

Definition 11 *A collective choice correspondence C on $D \times \mathcal{A}$ is **transitively rationalizable** if for any $R \in \mathcal{D}$, there exists a transitive binary relation on A, say $\mathbf{R}_R \in \mathcal{R}$, that rationalizes C (that is, for any agenda $B \in \mathcal{A}$, $C(R, B) = t(\mathbf{R}_R, B)$).*

We identify collective choice correspondences defined on all possible agendas of size two or larger which are transitively rationalizable with social welfare functions. We then observe that each social welfare function uniquely defines a transitively rationalizable collective choice correspondence, and vice versa. The connection between these two objects is finally established in the following proposition by the fact that any pairwise justifiable collective choice correspondence is transitive rationalizable. The proof is in the Appendix.

Proposition 3 *Any collective choice correspondence C on $\mathcal{D} \times \mathcal{A}$ satisfying pairwise justifiability on $\mathcal{D} \times \mathcal{A}$ is transitively rationalizable on $\mathcal{D} \times \mathcal{A}$.*

In what follows, abusing of the language, when we say that a social welfare function F satisfies either pairwise justifiability or weak decisiveness, we mean that the associated collective choice correspondence satisfies either of them.

After having identified collective choice correspondences as social welfare functions, note that by applying Theorem 2 for $\mathcal{B} = \mathcal{A}$, we get Arrow's impossibility result for the universal domain: any collective choice correspondence satisfying pairwise justifiability and weak decisiveness must be dictatorial. Since a collective choice correspondence on $\mathcal{P}^n \times \mathcal{A}$ satisfying weak decisiveness is a collective choice function, by Theorem 3 for $\mathcal{B} = \mathcal{A}$ we get Arrow's impossibility result for the strict universal domain.

Propositions 4 and 5 proven in the Appendix, clarify why Arrow's results can be obtained as corollaries of our Theorems.

Proposition 4 *If a social welfare function F on $\mathcal{D}$ for $\mathcal{D} = \{\mathcal{P}^n, \mathcal{R}^n\}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then C on $\mathcal{D} \times \mathcal{A}$ satisfies pairwise justifiability.*

Proposition 5 *If a social welfare function F on $\mathcal{D}$ for $\mathcal{D} = \{\mathcal{P}^n, \mathcal{R}^n\}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then C on $\mathcal{D} \times \mathcal{A}$ satisfies weak decisiveness.*

See their proofs in the Appendix. As noted by Mossel and Tamuz (2012), the connection between Arrowian social welfare functions and dictatorial rules is no longer one of equivalence when individuals are allowed to be indifferent among alternatives. Some dictatorial rules may not satisfy the conditions of Arrow's theorem. Likewise, it is also possible to construct a dictatorial social welfare function that violates pairwise justifiability when the dictator is indifferent between two alternatives.

6 Connections between pairwise justifiability and other conditions

Pairwise justifiability resembles several properties that have been proposed in the social choice literature. Nevertheless, most of them apply only to preference domains that do not admit indifferences. Among the properties that defined on preference domains admitting individual indifferences, we deem important to compare pairwise justifiability with strategy-proofness and Maskin monotonicity, two well-known properties for social choice functions.

The following two examples prove that our property and strategy-proofness are in general independent.

Example 1 *Let* $N = \{1, 2\}$ *and* $A = B = \{x, y, z, w\}$. *The set of admissible preference profiles is* $\mathcal{D} = \{R_1, R_1'\} \times \{R_2, R_2'\}$ *where* R_1*:* $yI_1wP_1xI_1z$ *and* R_1'*:* $yP_1'wP_1'xI_1'z$, R_2*:* $zI_2yP_2xI_2w$ *and* R_2'*:* $xI_2'wP_2'zI_2'y$. *Define* f *such that* $f(R_1, R_2) = x$, $f(R_1, R_2') = w$, $f(R_1', R_2) = z$, *and* $f(R_1', R_2') = y$. *It is easy to check that* f *is strategy-proof. However,* f *violates pairwise justifiability. To show the latter, take* (R_1, R_2) *and* (R_1', R_2') *and observe that no alternative improves with respect to* $f(R_1, R_2) = x$ *for no agent (equivalently, for any agent* i*, the lower contour set at* x *from* R_i *to* R_i' *weakly increases). Moreover, no agent is indifferent between* x *and* y *under* (R_1, R_2)*:* yP_1x *and* yP_2x.

Example 2 *Let* $N = \{1, 2\}$ *and* $A = B = \{x, y, z, w\}$. *The set of admissible preference profiles is* $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, $\mathcal{D}_1 = \mathcal{D}_2 = \{R, R'\}$ *where* R*:* $xIzPyPw$ *and* R'*:* $wI'xP'yI'z$. *Define* f *such that* $f(R_1, R_2) = y$, $f(R_1, R_2') = f(R_1', R_2) = x$, *and* $f(R_1', R_2') = w$. *We can check that* f *satisfies pairwise justifiability. However,* f *violates strategy-proofness. To show the latter, take* R *and observe that agent 1 would strictly gain by saying* R_1' *instead of* R_1 *since* $f(R_1', R_2) = xP_1y = f(R)$. *To show that* f *satisfies pairwise justifiability one must consider any pair of profiles with different outcome and check that either part (1) or (2) of the condition holds. In this example ten comparisons are required. For the sake of illustration, we prove it only for one pair of profiles:* $R = (R_1, R_2)$ *and* $\widetilde{R} = (R_1', R_2)$. *Observe that from* R *to* $\widetilde{R}$ *there is an agent, 1, and an alternative,* w*, such that* $f(R) = yP_1w$ *and* $wP_1'y$.

Maskin monotonicity is a property that plays a fundamental role in the implementation literature (see Maskin, 1999).

Definition 12 *Let $B \in \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{R}^n$ be the domain. A social choice function $f : \mathcal{D} \to B$ satisfies* ***Maskin monotonicity*** *on $\mathcal{D}$ if for any pair of preference profiles $R, R' \in \mathcal{D}$ such that for each agent $i \in N$, $[f(R)R_i z \Rightarrow f(R)R'_i z]$ then $f(R') = f(R)$.*

When indifferences are allowed Maskin monotonicity implies pairwise justifiability but the converse does not hold. Proposition 6, proved in the Appendix, and Example 3 show that our property is weaker than Maskin Monotonicity.

Proposition 6 *Let $B \in \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{R}^n$. Any social choice function $f : \mathcal{D} \to B$ satisfying Maskin monotonicity on $\mathcal{D}$ satisfies pairwise justifiability on $\mathcal{D}$.*

Example 3 *Let $N = \{1, 2, ..., n\}$ and $A = B = \{x, y, z\}$. There are only two admissible preference profiles and indifferences are allowed: $\mathcal{D} = \{(R_1, R_{-1}), (R'_1, R'_{-1})\}$ where R_1: xP_1yI_1z, R'_1: $xI'_1yP'_1z$, and for any $j \in N \backslash \{1\}$, $R_j = R'_j \in \mathcal{R}$. Let f be such that $f(R) = x$ and $f(R') = y$. It is easy to check that f satisfies pairwise justifiability (from R to R', xP_1y and yR'_1x while from R' to R, xI'_1y). However, f violates Maskin monotonicity: for both R and R', all alternatives are worse or indifferent to x but their outcome differ.*

When only strict preferences are admissible, additional properties have been defined. In the strict universal domain of preferences, pairwise justifiability is not only equivalent to strategy-proofness and Maskin monotonicity but also to other well-known properties that have been defined in the literature, like strong positive association (see Muller and Satterthwaite, 1977) or strong monotonicity (see Moulin, 1988).[15]. All these properties bring to dictatoriality, and Theorem 3 states their equivalence with pairwise justifiability.[16] Among the properties proposed for the strict universal domain, the

[15]See a summary in Section 5 in Barberà, Berga, and Moreno (2012).

[16]Proposition 4 in Barberà, Berga, and Moreno (2012) shows that these equivalences break down when considering smaller domains of strict preferences.

preference reversal property proposed by Eliaz (2004) is of upmost importance for our setting because he presents a framework that encompasses, like our own, social choice functions and social welfare functions . Preference reversal can be rephrased as follows to facilitate comparison with ours: “If a rule chooses x to be socially better than y in situation 1, and y better than x in situation 2, it must be that at least one member of society prefers x to y in 1 and y to x in 2.” Formally, for social choice functions:

Definition 13 *Let $\mathcal{D} \subseteq \mathcal{P}^n$. A social choice function $f : \mathcal{D} \to B$ satisfies **preference reversal** on $\mathcal{D}$ if for any pair of preference profiles $R, R' \in \mathcal{D}$ such that $f(R) = x$ and $f(R') = y$, then there must exist one agent $i \in N$ such that xP_iy and $yP_i'x$.*

And for social welfare functions:

Definition 14 *A social welfare function $F : \mathcal{D} \to \mathcal{R}$ satisfies **preference reversal** if for any pair of preference profiles R and $R' \in \mathcal{D}$ and for any pair of alternatives $x, y \in A$, such that $xF(R)y$ and $yF(R')x$, there is some agent $i \in N$ such that xP_iy and $yP_i'x$.*

Eliaz proved his condition to imply dictatorship if there are at least three alternatives, as we also do. Hence, his principle and ours, when applied in a universal domain of preferences profiles, turn out to be equivalent, as they also are to Gibbard and Satterthwaite’s and to Arrow’s conditions. However, our notion of pairwise justifiability is strictly weaker, in general, than preference reversal, and just enough to imply, in addition, the positive results about Condorcet consistency as presented in Section 5. We also enlarge the scope of our analysis to include the case in which individual indifferences are allowed.

It is straightforward to notice that, by definition, preference reversal implies pairwise justifiability. However, the converse does not always hold as shown for social choice functions and for social welfare functions in the following example.

Example 4 *Let* $N = \{1, 2\}$*,* $B = A = \{x, y, z, w\}$*, and the set of admissible preference profiles is* $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$*,* $\mathcal{D}_1 = \mathcal{D}_2 = \{R^1, R^2, R^3, R^4\}$ *where* $xP^1wP^1yP^1z$*,* $wP^2xP^2zP^2y$*,* $yP^3zP^3xP^3w$*, and* $zP^4yP^4wP^4x$*.*[17]
Consider the Borda Count with the tie-breaking $w \succ z \succ y \succ x$ *which defines a social welfare function* F*.*[18]
We show that the social choice function f *rationalized by* F *on* A *violates preference reversal, but it satisfies pairwise justifiability. To show the latter, consider* $R = (R_1^1, R_2^3)$*, and* $R' = (R_1^1, R_2^4)$*. Observe that* $f(R) = y$ *and* $f(R') = w$*. Note that pairwise justifiability from* R *to* R' *is satisfied because* yP_2^3z *and* zP_2^4y*, and from* R' *to* R *is satisfied because* wP_2^4x *and* xP_2^3w*. A similar argument can be repeated for each pair of preference profiles, which would prove that pairwise justifiability holds. To check that* f *violates preference reversal, note that no agent has changed her preferences between* w *and* y *from* R *to* R'*.*

Now, we show that the social welfare function F *violates preference reversal and satisfies pairwise justifiability. The score at* R *of* w *is 2 while that of* y *is 4, and therefore,* $yF(R)w$*. The score at* R' *of* w *and* y *is 3, thus* $wF(R')y$*. Since no agent has changed her preferences from* R *to* R' *between* w *and* y*, this is a violation of preference reversal. Note that pairwise justifiability from* R *to* R' *is satisfied because* yP_2^3z *and* zP_2^4y*, and from from* R' *to* R *is satisfied because* xP_2^3w *and* wP_2^4x*. A similar argument can be repeated for each pair of preferences profiles and alternatives, which would prove that pairwise justifiability holds. Therefore, the Borda Count applied to each feasible agenda defines a collective choice function that satisfies pairwise justifiability.*

[17]Notice that these preference profiles satisfy one of the three forms of value restriction defined by Sen and Pattanaik (1969), called intermediate. Namely, for any triple of alternatives, there is one that never appears in the second place.

[18]This rule F is defined as follows: for each agent it allocates 3 points to the alternative at the top of the agent's preference order, 2 points to the alternative in the second place, and 1 point to the alternative in the third place. Then, the social welfare function is constructed by ranking alternative a over b when a's total score (summing points for a over all agents) is greater than b's total score, and use the tie-breaking rule when the two scores coincide.

We point out that the scope of our property goes beyond this framework, but this non-exhaustive set of comparisons suggests that, in addition to the applications that we discuss in this paper, an analysis of the condition's implications in different cases, preference domains and combinations of agendas will be worthy.

Given the equivalences among properties mentioned above, some of our results on Condorcet consistency are satisfied with conditions other than ours, or not, depending on the case. For example, when indifferences are allowed Condorcet consistency implies strategy-proofness under a Cartesian product domain but it does not imply Maskin monotonicity (strong positive association has no bite with indifferences).[19] Under the strict universal domain, the result in Theorem 1 holds by replacing pairwise justifiability by either Maskin monotonicity, strong positive association, or preference reversal since as above mentioned all of them are equivalent under this domain.

7 Final Remarks

We propose a novel property that provides a foundation for Condorcet consistent rules. It also offers a unifying result for the two most important impossibility results in social choice theory, in a general setting in which individual preferences are allowed to admit indifferences. In addition to the extreme cases where only one agenda is considered, or all possible agendas are deemed relevant, our collective choice rules can be used to analyze many interesting real-life situations in which specific subsets of alternatives are faced by society, and others are not.

This adaptability to variation in feasible agendas allows to introduce a novel definition of dictatorship, when the dictator changes depending on which agenda is under scrutiny.

Definition 15 *A collective choice correspondence C on $\mathcal{D} \times \mathcal{B}$ is a* ***dicta-***

[19]See Remark 2 and Example 5 in the Appendix.

torship collection *(on $\mathcal{D} \times \mathcal{B}$) if for each $B \in \mathcal{B}$ there exists an agent $i \in N$, the local dictator on $\mathcal{D} \times \{B\}$, such that for any $R \in \mathcal{D}$, $C(R, B) \subseteq t(R_i, B)$.*

In words, a collective choice correspondence is a dictatorship collection if there is a dictator for each agenda, though not necessarily the same for different agendas. When the dictator is the same for all agendas we obtain the usual definition of dictatorship that we also use in the paper.

We could have weakened the notion of pairwise justifiability by only requiring that the conditions it imposes should apply when comparing situations that involve the same agenda.[20] Under this weaker form of our condition and considering collections of agendas, we obtain the following theorem (which is proved as Step 1 of Theorem 3 in the Appendix).

Theorem 4 *Any full range collective choice rule C on $\mathcal{P}^n \times \mathcal{B}$ satisfies weak pairwise justifiability on $\mathcal{P}^n \times \mathcal{B}$ for any $\mathcal{B} \subseteq \{B \in \mathcal{A}$ such that $\#B \geq 3\}$ if and only if C is a dictatorship collection.*

For each given agenda, a single agent would have to be a dictator, however, this agent would no longer have to be the same as the agenda changed. Cases where, depending on the subject matter under discussion, citizens delegate the collective decision to someone with specific expertise could, then, enter as candidates to satisfy this weak version of pairwise justifiability.

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[20]Formally: We say that a *collective choice correspondence* C satisfies **weak pairwise justifiability** on $\mathcal{D}' \times \mathcal{B}$ if if, for any two situations with fixed agenda (R, B), $(R', B) \in \mathcal{D}' \times \mathcal{B}$ such that $x \in C(R, B)$, $x \notin C(R', B)$, and there exists $y \in C(R', B)$ such that $x, y \in B$, then either (1) there is some agent $i \in N$ and some alternative $z \in A \backslash \{x\}$ such that xP_iz and zR'_ix or (2) there is some agent $i \in N$ such that xI_iy and $R_i \neq R'_i$.

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Appendix

7.1 Proofs of results in Section 3

Before proving Theorem 1, we provide a useful remark, whose proof involves the construction of a specific strict order $\mathbf{R}_R^{\mathcal{B}}$ that rationalizes the choices of C at all situations given by a fixed R, as the agendas vary over all of those in $\mathcal{B}$ while the Condorcet winner exists. This order is heavily used along the "only if" part of the proof of Theorem 1.

Remark 1 *Let C be any collective choice function that is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, where $\mathcal{B}$ is a collection of agendas such that for all $B, B' \in \mathcal{B}, \langle B \cup B' \rangle \in \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{R}^n$ a subset of preference profiles. Then C on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ is transitively rationalizable.*

Proof of Remark 1. Note that C restricted to $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ chooses the Condorcet winner at any situation $(R, B) \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Given a collection of agendas $\mathcal{B}$, take any $R \in \mathcal{D}_{C\mathcal{B}}$. To construct the rationalization we identify and we rank those alternatives that will be the Condorcet winner at some situation (R, B) for $B \in \mathcal{B}$. We construct the binary relation $\mathbf{R}_R^{\mathcal{B}} \in \mathcal{P}$ as follows. Define B^1 as the union of all elements in $\mathcal{B}$. By assumption, $B^1 \in \mathcal{B}$ and therefore there exists a strong Condorcet winner at (R, B^1), denoted x^1. Let x^1 be the top alternative of $\mathbf{R}_R^{\mathcal{B}}$. Next, define $B^2 = \cup\{B \in \mathcal{B} : x^1 \notin B\}$. If B^2 is empty we stop. Otherwise, by assumption $B^2 \in \mathcal{B}$ and denote x^2 the strong Condorcet winner at (R, B^2). Then, let x^2 be the second ranked alternative of $\mathbf{R}_R^{\mathcal{B}}$. In general, we could define $B^k = \cup\{B \in \mathcal{B} : \{x^1, ..., x^{k-1}\} \notin B\}$ and if B^k is not empty, we define x^k to be the strong Condorcet winner at (R, B^k), and let x^k be the top k-ranked alternative of $\mathbf{R}_R^{\mathcal{B}}$. Proceed until B^{k+1} is empty. Our rationalization $\mathbf{R}_R^{\mathcal{B}}$ is a strict order where the identified alternatives are ranked in the first places: x^1 is first, x^2 second, until x^k, and then the rest of alternatives are ordered acording to a fixed exogenous strict order $\succ$ on A. Clearly, for any agenda $B \in \mathcal{B}$ and any $R \in \mathcal{D}_{C\mathcal{B}}$, the Condorcet consistent rule on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ is such that $C(R, B) = t(\mathbf{R}_R^{\mathcal{B}}, B)$. ■

Observe that the order $\mathbf{R}_R^{\mathcal{B}}$ in the proof of Remark 1 is a strict order over A and if alternative y is the strong Condorcet winner at (R, B), with $B \in \mathcal{B}$, then y is ranked above z according to $\mathbf{R}_R^{\mathcal{B}}$ for all $z \in B \setminus \{y\}$.

Theorem 1 Let $\mathcal{B}$ be a collection of agendas such that for all $B, B' \in \mathcal{B}, \langle B \cup B' \rangle \in \mathcal{B}$ and $\mathcal{D} = \mathcal{R}^n$. A collective choice function C on $\mathcal{D} \times \mathcal{B}$ is anonymous, neutral, and satisfies pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ if and only if C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$.

Proof of Theorem 1. Note that since C is defined on $\mathcal{R}^n \times \mathcal{B}$, for any $R = (R_1, R_2, ..., R_n) \in \mathcal{D}_{C\mathcal{B}}$ we have that $\mu(R_{\rho(1)}, R_{\rho(2)}, ..., R_{\rho(n)}) \in \mathcal{D}_{C\mathcal{B}}$ for any permutations ρ of N and μ of A.[21]
First, we prove the "if" implication. Since C is Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, then C is anonymous and neutral on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. We now prove that C satisfies pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, showing that for any two situations (R, B), $(R', B') \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ where $C(R, B) = x$, $C(R', B') = y$, and $x, y \in B \cap B'$ there is some agent i and some alternative $z \in A \setminus \{x\}$ such that either (1) xP_iz and zR'_ix or (2) xI_iy for some $i \in N$ for whom $R_i \neq R'_i$ holds. Since $C(R, B) = x$, the number of agents who strictly prefer x over y is greater than that of those who strictly prefer y over x at R. Also since $C(R', B') = y$, the number of agents who strictly prefer y over x is greater than that of those who strictly prefer x over y at R'. Therefore, there exists some agent i such that xP_iy and yR'_ix.

We now prove the "only if" statement by contradiction. Let C satisfy anonymity, neutrality, and pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Suppose that C is not Condorcet consistent on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$. Then, there exists a situation

[21] We use the fact that these permutations do not take us outside the preference domain repeatedly along the proof, without making it explicit all the time.

$(R^0, B) \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ such that y is the strong Condorcet winner at (R^0, B) and $C(R^0, B) = x^0 \neq y$.

From here on, the proof that this cannot happen under our assumptions proceeds in two differentiated parts. In part 2 we show that if the strong Condorcet winner is not chosen at a situation which has a specific structure, then C cannot satisfy the conjunction of anonymity, neutrality, and pairwise justifiability. Part 1 is devoted to show that, starting from any situation (R^0) where Condorcet consistency is violated, we can identify another situation in the Condorcet domain of C where this violation also arises, and which has the characteristics needed to validate Step 2. The construction of such situation may involve several iterations, and the delicate part of the arguments is to show that each of the new situations that are proposed at each one of them still belongs to our Condorcet domain.

Part 1.

Step 1. Consider the following partition of N: $Y^0 = \{i \in N : yP_i^0 x^0\}$, $X^0 = \{i \in N : x^0 P_i^0 y\}$, and $L^0 = \{i \in N : x^0 I_i^0 y\} = N \backslash (Y^0 \cup X^0)$. Since y is the strong Condorcet winner at (R^0, B), then $\#Y^0 > \#X^0$. Let R^1 be a preference profile defined as follows: for every agent $i \in L^0$, $R_i^1 = R_i^0$, for every $i \in Y^0 \cup X^0$, every alternative except x^0 is ranked according to the strict order $\mathbf{R}_{R^0}^{\mathcal{B}}$ over A induced by R^0, as defined in Remark 1 above. For all $i \in Y^0 \cup X^0$ alternative x^0 is ranked adjoining y: for every agent $i \in Y^0$ the alternative x^0 is ranked just below y, while for every agent $i \in X^0$ the alternative x^0 is ranked just above y.

We now prove that for every $B' \in \mathcal{B}$ there exists a strong Condorcet winner at (R^1, B') and therefore $R^1 \in \mathcal{D}_{C\mathcal{B}}$. We distinguish some cases.

Case 1: $x^0 \notin B'$.

Suppose first that y is the strong Condorcet winner at (R^0, B'). Since for all $i \in N$ and for all $a \in B'$, $yP_i^0 a$ implies $yP_i^1 a$ and $yI_i^0 a$ implies $yR_i^1 a$, then y is still the strong Condorcet winner at (R^1, B'). Suppose that $z \neq y$

is the strong Condorcet winner at (R^0, B'). This implies that either $y \notin B'$ or $z \notin B$. If $y \notin B'$, then z is still the strong Condorcet winner at (R^1, B'), because z is ranked above every $w \in B' \setminus \{z\}$ according to $\mathbf{R}^{\mathcal{B}}_{R^0}$. If $y \in B'$, then it follows that $z \notin B$ and z is ranked above y according to $\mathbf{R}^{\mathcal{B}}_{R^0}$ and therefore is still the strong Condorcet winner at (R^1, B').

Case 2: $x^0 \in B'$.

First notice that if either y or x^0 is the strong Condorcet winner at (R^0, B'), then it immediately follows that the strong Condorcet winner at (R^1, B') is the same (note that if x^0 is the strong Condorcet winner at (R^0, B') then $y \notin B'$). Let $z \in B' \backslash \{x^0, y\}$ be the strong Condorcet winner at (R^0, B'). We distinguish two cases. Suppose first that z is the strong Condorcet winner at $(R^0, B' \cup B)$. It follows that $z \notin B$ and z is ranked above y according to $\mathbf{R}^{\mathcal{B}}_{R}$ and thus, z is the strong Condorcet winner at (R^1, B'). Suppose now that y is the strong Condorcet winner at $(R^0, B' \cup B)$. It follows that there is a majority of agents at R^0 who prefers y to z and y is ranked above z according to $\mathbf{R}^{\mathcal{B}}_{R}$. Let H^0 be the set of agents who strictly prefer y to z at R^0: if $i \in H^0 \cap L^0$ then, by definition, $R^0_i = R^1_i$, if $i \in H^0 \cap (Y^0 \cup X^0)$ then again, by definition, $x^0 P^1_i z$. Therefore, x^0 is the strong Condorcet winner at (R^1, B') and $(R^1, B') \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$.

By pairwise justifiability from (R^0, B) to (R^1, B), $C(R^1, B) = x^1$, such that $x^1 I^0_i x^0 \neq y$ for some $i \in Y^0 \cup X^0$, since there is no alternative $z \in B \backslash \{x^0\}$ and no agent $i \in N$ such that z has improved with respect to x^0.

Let $Y^1 = \{j \in N : y P^1_j x^1\}$, $X^1 = \{k \in N : x^1 P^1_k y\}$, and $L^1 = \{\ell \in N : y I^1_\ell x^1\}$ and note that $\#Y^1 > \#X^1$.

Observe that for all $i \in Y^0 \cup X^0$, R^1_i is a strict order over A, and therefore $L^1 \subseteq L^0$. If $L^1 = L^0$, Part 1 ends, let $T = 1$ and go to Part 2. Otherwise, if $L^1 \subsetneq L^0$ continue to the next step.

<u>*Step t for $t \in [2, ..., T]$*</u>: Let R^t be a preference profile defined as follows: for every agent $i \in L^{t-1}$, $R^t_i = R^{t-1}_i$, for every $i \in Y^t \cup X^t$, every alternative except x^{t-1} is ranked according to the strict order $\mathbf{R}^{\mathcal{B}}_{R^{t-1}}$ over A induced by

R^{t-1}, as defined in Remark 1 above. Note that y is the strong Condorcet winner at (R^t, B). For all $i \in Y^{t-1} \cup X^{t-1}$ alternative x^{t-1} is ranked adjoining y: for every agent $i \in Y^{t-1}$ the alternative x^{t-1} is ranked just below y, while for every agent $i \in X^{t-1}$ the alternative x^{t-1} is ranked just above y. Applying the same arguments as in Step 1 we can prove that there exists a strong Condorcet winner at (R^t, B') for all $B' \in \mathbf{R}^{\mathcal{B}}_{R^{t-1}}$.

By pairwise justifiability from (R^{t-1}, B) to (R^t, B), $C(R^t, B) = x^t I_i^{t-1} x^{t-1} \neq y$ for some $i \in Y^{t-1} \cup X^{t-1}$, since there is no alternative $z \in A \backslash \{x^{t-1}\}$ and no agent $i \in N$ such that z has improved with respect to x^{t-1}.
Let $Y^t = \{j \in N : y P_j^t x^t\}$, $X^t = \{k \in N : x^t P_k^t y\}$, and $L^t = \{\ell \in N : y I_\ell^t x^t\}$ and note that $\# Y^t > \# X^t$.

Observe that $L^t \subseteq L^{t-1}$. If $L^t = L^{t-1}$, then Part 1 ends. Let $t = T$ and go to Part 2. Otherwise, if $L^t \subsetneq L^{t-1}$ continue to the next step and repeat the same argument as in Step t as many times as necessary till we get either $L^T = L^{T-1}$ or $L^T = \varnothing$. Define R^T as usual and go to Part 2 below. Notice that $C(R^T, B) = x^T$, y is the strong Condorcet winner at (R^T, B) and for all i in L^T, $x^T I_i^T y$.

Part 2. Let $\{Y_1^T, Y_2^T\}$ be a partition of Y^T such that $\# Y_1^T = \# X^T$**.** Let $R'' = (R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T})$ be obtained from R^T where agents in Y_1^T and X^T exchange their preferences. By anonymity, $C\left((R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T}), B\right) = x^T$. Take μ a permutation of A such that $\mu(x^T) = y$, $\mu(y) = x^T$, and $\mu(z) = z$ for all $z \in A \backslash \{x^T, y\}$. By neutrality, $\mu(C\left((R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T}), B\right)) = C(\mu(R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T}), B)$. Thus, $C(\mu(R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T}), B) = y$. Note that in profile $\mu(R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T})$, agents in Y_1^T and in X^T have the same preferences as in R^T. We now change the preferences of all agents in Y_2^T to $R^T_{Y_2^T}$ and the new preference profile is R^T. By pairwise justifiability applied from $(\mu(R''_{Y_1^T}, R^T_{Y_2^T}, R''_{X^T}, R^T_{L^T}), B)$ to (R^T, B), then $C(R^T, B) = y$ since there is no alternative $z \in A \backslash \{y\}$ and no agent $i \in N$ such that z has improved with respect to y and y is not indifferent to any alternative for agents in Y_2^T. Thus, we get the desired contradiction.

Proposition 1 Let $\mathcal{B}$ be a collection of agendas, $\mathcal{D} \subseteq \mathcal{R}^n$ a subset of preference profiles, and C be a collective choice function on $\mathcal{D} \times \mathcal{B}$. If C is Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$, then C satisfies pairwise justifiability on $\mathcal{D}' \times \mathcal{B}$ for any $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$.

Proof of Proposition 1. Let C be Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$. We prove that C satisfies pairwise justifiability on $\mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$, showing that for any two situations (R, B), $(R', B') \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ where $C(R, B) = x$, $C(R', B') = y$, $x, y \in B \cap B'$, there is some agent i and some alternative $z \in A \backslash \{x\}$ such that either (1) xP_iz and zR'_ix or (2) xI_iy for some $i \in N$ for whom $R_i \neq R'_i$ holds. Since $C(R, B) = x$, the number of agents who strictly prefer x over y is greater than that of those who strictly prefer y over x at R. Also since $C(R', B') = y$, the number of agents who strictly prefer y over x is greater than that of those who strictly prefer x over y at R'. Therefore, there exists some agent i such that xP_iy and yR'_ix.

7.2 Proofs of results in Section 4

We first prove Theorem 3 because we use it to prove Theorem 2.

Theorem 3 If $A \in \mathcal{B}$, a full range collective choice function C on $\mathcal{P}^n \times \mathcal{B}$ satisfies pairwise justifiability on $\mathcal{P}^n \times \mathcal{B}$ if and only if it is dictatorial.

Proof of Theorem 3. Note that any dictatorial collective choice function satisfies full range and pairwise justifiability on $\mathcal{P}^n \times \mathcal{B}$. We prove the converse implication in two steps. In the first step we fix an agenda with at least three alternatives (which exists since $A \in \mathcal{B}$) and show that our property implies the existence of a dictator on such fixed agenda. In the second step we show that this dictator is the same for all possible agendas.

Step 1: Let $B \in \mathcal{B}$. For any full range collective choice function C on $\mathcal{P}^n \times \{B\}$ satisfying pairwise justifiability on $\mathcal{P}^n \times \{B\}$ where $\#B \geq 3$, there is an agent that is a dictator on $\mathcal{P}^n \times \{B\}$.

Since this statement refers to the case where individual preferences are strict, we only need to use the first part of Definition 3 of pairwise justifiability.

In this step we concentrate on the consequences of pairwise stability when individual preferences change but the agenda remains the same.

Claim 1. *Let R be a preference profile where alternative x is the top of R_i in B for each agent i. Then, $C(R,B) = x$.*

Proof of Claim 1: Since all alternatives in B are in the range of C, there exists a profile $\widetilde{R}$ for which $C(\widetilde{R}, B) = x$.

Consider now the profile $\widehat{R}$ where all agents place x at their top in A, thus also in B, while keeping the same ordering among the rest of alternatives as in $\widetilde{R}$. No alternative in A has improved, for no agent, its position relative to x when society's profile changes from $\widetilde{R}$ to $\widehat{R}$, hence $C(\widehat{R}, B) = x$ by pairwise justifiability. Since in all profiles where all agents have x as their top alternative in A, no alternative in A has improved, for no agent, its position relative to x, again by pairwise justifiability, the proof is complete.

Claim 2. *Let R be a preference profile where all agents have either z or w as their top alternative in B. Then, the choice at this profile in B must be either z or w.*

Proof of Claim 2: Let J be the set of agents whose top in B is z and K be the set of those whose top in B is w at profile R, and assume that $C(R,B) = x \notin \{z, w\}$. Consider now the profile $\widehat{R}$ where all agents in J place z at their top in A and all agents in K place w at their top in A, while keeping the same ordering among the rest of alternatives as in R. Since no alternative in A has improved, for no agent, its position relative to $C(R,B)$, by pairwise justifiability, $C(\widehat{R}, B) = x$. Now, we shall arrive at a contradiction through several assertions.

(2.1) It cannot be that $z\widehat{R}_j w\widehat{R}_j x$ for all $j \in J$, nor that $w\widehat{R}_k z\widehat{R}_k x$ for all $k \in K$.

Suppose that $z\widehat{R}_j w\widehat{R}_j x$ for all $j \in J$. Consider the profile $\widetilde{R}$ where all agents in J have w as their top in B, keeping the rest of their ranking unchanged as in $\widehat{R}$, and agents in K have not changed preferences. Then, the choice at profile $\widetilde{R}$ in B should be w by Claim 1, because w is the top in B for all agents in $\widetilde{R}$. Yet, no alternative has improved, for no agent, its position relative to x when society's profile changes from $\widehat{R}$ to $\widetilde{R}$, hence $C(\widetilde{R}, B) = x$ by pairwise justifiability. A contradiction. By a similar argument, it cannot be the case that $w\widehat{R}_k z\widehat{R}_k x$ for all $k \in K$.

(2.2) Now, consider the partition of profile $\widehat{R}$ into four sets of preferences, corresponding to agents whose preferences rank x, z, w as follows taking into account that, without loss of generality, by pairwise justifiability, we can assume that x is ranked as the alternative in the second place in A in those profiles for J where it is above w, and in those for K where it is above z.

J_1 are agents who rank z as the top, followed by x as the second alternative in A (thus, in B).

K_1 are agents who rank w as the top, followed by x as the second alternative in A (thus, in B).

$J \backslash J_1$ is the set of those agents i for whom z is the top and $w\widehat{R}_i x$.

$K \backslash K_1$ is the set of those agents i for whom w is the top and $z\widehat{R}_i x$.

If J_1 or K_1 are empty, we would be in Claim (2.1). Our starting assumption is that $C(\widehat{R}, B) = C(\widehat{R}_{J_1}, \widehat{R}_{K_1}, \widehat{R}_{J\backslash J_1}, \widehat{R}_{K\backslash K_1}, B) = x$.

We shall now consider the possible choices in B under several profiles. In all of them, the preferences of $J \backslash J_1$ and $K \backslash K_1$ remain unchanged.

Let R' be such that, all the rest being unchanged with respect to $\widehat{R}$, agents in J_1 have w as the alternative in the second place, between z and x.

Let R'' be such that, all the rest being unchanged with respect to $\widehat{R}$, agents in K_1 have z as the alternative in the second place, between w and x.

Let R''' be such that both agents in J_1 and K_1 have changed in the way described when defining R' and R''. That is, those in J_1, have w as the alternative in the second place, between z and x, and those in K_1, have z as the alternative in the second place, between w and x.

Remark that, $C(R', B) \neq x$, by the argument we used in (2.1), and that w is the only alternative whose ranking has improved over some alternative and for some agent from $\widehat{R}$ to R' (equivalently, w is the unique alternative that gets worse from R' to $\widehat{R}$ for some agent). Hence, by pairwise justifiability $C(R', B) = w$. For the same reasons, it must be that $C(R'', B) = z$. But then, $C(R''', B)$ must be w, because passing from R' to R''', the relationship between w and all other alternatives has not changed for any agent. And, for the same reasons, passing from R'' to R''', $C(R''', B)$ must be z. Since C is a collective choice function, that is a contradiction.

Remark Before we develop our third claim, let us introduce some notation and definitions that will be useful. For $B' \subseteq B$, we will say that an *agent i is B'-determinant at profile R* if and only if $C\left((R'_i, R_{N\setminus\{i\}}), B'\right) = t(R'_i, B')$ for all R'_i. Also remark that if i is B'-determinant at R, it is also B'-determinant at all profiles $\widetilde{R}$ such that $\widetilde{R}_j = R_j$ for all $j \neq i$.

Given Claims 1 and 2, there will exist a profile where agents' preferences have alternative z and w as the only top in B and one of the agents is (z, w)-determinant.

Claim 3. *If agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then agent i is (z, w)-determinant at any profile $R' = (R_i, R_J, R'_K)$ where all agents in K have w as the top in B and $x \in A\setminus\{w\}$ as the alternative in the second place in B.*

Proof of Claim 3: By pairwise justifiability, without loss of generality, we can assume that agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$ where agents in J have z as the top in A, and those in K have w as the top in A. Since i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$, by pairwise justifiability, $C\left((R_i^w, R_J, R_K), B\right) = w$ where R_i^w is such that w is the top and z as the alternative in the second place in A. Again, by pairwise justifiability, $C\left((R_i^w, R_J, R'_K), B\right) = w$ where R'_K is such that w is the top and x as the alternative in the second place in A. By Claim 2, $C\left((R_i^z, R_J, R'_K), B\right) \in \{z, w\}$ where R_i^z is such that z is the top and w as the alternative in the second place in A. If $C\left((R_i^z, R_J, R'_K), B\right) = w$, by pairwise justifiability, $C((R_i^z, R_J, R_K), B) = w$ which is a contradiction to agent i being (z, w)-determinant at (R_i, R_J, R_K).

Claim 4. *If agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R'_K)$ where agents in J have z as the top in B, and those in K have w as the top in B and $x \in A\backslash\{w\}$ as the alternative in the second place in B, then agent i is also (x, z, w)-determinant at profile $R = (R_i, R_J, R'_K)$.*

Proof of Claim 4: We show that agent i is (x, z, w)-determinant at profile $R = (R_i, R_J, R'_K)$.

We first prove that $C((R'_i, R_J, R'_K), B) = x$ where R'_i ranks x as the first, z as the second in B. Suppose not. By Claim 3, agent i is (z, w)-determinant at (R_i, R_J, R'_K). Then, $C((R_i^z, R_J, R'_K), B) = z$, where R_i^z ranks z as the first, x as the second in B, and the relative order of the rest of alternatives in A as in R'_i. Note that if $C((R'_i, R_J, R'_K), B)$ was $y \neq z$, since y has the same relative order with respect to all alternatives in R_i^z and in R'_i, by pairwise justifiability, $C((R_i^z, R_J, R'_K), B) = y$ which is a contradiction to what we obtained above. Then, $C((R'_i, R_J, R'_K), B) = z$.

For the same reason, when R''_i ranks x as the first, w as the second in B, and the rest of alternatives in A as in R'_i, it should be that $C((R''_i, R_J, R'_K), B) = w$.

Now consider the profile where agents in K switch the positions of x and w, and the relative order of the rest of alternatives in A as in R'_K, so that x is the top in B for all of them. At that new profile (R''_i, R_J, R''_K), the only two alternatives in top positions in B are z and x. Hence, by Claim 2, the choice must be either z or x. But z cannot be, because this would violate pairwise justifiability, because z has not improved for any alternative by any agent. Thus, $C((R''_i, R_J, R''_K), B) = x$.

Now, starting from this last profile, consider the one, say R'''_i, where i changes preferences so that z becomes the second to x in B, and the relative order of the rest of alternatives in A as in R''_i. Again, by pairwise justifiability, $C((R'''_i, R_J, R''_K), B) = x$ since no alternative and for no agent has improved

relative to x. Finally, let agents in K change preferences to rank w as the first, x as the second, and the relative order of the rest of alternatives in A as in R''_K, say R'''_K. By pairwise justifiability, the choice cannot be z, and yet (R'''_i, R_J, R'''_K) is the same profile we start with, where z was to be chosen. This contradiction proves that $C((R'_i, R_J, R'_K), B) = x$.
We now show that for any R_i whose top is $x \in B$, then $C((R_i, R_J, R'_K), B) = x$. Suppose not. We have shown that $C((R'_i, R_J, R'_K), B) = x$ where R'_i ranks x the first, z the second in B. By pairwise justifiability, from (R'_i, R_J, R'_K) to (R''_i, R_J, R'_K) with x as the top in A and the relative order of the rest of alternatives in A as in R_i, then $C((R''_i, R_J, R'_K), B) = x$. Again, by pairwise justifiability, from (R_i, R_J, R'_K) to (R''_i, R_J, R'_K) we have that $C((R''_i, R_J, R'_K), B) \neq x$ which is a contradiction.

Claim 5. If an agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then this agent i is B-determinant at that profile.
Proof of Claim 5: Suppose not. That is, there exists $\widehat{R}_i$ with x as the top alternative in B such that $C\left((\widehat{R}_i, R_J, R_K), B\right) \neq x$.
Let R'_i with x as the first and w as the second in B and the relative order of the rest of alternatives in A is as in $\widehat{R}_i$. By the same argument as the one at the end of Claim 4, since x is the top in B of $\widehat{R}_i$ and R'_i, $C((R'_i, R_J, R_K), B) \neq x$. Since agent i is (z, w)-determinant at a profile $R = (R_i, R_J, R_K)$, then $C((R^w_i, R_J, R_K), B) = w$, where R^w_i ranks w as the first, x as the second in B, and the relative order of the rest of alternatives in A as in $\widehat{R}_i$. If $C((R'_i, R_J, R_K), B)$ was $y \neq w$, since y has the same relative order with respect to all alternatives in R^w_i and in R'_i, by pairwise justifiability, $C((R^w_i, R_J, R_K), B) = y$ which is a contradiction to agent i being (z, w)-determinant at (R'_i, R_J, R_K). Thus, $C((R'_i, R_J, R_K), B) = w$. Let R'_K where all agents in K have w as the top in B and $x \in A \backslash \{w\}$ as the alternative in the second place in B, and the relative order of the rest of alternatives is as in R_K. Since w has the same relative order with respect to all alternatives and

all agents, by pairwise justifiability from (R_i', R_J, R_K) to (R_i', R_J, R_K'), we have $C((R_i', R_J, R_K'), B) = w$. But, this is a contradiction since by Claims 3 and 4, agent i is also (x, z, w)-determinant at profile $R = (R_i, R_J, R_K')$ and thus $C((R_i', R_J, R_K'), B) = x$.
Therefore, $C((R_i', R_J, R_K), B) = C\left((\widehat{R}_i, R_J, R_K), B\right) = x$.

Claim 6. If an agent i is B-determinant at a profile $R = (R_i, R_J, R_K)$ where all agents in J have z as the top in B, and all those in K have w as the top in B, then i is B-determinant at all profiles.

Proof of Claim 6: We first show that agent i is B-determinant at all profiles. Consider R in the statement where agent i is B-determinant. Thus, i is also (z, w)-determinant at R. Change the preferences of all agents for $y \in B \backslash \{z, w\}$ so that y is the worst alternative, keeping the relative ordering of the rest of the alternatives. The choice is either z or w by Claim 2, depending on agent i's preferences. Then, by Claim 5, the modified profile still leaves i as being B-determinant. Now let y become the top alternative in B for i. The choice will be y, even if it is worse for all other agents since agent i is B-determinant at the modified profile. By pairwise justifiability, all profiles where y is the top in B for agent i gives y. This argument can be repeated for all alternatives y in B. Thus, agent i is B-determinant at all profiles.

Claim 7. For any $B \in \mathcal{B}$, there is an agent that is B-determinant at all profiles in $\mathcal{P}^n$. Thus, there is an agent that is a dictator on B at all profiles in $\mathcal{P}^n$.

Claim 7 follows from all previous claims. This ends the proof of Step 1.

Step 2. For any B, $B' \in \mathcal{B}$ such that $B' \neq B$ there exists an agent who is both a dictator on $\mathcal{P}^n \times \{B\}$ and on $\mathcal{P}^n \times \{B'\}$.

Take any two sets B, $B' \in \mathcal{B}$ such that $B' \neq B$. Let $B'' \in \mathcal{B}$ with at least three alternatives and such that $B \cup B' \subseteq B''$ (note that B'' exists since $A \in \mathcal{B}$). By Step 1 there is an agent, say i, who is a dictator on B''. We show that i is also a dictator on B and B'. Note that this is straightforward if an agenda B has only one alternative. Consider two cases.

Case 1: B and B' have both only 2 alternatives.

Take one of them, without loss of generality, $B = \{z, w\}$. Suppose, to get a contradiction, that agent i is not a dictator on B.

Subcase 1.1. There is a dictator on B, say agent $1 \neq i$. Let $R \in \mathcal{P}^n$ be such that z is the top of R_1 in A, w is the top of R_i in A, and any preference for other agents. Note that $C(R, B) = t(R_1, B) = z$ and $C(R, B'') = t(R_i, B'') = w$. By pairwise justifiability, from (R, B) to (R, B'') the choice must be the same for the two situations since agents' preferences do not change. Then, the dictator must be the same on B and on B'' which is the contradiction.

Subcase 1.2. There is no dictator on B. Let $R \in \mathcal{P}^n$ be such that for each agent $j \in N \backslash \{i\}$, $t(R_j, A) = z$ and $t(R_i, A) = w$. If $C(R, B) = w$ and since by Claim 1 in the proof of Step 1, when all agents have z as top, the choice is z, then agent i is (z, w)-determinant at R. By Claim 6 in the proof of Step 1, agent i is (z, w)-determinant at all profiles, meaning that i is a dictator on B, which is a contradiction. Therefore, $C(R, B) = z$. Since agent i is a dictator on B'', $C(R, B'') = w$. By pairwise justifiability, from (R, B) to (R, B'') the choice must be the same for the two situations, which is the contradiction.

Therefore, agent i must be the dictator on B.

Case 2: B and B' where at least one of them has three or more alternatives. Suppose, without loss of generality, that B' has at least three alternatives. We first show that i is a dictator on B'.

By Step 1 there is an agent, say 1, who is a dictator on B'. Suppose to get a contradiction that $i \neq 1$. Let $z \in B' \cap B''$ be such that $z \neq w$. Let $R \in \mathcal{P}^n$ be such that z is the top of R_1 in A, w is the top of R_i in A, and any preference for other agents. Note that $C(R, B') = t(R_1, B') = z$, $C(R, B'') = t(R_i, B'') = w$. By pairwise justifiability, from (R, B') to (R, B'') the choice must be the same for the two situations. Then, the dictator must be the same on B' and on B'' which is the contradiction.

We now show that i is also a dictator on B.
If $\#B \geq 3$, repeat the same argument as for B'. Otherwise, if $\#B = 2$, repeat the same argument as in Case 1.
This ends the proof of Theorem **3.**

Theorem 2 Let $A \in \mathcal{B}$, any full range collective choice correspondence C on $\mathcal{R}^n \times \mathcal{B}$ satisfying weak decisiveness and pairwise justifiability on $\mathcal{R}^n \times \mathcal{B}$ is dictatorial.

The next two lemmas will help to develop the proof of Theorem 2.

Lemma 1 If C on $\mathcal{R}^n \times \{B\}$ is a full range collective choice correspondence satisfying pairwise justifiability and weak decisiveness on $\mathcal{R}^n \times \{B\}$ then C on $\mathcal{P}^n \times \{B\}$ has full range.

Proof of Lemma 1. We show that for any $x \in B$ there exists $R \in \mathcal{P}^n$ such that $x \in C(R, B)$. Since all alternatives in B are in the range of C, there exists a profile $\widetilde{R} \in \mathcal{R}^n$ for which $x \in C(\widetilde{R}, B)$. Consider now the profile $\widehat{R}$ where all agents place x as the top in A, thus also in B, while keeping the same ordering among the rest of alternatives as in $\widetilde{R}$. Distinguish two cases:
(1) $x \in C(\widehat{R}, B)$. Construct $R'' \in \mathcal{P}^n$ such that for any $i \in N$, $t(R''_i, B) = x$. When agents' preferences change from $\widehat{R}$ to R'', no alternative in A has improved, for no agent, its position relative to x and no alternative is indifferent to x by any agent at $\widehat{R}$. Hence, by pairwise justifiability, $x \in C(R'', B)$.
(2) $x \notin C(\widehat{R}, B)$. When society's profile changes from $\widetilde{R}$ to $\widehat{R}$, no alternative in $A \backslash \{x\}$ has improved, for no agent, its position relative to any alternative in A. By pairwise justifiability and the fact that $x \in C(\widetilde{R}, B)$, then for some $y \in C(\widehat{R}, B)$ there exists some $j \in N$ such that $y \widetilde{I}_j x$.
Let $y \in C(\widehat{R}, B) \backslash \{x\}$, and construct $R'' \in \mathcal{P}^n$ such that for any $i \in N$, $t(R''_i, B) = x$ and $y P''_i z$ for any $z \in A \backslash \{x, y\}$ and keeping the same order among the rest of alternatives as in $\widehat{R}$. When agents' preferences change from $\widehat{R}$ to R'', no alternative in A has improved, for no agent, its position relative to y. Hence, by pairwise justifiability, either $y \in C(R'', B)$ or else

$w \in C(R'', B)$ if there is some agent $j \in N$ such that $y\widehat{I}_j w$. By definition of $\widehat{R}$, $w \neq x$. By weak decisiveness and Theorem 1, there is an agent i that is a dictator on $\mathcal{P}^n \times \{B\}$. Thus, $C(R'', B) = x$ which is a contradiction.

Lemma 2 Let C be a collective choice correspondence satisfying pairwise justifiability on $\mathcal{R}^n \times \{B\}$. If there is a dictator i for the restriction of C to $\mathcal{P}^n \times \{B\}$, then C is dictatorial and i is the dictator on $\mathcal{R}^n \times \{B\}$.

Proof of Lemma 2. Let i be a dictator for the restriction of C to $\mathcal{P}^n \times \{B\}$. Take $R \in \mathcal{P}^n$ such that $t(R_i, B) = x$ and x is the worst alternative for the rest of the agents. Since i is the dictator on $\mathcal{P}^n \times \{B\}$, then $C(R, B) = x$. Now, let $R' \in \mathcal{R}^n$ be any preference profile such that each agent's preferences keep the same relative order of x with respect to the rest of alternatives as in R, that is, x is the unique top alternative of i while x is the unique worst alternative for the rest of agents. By pairwise justifiability, no alternative in A has improved, for no agent, its position relative to x and x is not indifferent under R to any other alternatives for any agent. Then, $x \in C(R', B)$. We now show that $C(R', B) = x$. By contradiction, let $y \in C(R', B) \backslash \{x\}$, and construct $R'' \in \mathcal{P}^n$ such that $t(R''_i, B) = x$, $yP''_i z$ for any $z \in A \backslash \{x, y\}$, y is the unique top alternative and x is the worst alternative for the rest of the agents. When agents' preferences change from R' to R'', no alternative in A has improved, for no agent, its position relative to y. Hence, by pairwise justifiability, either $y \in C(R'', B)$ or else $w \in C(R'', B)$ if there is some agent $j \in N$ such that $yI'_j w$. By definition of R', $w \neq x$. Since i is a dictator on $\mathcal{P}^n \times \{B\}$, $C(R'', B) = x$ which is a contradiction. Thus, $C(R', B) = x$.
Now change the preferences of all agents different from i to any preference in $\mathcal{R}$. Again, by pairwise justifiability, the choice is still x since no alternative is weakly worse than x under R_j for no agent different from i. We have shown that agent i is a dictator if her top is unique.
Finally, suppose now to get a contradiction that agent i is not a dictator when her top is not unique. Then, there exists $\overline{R} \in \mathcal{R}^n$ such that $y \in C(\overline{R}, B)$ but $y \notin t(\overline{R}_i, B)$. Suppose, without loss of generality, that $x \in t(\overline{R}_i, B)$. Let R'''_i

be such that x is the unique top alternative keeping the relative order of the rest of alternatives. By pairwise justifiability from $\overline{R}$ to $(R_i''', \overline{R}_{N\setminus\{i\}})$, $y \in C((R_i''', \overline{R}_{N\setminus\{i\}}), B)$ or else $w \in C((R_i''', \overline{R}_{N\setminus\{i\}}), B)$ if $w\overline{I}_i y$. Since $x\overline{P}_i y$, $w \neq x$. Given that agent i is the dictator when she has a unique top alternative, we get $C((R_i''', \overline{R}_{N\setminus\{i\}}), B) = x$ which is a contradiction. This completes the proof of Lemma 2.

Proof of Theorem 2. Let $A \in \mathcal{B}$, and let C be any full range collective choice correspondence on $\mathcal{R}^n \times \mathcal{B}$ satisfying weak decisiveness and pairwise justifiability on $\mathcal{R}^n \times \mathcal{B}$. As a first step, for any fixed agenda $B \in \mathcal{B}$ we obtain that C has full range on $\mathcal{P}^n \times \{B\}$ by Lemma **1**. Since pairwise justifiability and weak decisiveness are inherited in subdomains, then C is a full range collective choice function on $\mathcal{P}^n \times \mathcal{B}$ satisfying pairwise justifiability on $\mathcal{P}^n \times \mathcal{B}$. As a second step, we apply Theorem **3** and obtain that there is a dictator i for the restriction of C to $\mathcal{P}^n \times \mathcal{B}$. Finally observe that by Lemma **2** applied to any $B \in \mathcal{B}$, C is dictatorial and i is the dictator on $\mathcal{R}^n \times \mathcal{B}$. This ends the proof.

7.3 Proofs of results in Section 5

Proposition 2 Let $B \in \mathcal{B}$ and $\mathcal{D} \in \{\mathcal{P}^n, \mathcal{R}^n\}$. A *social choice function* $f : \mathcal{D} \to B$ is strategy-proof on $\mathcal{D}$ if and only if it satisfies pairwise justifiability on $\mathcal{D} \times \{B\}$.

Proof of Proposition 2. As mentioned in the text, properties on f can be trivially translated as properties on C, and viceversa and we use f and C indistinctly

We first show the "if part". Let $\mathcal{D} = \mathcal{P}^n$ and by contradiction, suppose that f violates strategy-proofness on $\mathcal{P}^n$, that is, there exist $R \in \mathcal{P}^n$, $i \in N$, and $R_i' \in \mathcal{P}$ such that $y = f(R_i', R_{N\setminus\{i\}}) P_i f(R) = x$. Define $\widehat{R}_i$ and $\widetilde{R}_i$ such that y is the best alternative and the rest of alternatives are ordered as in R_i and R_i', respectively. By pairwise justifiability from R to $\widehat{R} = (\widehat{R}_i, R_{N\setminus\{i\}})$,

$f(\widehat{R}) = x$. Similarly, by pairwise justifiability from R to $\widetilde{R} = (\widetilde{R}_i, R_{N\setminus\{i\}})$, $f(\widetilde{R}) = y$. Then, by pairwise justifiability from $\widetilde{R}$ to $\widehat{R}$, $f(\widehat{R}) = y$ which is the desired contradiction.

Let now $\mathcal{D} = \mathcal{R}^n$. By contradiction suppose that f violates strategy-proofness on $\mathcal{R}^n \times \{B\}$, that is, there exist $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$ such that $y = f(R'_i, R_{N\setminus\{i\}}) P_i f(R) = x$. Let $\widetilde{R}_i \in \mathcal{R}^n$ be such that (i) the set of alternatives indifferent to x at $\widetilde{R}_i$ is the lower contour set at x of R_i (denoted as $L(R_i, x)$), (ii) $[z\widetilde{P}_i w \iff zP'_i w]$ for all $z \in A\setminus L(R_i, x)$ and $w \in L(R_i, x)$, and (iii) $[z_1\widetilde{R}_i z_2 \iff z_1 R'_i z_2]$ for all $z_1, z_2 \in A\setminus L(R_i, x)$. Thus, by pairwise justifiability from R to $\widetilde{R} = (\widetilde{R}_i, R_{N\setminus\{i\}})$, then $f(\widetilde{R}) = s$ where $s \in L(R_i, x)$. Since condition (1) in pairwise justifiability can not hold from $\widetilde{R}$ to $R' = (R'_i, R_{N\setminus\{i\}})$, condition (2) must hold which imposes that $f(R'_i, R_{N\setminus\{i\}}) \in L(R_i, x)$ which is the desired contradiction since $y \notin L(R_i, x)$.

We show the "only if part" by contradiction: suppose that f violates pairwise justifiability on $\mathcal{R}^n \times \{B\}$, that is, there exist two preference profiles R, $R' \in \mathcal{R}^n$ such that $f(R) = x$, $f(R') = y$, $x, y \in B$, and for no agent $i \in N$ and no alternative $z \in A\setminus\{x\}$, $xP_i z$ and $zR'_i x$, and for no agent $i \in M$ where $M = \{i \in N : R_i \neq R'_i\}$, $xI_i y$. Therefore, at R either $xP_i y$ or $yP_i x$ for all agents $i \in M$. Define $\widetilde{R}$ as follows: for each agent $j \in M$ such that $xP_j y$, x is the top in A of $\widetilde{R}$ and y in the second place, and for each agent $k \in M$ such that $yP_k x$, y is the top in A of $\widetilde{R}$ and x in the second place. Start from R and change, one by one, the preference of each agent $j \in M$ such that $xP_j y$ from R_j to $\widetilde{R}_j$. In each step, strategy-proofness implies that the outcome is x (otherwise, agent $j \in M$ would gain by saying R_j instead of $\widetilde{R}_j$). Now, change the preference of each agent $k \in M$ such that $yP_k x$ from R_k to $\widetilde{R}_k$. In each step, strategy-proofness implies that the outcome is x (otherwise, if $xP_k f(\widetilde{R})$ agent $k \in M$ would gain by saying R_k instead of $\widetilde{R}_k$ and if $f(\widetilde{R}) = y$ agent $k \in M$ would gain by saying $\widetilde{R}_k$ instead of R_k). Thus, $f(\widetilde{R}) = x$. Note that since pairwise justifiability is violated, for each agent $j \in M$ such that $xP_j y$, $xP'_j y$, while for each agent $k \in M$ such that

yP_kx, either xP'_ky, yP'_kx, or xI'_ky. Define $\widehat{R}$ as follows: for each agent $j \in M$ such that xP_jy and xP'_jy, x is the top in A of $\widehat{R}$ and y in the second place, for each agent $k \in M$ such that yP_kx and xP'_ky, x is the top in A of $\widehat{R}$ and y in the second place, for each agent $k \in M$ such that yP_kx and yR'_kx, y is the top in A of $\widehat{R}$ and x in the second place. Start from R' and change, one by one, the preference of each agent $j \in M$ such that xP_jy and xP'_jy from R_j to $\widehat{R}_j$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $j \in M$ would gain by saying R_j instead of $\widehat{R}_j$ or the converse). Now, change the preference of each agent $k \in M$ such that yP_kx and xP'_ky from R'_k to $\widehat{R}_k$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $k \in M$ would gain by saying R_k instead of $\widehat{R}_k$, or the converse). Finally, change the preferences of agents $k \in M$ such that yP_kx and yR'_kx from R'_k to $\widehat{R}_k$. In each step, strategy-proofness implies that the outcome is y (otherwise, agent $k \in M$ would gain by saying R'_k instead of $\widehat{R}_k$). Thus, $f(\widehat{R}) = y$. Finally, change the preference of each agent $i \in N$ from $\widetilde{R}_i$ to $\widehat{R}_i$ starting first with type j agents in M. Remember that all such agents have x as top and y as second in both preferences**.** Therefore, strategy-proofness implies that the outcome is x (otherwise, agent $j \in M$ would gain by saying $\widetilde{R}_j$ instead of $\widehat{R}_j$). We now change type k agents in M. Remember that all such agents have y as top and x is second in $\widetilde{R}$ and x and y in the first and second position in $\widehat{R}$. By strategy-proofness, the outcome must be either x or y. Note that if the outcome is y, then this agent k would gain by saying $\widehat{R}_k$ instead of $\widetilde{R}_k$). Therefore, $f(\widehat{R}) = x$ which is the desired contradiction.

Proposition 3 Any collective choice correspondence C on $\mathcal{D} \times \mathcal{A}$ satisfying pairwise justifiability on $\mathcal{D} \times \mathcal{A}$ is transitively rationalizable on $\mathcal{D} \times \mathcal{A}$.

Proof of Proposition 3. Let C be a collective choice correspondence on $\mathcal{D} \times \mathcal{A}$ satisfying pairwise justifiability on $\mathcal{D} \times \mathcal{A}$. Let $R \in \mathcal{D}$ be any preference profile. Since $\mathcal{A}$ contains every pair of alternatives as a possible agenda, define a binary relation $\mathbf{R}_R$ on A as follows: for any $x, y \in A$, $x\mathbf{P}_Ry$ if and only if $C(R, \{x, y\}) = x$ and $x\mathbf{I}_R y$ if and only if $C(R, \{x, y\}) = \{x, y\}$ $(*)$. Note that $\mathbf{R}_R$ is complete. We now show that $\mathbf{R}_R$ transitively rationalizes C, that is, for any agenda $\overline{B} \in \mathcal{A}$, $C(R, \overline{B}) = t(\mathbf{R}_R, \overline{B})$ and $\mathbf{R}_R$ is transitive. We first show that for any agenda $\overline{B} \in \mathcal{A}$, $C(R, \overline{B}) = t(\mathbf{R}_R, \overline{B})$ (**). Consider any agenda $\overline{B}$ containing at least two alternatives (otherwise, the choice is unique). First, we show $C(R, \overline{B}) \subseteq t(\mathbf{R}_R, \overline{B})$: take any $x \in C(R, \overline{B})$ and show that $x \in t(\mathbf{R}_R, \overline{B})$. Take any $y \in \overline{B} \setminus \{x\}$ such that $\{x, y\} \subseteq \overline{B}$ and observe that the following holds: if $x \in C(R, \overline{B})$ then $x \in C(R, \{x, y\})$ by pairwise justifiability since agents' preferences do not change. Thus, $x \in t(\mathbf{R}_R, \{x, y\})$ by $(*)$. Repeating the same argument for all $y \in \overline{B} \setminus \{x\}$, we obtain that $x \in t(\mathbf{R}_R, \overline{B})$.

Second, we prove $t(\mathbf{R}_R, \overline{B}) \subseteq C(R, \overline{B})$: take any $x \in t(\mathbf{R}_R, \overline{B})$ and suppose, by contradiction, that $x \notin C(R, \overline{B})$. Consider $\{x, y\}$ such that $y \in C(R, \overline{B})$ (which always exists). By pairwise justifiability, since $y \in C(R, \overline{B})$ then $y \in C(R, \{x, y\})$. Moreover, since $x \notin C(R, \overline{B})$ then $x \notin C(R, \{x, y\})$. By definition of $\mathbf{R}_R$ on pairs of alternatives $(*)$, $y\mathbf{P}_R x$ which is a contradiction to the fact that $x \in t(\mathbf{R}_R, \overline{B})$.

Now, we prove that $\mathbf{R}_R$ transitively rationalizes C, that is $\mathbf{R}_R$ is transitive: take any triple of alternatives $x, y, z \in A$ we have to show that if $x\mathbf{R}_R y$ and $y\mathbf{R}_R z$ then $x\mathbf{R}_R z$. Observe that by definition of $\mathbf{R}_R$ on pairs stated in $(*)$, each one of the three relationships can be written using $C(R, \cdot)$, where $\cdot$ refers to the corresponding pair of compared alternatives being $B = \{x, y\}$, $B'' = \{y, z\}$, or $B' = \{x, z\}$. Distinguish the two cases concerning the choice in B:

(1) $C(R, B) = x$ $(x\mathbf{P}_R y)$ or (2) $C(R, B) = \{x, y\}$ $(x\mathbf{I}_R y)$.

For each one of the two cases we distinguish subcases depending on the choices in B'', that is, $C(R, B'') \in \{\{y\}, \{y, z\}\}$ $(y\mathbf{R}_R z)$. Define $\widetilde{B} = \{x, y, z\}$ and start with case (1):

(1.1) $C(R, B) = x$, $C(R, B'') = y$, then we show that $C(R, B') = x$. Since $y \notin C(R, B)$, $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$

to (R, B), by pairwise justifiability we obtain $y \notin C(R, \widetilde{B})$. Similarly, since $z \notin C(R, B'')$, $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B''), by pairwise justifiability we obtain $z \notin C(R, \widetilde{B})$. Thus, $x = C(R, \widetilde{B})$. By (**), $x = t(\mathbf{R}_R, \widetilde{B})$ and thus $x = t(\mathbf{R}_R, B')$ and by $(*)$, $x = C(R, B')$.

(1.2) $C(R, B) = x$, $C(R, B'') = \{y, z\}$, then we show that $C(R, B') = x$. We prove it by contradiction. First observe that since $y \notin C(R, B)$, $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B), by pairwise justifiability we obtain $y \notin C(R, \widetilde{B})$ and by (**) $y \notin t(\mathbf{R}_R, \widetilde{B})$. Now suppose, by contradiction, that $z \in C(R, B')$. By (**), $z \in t(\mathbf{R}_R, B')$ and since $y \notin t(\mathbf{R}_R, \widetilde{B})$, we obtain that $z \in t(\mathbf{R}_R, \widetilde{B})$. Again by (**), $z \in C(R, \widetilde{B})$. Since $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from (R, B'') to $(R, \widetilde{B})$, by pairwise justifiability we obtain $y \in C(R, \widetilde{B})$ which is a contradiction to what we have previously obtained. Therefore, we have proved that $C(R, B') = x$.

We now consider case (2):

(2.1) $C(R, B) = \{x, y\}$, $C(R, B'') = y$, then we show that $C(R, B') = x$. We prove it by contradiction. First observe that since $z \notin C(R, B'')$, $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B''), by pairwise justifiability we obtain $z \notin C(R, \widetilde{B})$. By (**) $z \notin t(\mathbf{R}_R, \widetilde{B})$. Moreover, since $C(R, B) = \{x, y\}$, we get that $t(\mathbf{R}_R, \widetilde{B}) = \{x, y\}$ and by (**) $C(R, \widetilde{B}) = \{x, y\}$. Now suppose, by contradiction, that $z \in C(R, B')$. Since $x, z \in B' \cap \widetilde{B}$, and agents' preferences do not change from (R, B') to $(R, \widetilde{B})$, by pairwise justifiability we obtain $z \in C(R, \widetilde{B})$ which is a contradiction to what we have previously obtained. Therefore, we have proved that $C(R, B') = x$.

(2.2) $C(R, B) = \{x, y\}$, $C(R, B'') = \{y, z\}$, then we show that $C(R, B') = \{x, z\}$. By contradiction, if $C(R, B') = x$ then $z \notin C(R, \widetilde{B})$. If $z \in C(R, \widetilde{B})$, since $x, z \in B' \cap \widetilde{B}$, and agents' preferences do not change from $(R, \widetilde{B})$ to (R, B'), by pairwise justifiability we obtain $z \in C(R, B')$ which is a contradiction to our hypothesis. Suppose that $y \in C(R, \widetilde{B})$, then since $y, z \in B'' \cap \widetilde{B}$, and agents' preferences do not change from (R, B'') to $(R, \widetilde{B})$, by pairwise justifiability we obtain $z \in C(R, \widetilde{B})$ which is a contradiction, thus $y \notin C(R, \widetilde{B})$. Then, $x \in C(R, \widetilde{B})$. Since $x, y \in B \cap \widetilde{B}$, and agents' preferences do not change from (R, B) to $(R, \widetilde{B})$, by pairwise justifiability we obtain $y \in C(R, \widetilde{B})$ which is a contradiction. Thus, $C(R, \widetilde{B})$ is not well-defined. A similar argument holds and non-definiteness of $C(R, \widetilde{B})$ would be obtained if we suppose that $C(R, B') = z$. This ends the proof.

Proposition 4 If a social welfare function F on $\mathcal{D}$ for $\mathcal{D} = \{\mathcal{P}^n, \mathcal{R}^n\}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then C on $\mathcal{D} \times \mathcal{A}$ satisfies pairwise justifiability.

Proof of Proposition 4. Let F on $\mathcal{D}$ be a social welfare function satisfying the weak Pareto condition (WP) and independence of irrelevant alternatives (IIA). Suppose, to get a contradiction, that F violates pairwise justifiability. Therefore, there exist two profiles $R, R' \in \mathcal{R}^n$ and a pair of alternatives $x, z \in A$ such that either (1) $x \succeq_R z$ and $z \succ_{R'} x$, or (2) $x \succ_R z$ and $z \succeq_{R'} x$, or (3) $x \succ_R z$ and $z \succ_{R'} x$, and pairwise justifiability is violated when applied to R and R'. If case (1) holds, the violation of pairwise justifiability requires that there is no agent i and no alternative $y \in A \backslash \{x\}$ such that either xP_iy and yR'_ix or xI_iz. This means that for any agent i either xP_iz or zP_ix. This also implies that for those agents i such that xP_iz then xP'_iz. By WP, there must be at least one agent $j \in N$ such that xP_jz. By IIA, there must be an agent changing the relative position between x and z when going from R to R'. Therefore, by (1), there must be an agent $k \in N$ such that zP_kx and xR'_kz. We now define $\widehat{R}$ as follows: for any agent $j \in N$ such that xP_jz we have $x\widehat{P}_jz$ and $z\widehat{P}_jy$, and for agent $k \in N$ such that zP_kx we have $z\widehat{P}_ky$ and $y\widehat{P}_kx$. By IIA, $x \succeq_{\widehat{R}} z$ and by WP, $z \succ_{\widehat{R}} y$. We now define $\widetilde{R}$ as follows: for any agent $j \in N$ such that xP'_jz we have $x\widetilde{P}_jy$ and $y\widetilde{P}_jz$, and for agent $k \in N$ such that $z\widetilde{R}_kx$ if and only if zR'_kx and we have $y\widetilde{P}_kz$ and $y\widetilde{P}_kx$. By IIA, $z \succ_{\widetilde{R}} x$ and by WP, $y \succ_{\widetilde{R}} z$. Since $x \succ_{\widehat{R}} y$ and $y \succ_{\widetilde{R}} x$, but when going from $\widehat{R}$ to $\widetilde{R}$ no agent has changed the relative position between x and y we get a contradiction to IIA. Case (2) follows a symmetric argument exchanging the roles of R and R' and of x and z. Concerning case (3) the same argument as in Case (1) works.

Proposition 5 If a social welfare function F on $\mathcal{D}$ for $\mathcal{D} = \{\mathcal{P}^n, \mathcal{R}^n\}$ satisfies the weak Pareto condition and independence of irrelevant alternatives, then C on $\mathcal{D} \times \mathcal{A}$ satisfies weak decisiveness.

Proof or Proposition 5. Let F on $\mathcal{D}$ be a social welfare function satisfying the weak Pareto condition (WP) and independence of irrelevant alternatives (IIA). Suppose, to get a contradiction, that F violates weak decisiveness. Therefore, for some situation (R, B) and some $x, z \in B$, $x \sim_R z$ and there is no agent i such that xI_iz. This means that for any agent i either xP_iz or zP_ix. We now define $\widehat{R}$ as follows: for any agent $j \in N$ such that xP_jz we have $x\widehat{P}_jy$ and $y\widehat{P}_jz$, and for agent $k \in N$ such that zP_kx we have $y\widehat{P}_kz$ and $z\widehat{P}_kx$. By IIA, $x \sim_{\widehat{R}} z$ and by WP, $y \succ_{\widehat{R}} z$. We now define $\widetilde{R}$ as follows: for any agent $j \in N$ such that xP_jz we have $z\widetilde{P}_jy$ and $x\widetilde{P}_jz$, and for agent $k \in N$ such that zP_kx, we have $z\widetilde{P}_ky$ and $y\widetilde{P}_kx$. By IIA, $z \sim_{\widetilde{R}} x$ and by WP, $z \succ_{\widetilde{R}} y$. Since $y \succ_{\widehat{R}} x$ and $x \succ_{\widetilde{R}} y$, but when going from $\widehat{R}$ to $\widetilde{R}$ no agent has changed the relative position between x and y we get a contradiction to IIA.

7.4 Proofs of results in Section 6

Proposition 6 Let $B \in \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{R}^n$. Any social choice function $f : \mathcal{D} \to B$ satisfying Maskin monotonicity on $\mathcal{D}$ satisfies pairwise justifiability on $\mathcal{D}$.

Proof of Proposition 6. By contradiction, if f violates pairwise justifiability on $\mathcal{D}$, there exist two preference profiles R, $R' \in \mathcal{D}$ such that $f(R) = x$, $f(R') = y$, $x, y \in B$, and for any $i \in N$ and any alternative $z \in A\backslash\{x\}$ such that $R_i \neq R'_i$, either (1) xP_iz and xP'_iz, or (2) zP_ix and zR'_ix, or (3) zP_ix and xP'_iz, holds. Start from R and change the preference of all agents R_i to R'_i.

Note that by (1), (2) and (3), for each agent $i \in N$, $[f(R)R_i z \Rightarrow f(R)R_i' z]$ thus, by Maskin monotonicity $f(R') = f(R)$ which is the desired contradiction.

Remark 2 Let $\mathcal{B}$ be a collection of agendas and $\mathcal{D} \subseteq \mathcal{R}^n$ a subset of preference profiles. Any Condorcet consistent collective choice function C on $\mathcal{D} \times \mathcal{B}$ satisfies strategy-proofness on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$ and with a Cartesian product structure.

Proof of Remark 2. Let C be Condorcet consistent on $\mathcal{D}' \times \mathcal{B}$ where $\mathcal{D}' \subseteq \mathcal{D}_{C\mathcal{B}}$ and $\mathcal{D}'$ has a Cartesian product structure. We prove that C is strategy-proof. By contradiction, suppose that there exist two situations (R, B), $(R', B) \in \mathcal{D}_{C\mathcal{B}} \times \mathcal{B}$ where $C(R, B) = x$, $C(R', B) = y$, and for some agent i, $yP_i x$. Since $C(R, B) = x$ is the strong Condorcet winner, then $C(R', B) \neq y$ since if x defeats y under R, it also defeats it under R'.

Example 5 Consider $N = \{1, 2, 3\}$, $\mathcal{B} = \{B\}$ where $B = \{x, y\}$, and $\mathcal{D} = \{R, R'\}$. Let R be such $xP_1 y$, $xP_2 y$, and $yP_3 x$ and R' such that $xI_1 y$, $xI_2 y$, and $yP_3 x$. The strong Condorcet winner at (R, B) and (R', B) are x and y, respectively. Any Condorcet consistent rule C must be such that $C(R, B) = x$ and $C(R', B) = y$. If C satisfied Maskin monotonicity, we should have that $C(R', B) = C(R, B)$ which is not the case.