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Limits to Local Insurance*

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Abstract

We study decentralized insurance when multiple risks are payoff-relevant, but each agent may only trade a (possibly different) subset of risks. Unless (at least) one agent can trade every risk, insurance markets remain incomplete, and the economy is not resilient to worst-case events. We also identify spill overs in any feasible allocation: others' inability to trade some risks restricts an agent's resilience to joint realizations. Unless an agent can trade a superset of i's risks, agent i is not resilient to them. In an application, we model constraints as risk-sharing networks and measure resilience in a Malawian village.

Keywords: Risk sharing, Incomplete markets, Market structure, Networks

JEL Codes: D11, D52, D53, D85, G52

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1 Introduction

Financial markets enable the transfer of risks across the global economy. However, some risks may not be traded by each and every market participant. An agent may be unable to sell insurance against a specific risk due to legal restrictions, cognitive constraints, or limited information. She may be constrained, for instance, by a mandate to manage a company's risk. She may lack access to timely information about a foreign sector or corporation, or she may simply be unaware that the option to trade this risk exists. In an economy where different trading constraints interact, does partial integration mean that some events cannot be insured at all? If so, are there sharp predictions on which events cannot be targeted? In particular, given the concern for systemic risk, is such an economy able to protect against events in which multiple shocks occur simultaneously?

In this paper we provide sharp bounds on insurance possibilities when risks must be shared this way. Our main model maintains the classical assumptions from efficient insurance in endowment economies (Arrow, 1964, 1971) except for two adjustments. First, rather than working on an abstract state space, we associate each state to a joint realization of payoff-relevant random variables. Each of these variables represents an underlying risk, such as the economic output of a firm, sector, or country, a specific weather event, inflation, exchange-rate volatility, or any other source of randomness. Second, we assume that agents may only transfer resources across states which differ by a subset of these risks, and we endow each agent with a (potentially) different subset.

The classical complete markets model is a special case of our environment in which every agent can condition on the entire set of payoff-relevant variables, and thus may access contingent claims on every state of the world. A constrained agent, on the other hand, can only obtain payoffs which result from combining coarser claims, which do not condition on variables she cannot trade. We refer to these constraints as *measurability constraints* and we call the corresponding claims *local* assets. In order to isolate the role of differential access from other trading frictions like short-selling constraints, we assume that each individual can freely combine all of her local assets.

We identify an economy by the collection of all individual constraints. Such an economy is two steps removed from the classical benchmark. On the one hand, insurance must be provided by means of a potentially restrictive set of instruments. On the other, agents have differential access to those instruments. In order to isolate the effect of the former, we first ask whether the entire set of available coarse instruments are collectively rich enough to replicate state-contingent claims. If true, the economy as a whole corresponds to an Arrow-Debreu economy, but where individuals differ in access. If false, then even if someone had the privilege to access the economy's full set of available instruments, she could not trade every joint risks. Our first main result shows that an economy is *globally complete* if and only if there exists an individual who can trade all risks. No matter how many instruments of arbitrarily restricted agents are combined, they can never compensate for the absence of an individual with direct access to all Arrow securities.

Next, consider which risks specific individuals may trade in this economy. Whenever an agents seeks to adjust her consumption across states, the remaining agents need to collectively take the opposite side of the trade. If $x_i(\omega)$ is the net-trade of agent i for a given state ω, the absolute value of her net-trade must be equal to the sum of net-trades of all other agents: $x_i(\omega) = -\sum_{j \neq i} x_j(\omega)$. Therefore, even if i's own measurabilty constraint allows her to trade across two events, such a trade is only feasible if the remaining agents can collectively generate the same trade. As a result – and in contrast to the predictions of the classical model – differential constraints may *spill over* from one agent to another, determining the set of joint shocks that can be feasibly insured.

In special cases, the spill-over effect is straightforward. For instance, suppose that agent 1 can trade all risks that agent 2 can trade, who in turn can trade all risks agent 3 can trade, and so on. Since resource constraints require $x_1(\omega) = -\sum_{j \geq 2} x_j(\omega)$, it follows immediately that agent 1's insurance possibilities are effectively reduced by others' restrictions. Indeed, agent 1 may only trade what is allowed by 2's constraint. All other agents are only limited by their own constraints. We say that an individual is *resilient* against a joint risk if, collectively, this risk can also be traded by rest of the economy. For instance, agent 1 above is not resilient against any set of risks which includes risks that 2 cannot trade. We say that an agent enjoys *local completeness* if she is resilient to the full set of risks that her constraint allows. Intuitively, local completeness fails when others' constraints spill over and further limit an agent's ability to transfer risks. Agent 1, for instance, is not locally complete, while all other agents in this example are.

In real world applications, where individuals face a variety of institutional, social, or cognitive constraints, the joint trading possibilities of $\sum_{j\neq i} x_j$ is not obvious. Unlike in the example, where there was a single most flexible agent 2, each agent's trading possibilities may complement others'. Which, then, are the relevant constraints that shape economic resilience? Which joint risks may be targeted by a specific group of individuals? Or by the entire economy? We provide answers to all these questions by characterizing the feasible payoff space of any individual in any economy. In doing so we develop necessary and sufficient conditions for completeness.

We show that an agent enjoys local completeness if and only if there exists another agent who can trade all risks that she has access to. This striking result implies that however rich the trading possibilities of a group, they cannot compensate for the absence of an agent who can complete i's market on her own. This result has several implications. Methodologically, it allows us to characterize spill overs, determine the joint insurance possibilities, and ultimately, understand how the interaction of constraints shape the space of feasible payoffs. More importantly, the result implies that when certain insurance instruments are missing, the system cannot transfer resources to worst case scenarios – those where multiple bad shocks coincide. To formalize this last result, we show that any economy can only be as resilient as its most resilient member. For instance, the economy described above is only as resilient as agent 2, since agent 1 is not more resilient than 2, even though she is less constrained.

To our knowledge, this is the first paper to characterize all feasible allocations for a broad class of economic environments with differential constrains. We identify those risks which can no longer be transferred, no matter the assumptions on beliefs, preferences, or trading protocols. Most importantly, we reveal a systematic tendency to under-provide insurance against extreme events in which multiple negative realizations coincide. That is, we argue that if certain events can no longer be targeted, these joint risk are the first to loose coverage. We show that, except for the special case where agents have preferences represented by constant absolute risk aversion, this loss in coverage unambiguously decreases welfare.

This paper is not the first to acknowledge that financial access may not be universal. Starting with Merton (1987), several strands of literature have shown that if agents must share risk by interacting across different submarkets, asset prices get distorted and risk can no longer be shared efficiently. Market fragmentation may even hinder diversification and give rise to market power if submarkets are small (Merton, 1987; Malamud and Rostek, 2017). Providing equilibrium predictions in such complicated environments typically requires reducing the individual decision problems into a lower-dimensional space under suitable choices of risk preferences and correlation structure of assets. For instance, because Gaussian environment with constant absolute risk aversion allow for a decision criterion which only depends on means and covariances, there is no meaningful concern for extreme events and tail risks. In this paper, we take a complementary view. Rather than fully describing the equilibrium, we focus on characterizing the set of feasible allocations. In taking this narrower approach we do not need to confine our analysis to special cases, and we can therefore analyze an economy's resilience against joint risks.

In line with the literature on incomplete markets, this paper explores how individuals cope with risks that cannot be easily shared with others. One strand in Macroeconomics and Finance is concerned with analyzing specific sources of excess risks, often to explain phenomena at odds with the classical predictions (Diamond, 1967; Aiyagari, 1994).[1] A second strand of the literature is concerned with exploring more fundamental properties of prices and risk allocations while making minimal assumptions on the source of incompleteness (Radner, 1972; Geanakoplos and Polemarchakis, 1986; Balasko, Cass, and Shell, 1995). We seek to minimally adapt the classical theory such that it accommodates a concrete model of decentralization through differential and overlapping risk sharing groups.

Measurability constraints are flexible enough to represent a broad class of frictions. This paper is not the first to adopt this modelling approach. In previous work, Guerdjikova and Quiggin (2019) propose such a model to explore when incorrect beliefs continue to drive competitive equilibria in the long run. In addition, they provide sufficient conditions for general trading restriction to be represented by measurabilty constraints. Although Guerdjikova and Quiggin (2019) also model financial constraints by a collection of individual partitions, these are not disciplined by a common source of frictions. As a result, their setting does not allow for a general method of aggregating individual constraints to determine collective payoff spaces. Indeed, most of the results on survival probabilities provided in their paper require partitions to relate by refinements, just as in our example above. Imposing nestedness would be too rigid for most of our applications. In contrast, we only assume a common structure in the inability to condition on a fixed set of payoff-relevant variables.

Our environment is flexible enough to accommodate several common frictions which have been repeatedly proposed as potential causes for constrained access. For example, our setting can represent limited awareness, where investors differ in their ability to assess the availability of particular class of assets. Guiso and Jappelli (2005), for instance, find that financial literacy rates vary widely across Italian households, while Van Rooij, Lusardi, and

[1]Prime applications include explaining cross-sectional behavior or assets price phenomena at odds with the classical predictions.

Alessie (2011) find that financial literacy affects financial decision-making in the Netherlands. Auster and Pavoni (2020) model the incentives of financial intermediaries in this context and find that the menus offered to less knowledgeable investors in equilibrium contain fewer products.

Even among equally sophisticated investors, the set of available financial instruments may vary substantially due to varying legal requirements. For instance, a hedge fund manager in Shanghai may be subject to a different set of regulations than the risk management department of a company in New York. Indeed, multinational companies manage third-party risk under strict corporate guidelines regularly approved by the sitting board of directors (Aebi, Sabato, and Schmid, 2012). In another example, tax-deductible retirement plans typically rule out investing in certain classes of risky assets (Atkins, 2011). We show in Section 2.2 that our environment also accommodates these type of institutional constraints.

Finally, we can accommodate cases where the different payoff-relevant variables correspond to the various individual incomes. In this mutual insurance application, our measurability constraints now imply that each individual's income risk can only be shared among a specific subset of the population, perhaps due to a lack of information, because of verifiability issues, or due to cultural constraints. Any set of constraints now induces a risk-sharing network between individuals: a link from i to j implies that agent i is able to trade j's income risk. Ambrus, Gao, and Milán (2021) consider a similar setting, but their results only hold for the special case of constant absolute risk aversion (CARA), for which they characterize constrained efficient sharing rules but not the set of feasible allocations.[2] We show in Section 4 how all our results on global and local completeness can be expressed as necessary and sufficient conditions on the link structure of the underlying network. We also

[2] A growing theoretical literature on risk sharing networks considers the role of connections in enforcing informal insurance (e.g. Bramoullé and Kranton, 2007; Bloch, Genicot, and Ray, 2008; Ambrus, Mobius, and Szeidl, 2014; Ambrus and Elliott, 2021). All these models assume that consumption allocations can respond to the economy's full set of income realizations. Risk Sharing networks have been documented and estimated extensively in the development literature (e.g. Fafchamps and Lund, 2003; Ligon, 1998)

describe systematic patterns of spill overs, for instance in core-periphery networks, where the hub's resilience to joint shocks from the periphery is limited.

We develop a network measure that captures the extent to which individuals' constraints spill over across neighbors in the network. Our *Resilience Measure* computes the proportion of a neighborhood's joint risks that can be traded by more than one agent. This measure is reminiscent of the notion of "supported links" introduced by Jackson, Rodriguez-Barraquer, and Tan (2012) in the context of favor-exchange networks, but with several important differences. We compute this centrality measure numerically on a real-world village network of households in rural Malawi. We find that only around 5.5% of households are resilient to all their joint shocks (i.e. complete), while 90% of households have a support centrality below 0.5, meaning they are not resilient to at least half of the joint shocks arising in their own neighborhoods.

A recent paper by Chandrasekhar, Townsend, and Xandri (2020) also considers risk sharing among small groups. But in their case, the friction comes from each agent having random access, where, at any given state, a subset of individuals form a centralized market and trade their random incomes. In contrast to our paper, conditional on access, all participants are assumed to be equally unconstrained while those without access remain in autarky. Therefore, upon participation, each group shares risk according to the classical properties. In the present paper, each individual's constraints are tied to a personal subset of risks. As a result, even if agents are able to trade with one another, their risk sharing patterns are at odds with the classical predictions.

The rest of the paper is organized as follows. Section 2 presents our main model. Section 3 characterizes which risks can be transferred in a feasible allocation, both globally and on an individual level. Section 4 applies the results to the case of risk-sharing networks. Section 5 concludes. All proofs are relegated to the Appendix. A supplementary Appendix shows how some of the main assumptions can be relaxed.

2 The Model

2.1 Set-up

Consider ex-ante trade in an economy where each agent $i \in N = \{1, 2, ..., n\}$ has a state-contingent endowment of the single consumption good, $y_i : \Omega \to \mathbb{R}$ and where Ω is a finite state space. All agents have identical expected utility preferences over final payoffs, $c_i : \Omega \to \mathbb{R}$ with $u : \mathbb{R} \to \mathbb{R}$ strictly increasing and strictly concave and a common belief $p : \Omega \to [0, 1]$.[3]

Let there be Q different payoff-relevant random variables, such as rain in a given location, a firm's profits, or an individual's income. Slightly abusing notation, we index each of these random variables by $q \in Q = \{1, ..., Q\}$, and we refer to each q as a *risk*. We assume that the set Q is exhaustive in the sense that any two distinct states differ in terms of the realization of at least one variable. Each state can therefore be identified by its characteristic vector of realized risks, $\omega = (z_1, \ldots, z_Q)$, where $z_q \in Z_q$ and $Z_q \subset \mathbb{R}$ is a finite set of possible values the q-th risk can take.

We assume that individuals may be unable to condition on all payoff-relevant variables in Q due to lack of information, funds, legal capacities, physical proximity, et cetera. To capture this, we endow each individual i with a set $M_i \subseteq Q$, which represents a personal subset of risks that i may condition on. Since i cannot condition on a risk $q \in Q \setminus M_i$, her consumption must be constant across any two states which differ only along these dimensions. In this case, we say that agent i cannot *distinguish* these pairs of states $\omega, \omega' \in \Omega$, in the sense that $c_i(\omega) = c_i(\omega')$ must hold for all feasible allocations.

Individual i's exogenous constraint, M_i, defines a personal partition of Ω, denoted by $\mathcal{L}_{M_i}$. Two states ω, ω' belong to the same element of $\mathcal{L}_{M_i}$ if and only if they are indistinguishable by individual i. Notice that c_i must therefore be measurable with respect to $\mathcal{L}_{M_i}$. More generally, starting from an arbitrary

[3]Our results do not require homogeneous preferences, and readily extend to non-expected utility.

set $M \subseteq Q$, define by $\mathcal{L}_M$ the partition of $\Omega = Z_1 \times \ldots \times Z_Q$ into cells that group states which are indistinguishable according to the risks in M. For each $\mathcal{L}_M$, denote by $L_M(\omega) \subseteq \Omega$ the cell which contains ω:

$$L_M(\omega) \equiv \{\omega' \in \Omega : z_q(\omega') = z_q(\omega) \text{ for all } q \in M\}. \tag{1}$$

We also call $L_M(\omega)$ an *M-local state*, i.e., the event that the the realized state is indistinguishable from ω in terms of M. By definition, $L_M(\omega) = L_M(\omega')$ for any $\omega' \in L_M(\omega)$.

Several constraints to insurance can be expressed in this way. For instance, setting $M = \emptyset$ the set $\mathcal{L}_\emptyset = \{\{\Omega\}\}$ corresponds to the trivial partition. Indeed, the standard dynamic self-insurance model in which every agent can only trade a riskless bond corresponds to $M_i = \emptyset$ for all i. At the other extreme, $\mathcal{L}_Q = \{\{\omega\}_{\omega\in\Omega}\}$ partitions Ω into singleton cells, representing an agent who is able to trade across all states. Therefore, setting $M_i = Q$ for all i corresponds to the standard complete markets model. Our environment is flexible enough to capture these benchmarks as well as many other intermediate cases of heterogeneous frictions.

We assume that the endowment y_i (and therefore i's net trade $x_i \equiv c_i - y_i$) must be measurable with respect to $\mathcal{L}_{M_i}$. In other words, i's constraint does not prevent her from trading her own income risk.[4] Moreover, we abstract from other potential frictions – like short-selling constraints or portfolio restrictions – which further limit *how* an agent can be exposed to the risks contained in M_i. In other words, we assume an otherwise frictionless economy where each individual i has access to a rich enough set of instruments to generate any payoff, as long as it is measurable with respect to $\mathcal{L}_{M_i}$.

An asset can be described by a function $a : \Omega \to \mathbb{R}$ which assigns a payoff of

[4]Intuitively, for every $M \subseteq M_i$, the partition $\mathcal{L}_M$ corresponds to a *coarsening* of the local partition $\mathcal{L}_{M_i}$ (i.e., $\mathcal{L}_{M_i} \leq \mathcal{L}_M$). The coarsening ignores information about the variables $M_i \setminus M$ (those which do not belong to M). Therefore, $\mathcal{L}_M$ can by obtained from $\mathcal{L}_{M_i}$ by merging all cells $L_{M_i}(\omega)$ which assign the same realizations in terms of the variables $M \subseteq M_i$.

$a(\omega)$ units of the consumption good in state ω. Equivalently, we can describe an asset by a column vector $\boldsymbol{a} \in \mathbb{R}^{|\Omega|}$ with coordinates $a_\omega = a(\omega)$. For any event $E \subseteq \Omega$, we denote by a_E the *unit claim* on E which is an asset that pays 1 in event E and 0 otherwise:

$$a_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The asset $\boldsymbol{a}_{L_M(\omega)}$ is of particular interest. It is a unit claim contingent on the event $L_M(\omega)$. From here on, we call $\boldsymbol{a}_{L_M(\omega)}$ an $M-$*local asset*. When setting $M = Q$, the local assets correspond to the standard Arrow security for state ω. The payoff matrix which collects all M-local assets is defined by

$$\boldsymbol{A}_M = [\boldsymbol{a}_L]_{L \in \mathcal{L}_M}. \tag{3}$$

We refer to the M_i-local assets simply as i's *local assets*.[5]

Proceeding analogously with payoff function c_i, we can equivalently describe it by the column vector $\boldsymbol{c}_i \in \mathbb{R}^{|\Omega|}$ of state-contingent consumption levels. Denote by C_i the subspace of payoff vectors which are compatible with i's measurability constraints:

$$C_i \equiv \{\boldsymbol{c}_i \in \mathbb{R}^{|\Omega|} \mid \text{for every } \omega' \in L_{M_i}(\omega) : c_i(\omega) = c_i(\omega'), \forall \omega\}. \tag{4}$$

The following lemma establishes that, regardless of the assignment mechanism, any $\boldsymbol{c}_i$ satisfies i's measurability condition defined in equation (4) if and only if net-transfers lie in the span of i's local assets.

[5]Note that the column span of $\boldsymbol{A}_{M_i}$ includes the payoff of any coarser asset whose payoff is measurable with respect to $\mathcal{L}_{M_i}$. We interpret the relevant list of available insurance $\boldsymbol{A}_{M_i}$ as a basis that characterizes her payoff space. Hence we do not impose a zero-net-supply condition. Alternatively, we could represent an appropriately expanded asset structure which includes for each i all the redundant columns for M-local asset that condition on a subset $M \subset M_i$. In this case, individuals with differential restrictions could engage in mutually measurable trade in appropriately coarse assets in zero net supply.

Lemma 1. *Given initial endowment $\boldsymbol{y}_i$, the vector $\boldsymbol{c}_i$ satisfies i's measurability constraints* (4) *if and only if the net-trade $\boldsymbol{x}_i = (\boldsymbol{c}_i - \boldsymbol{y}_i)$ lies in the column span of $\boldsymbol{A}_{M_i}$, defined in* (3).

A vector of final payoffs $\boldsymbol{c} = (\boldsymbol{c}_1, ..., \boldsymbol{c}_n)$ satisfies the resource constraints with equality if

$$\sum_i (c_i(\omega) - y_i(\omega)) = \sum_i x_i(\omega) = 0. \tag{5}$$

If the vector $\boldsymbol{c}$ also satisfies that each individual payoff is measurable (i.e., $\boldsymbol{c}_i \in C_i$), we then say that the consumption allocation is *feasible*. Recall that since endowments $\boldsymbol{y}_i \in \mathbb{R}^{|\Omega|}$ are assumed to be measurable (i.e., $\boldsymbol{y}_i \in C_i$) an individual's set of feasible net-trades X_i, defined by

$$\boldsymbol{x}_i \equiv \boldsymbol{c}_i - \boldsymbol{y}_i, \tag{6}$$

coincides with the set of feasible payoffs, $X_i = C_i$.

A wide range of economic models share the fundamental feature that different risks can only be shared among a specific subgroup of individuals. Our results are independent of any specific trading protocols, pricing mechanisms or equilibrium concepts. Concretely, section B.3 shows how we can adapt our model to apply the measurability constraints on assets rather than individuals. Our results extend naturally to economies where each agents trades different sets of these assets on distinct submarkets with heterogeneous access, akin to Malamud and Rostek (2017).

Before characterizing the trading possibilities and the set of feasible consumption allocations, we summarize a few well-studied applications that our framework is able to capture. We describe these applications concretely in the context of a simple example.

2.2 Applications

Measurability constraints capture a range of institutional and informal settings. From a normative perspective, the set of feasible allocations corresponds to the choice set for constrained efficient risk sharing rules when some individuals cannot be relied on to bear certain risks. Similarly, in informal arrangements where outcomes are reached through local bargaining, social norms, or other non-market insurance mechanisms, multilateral transfers may be limited to condition on set of locally relevant variables.

To fix ideas, it will be useful to provide a particular running example. Consider therefore a heterogeneous access model, similar to Guerdjikova and Quiggin (2019), which departs minimally from the classical competitive equilibrium. Our approach captures many alternative settings where financial incompleteness stem from heterogeneous contracting frictions, such as institutional constraints, transaction costs, information asymmetries, or even trust and social capital requirements. The constraints in this subsection describes three different instances of well-documented frictions, all of which presented in terms of a unified example with three agents $N = \{1, 2, 3\}$ and three payoff-relevant variables $Q = \{1, 2, 3\}$.

Application 1. ***(Limited Awareness)*** *Our setting can easily capture differences in investor sophistication coming from limited awareness. For instance, let $q = 1$ represent an aggregate index like the S&P 500, $q = 2$ a specific company's performance relative to the S&P 500, and let $q = 3$ represent a local weather index (such as the amount of rainfall in some area). Let agent $i = 1$ be an institutional investor. She is able to trade instruments which condition on all attributes $M_1 = \{1, 2, 3\} = Q$. Agent 2 is a retail investor who does not know about instruments which pay on specific weather events (the third variable), such that $M_2 = \{1, 2\}$. The assets that he trades only allow him to target all joint realizations of z_1 and z_2. Finally, let agent $i = 3$ be a farmer who, apart from buying weather insurance, is only aware of the instruments which condition on the aggregate index. She is not aware of instruments which*

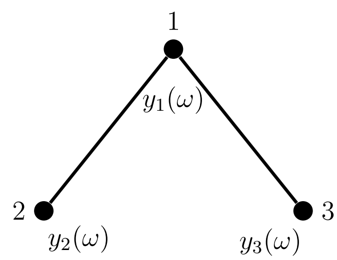

Figure 1: A Local Insurance Network

trade on the company's performance (the second attribute), hence $M_3 = \{1, 3\}$.

As mentioned above, even equally sophisticated investors may face differential legal constraints. The next example shows how they may be captured in terms of coarse partitions.

Application 2. ***(Institutional Constraints)*** *Let $q = 1$ represent a commodities index, $q = 2$ be a broad market index, and $q = 3$ be the local exchange rate vis-à-vis a foreign currency. Individual $i = 1$ can trade all variables. Individual $i = 2$ is the custodian of an IRA who commits not to trade on the exchange rate. Finally, let $i = 3$ be a risk manager for a company which may hedge supply risk on commodities and currency risk, but she cannot trade on the market index. Again, the relevant constraints correspond to $M_1 = \{1, 2, 3\}$, $M_2 = \{1, 2\}$ and $M_3 = \{1, 3\}$, respectively.*

Finally, consider a situation of mutual insurance relationships where households must be sufficiently "close" to trade their respective income risks. The variables Q coincide with the set of households N in terms of each of their random incomes. Any coarse partition comes from an inability to condition on the income shocks of others.

Application 3. ***(Risk-Sharing Networks)*** *Consider a group of three households, $i = 1, 2, 3$. Agent i can trade j's income risk if and only if i and j are neighbors on some underlying network. This network may capture information frictions that preclude informal insurance arrangements when monitoring* *is too costly and incentives to reveal income cannot be provided (see for instance Ambrus, Gao, and Milán (2021)).*

The network in Figure 1 corresponds to a situation where agents 2 *and* 3 *cannot reliably observe each others' income and this information is not available to them when sharing their respective risks with agent* 1*. In the context of our current setting, this example implies that agent* 1 *can condition on all incomes (i.e.,* $M_1 = N$*) while agents* 2 *and* 3 *cannot. In particular,* $M_2 = \{1, 2\}$ *and* $M_3 = \{1, 3\}$*. We consider the implications of this specification in more detail in Section 4*

While the above applications capture very different economic environments, they correspond to the same primitives of our general framework. To further fix ideas, assume that each attribute $q \in Q$ can only take a good or bad outcome: $z_q \in \{g, b\}$.[6] Without loss of generality, order the resulting states by $\{(b, b, b), (b, b, g), (b, g, b), ..., (g, g, g)\}$.

In all three cases, agent 1 can trade all risks and her personal partition $\mathcal{L}_{M_1} = \{\{\omega\}_{\omega \in \Omega}\}$ consists of Ω. Agent 2, on the other hand, cannot trade $q = 3$, so his partition groups any two states which only differ by this variable: $\mathcal{L}_{M_2} = \{\{(b, b, b), (b, b, g)\}, ..., \{(g, g, b), (g, g, g)\}\}$. Agent 3's situation is analogous to agent 2's, except that she can't trade $q = 2$. Her partition is therefore, $\mathcal{L}_{M_3} = \{\{(b, b, b), (b, g, b)\}, ..., \{(g, b, g), (g, g, g)\}\}$.

We can also represent each agent's contingent claims space through the payoff matrix $\boldsymbol{A}_{M_i}$. Without loss of generality, we can write $\boldsymbol{A}_{M_1} = (\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_8)$ where $\mathbf{e}_i \in \mathbb{R}^8$ denotes the standard vector with a 1 in the i-th coordinate and 0's elsewhere. In other words, because agent 1 can trade all variables, her local assets correspond to the standard basis of $\mathbb{R}^{|\Omega|}$. On the other hand, $\boldsymbol{A}_{M_2} = (\mathbf{e}_1 + \mathbf{e}_2, \ldots, \mathbf{e}_7 + \mathbf{e}_8)$ and $\boldsymbol{A}_{M_3} = (\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4, \ldots, \mathbf{e}_6 + \mathbf{e}_8)$, represent how agents 2 and 3 each face four different local assets that pay along different M-local states.

[6] A state ω can therefore be described by a characteristic vector $\omega = (z_1, z_2, z_3) \in \{g, b\}^3$, and $|\ \Omega\ | = 8$.

Finally, in order to fix ideas on a potential implementation, consider a suitably adapted version of the competitive equilibrium. Denote by $\Pi : \Omega \to \mathbb{R}$ a state-price vector such that the price Π_x of any payoff $x \in \mathbb{R}$ satisfies $\Pi_x = \sum_\omega \Pi(\omega)x(\omega)$. By our assumptions on preferences, the law of one price holds in any competitive equilibrium (LeRoy and Werner, 2014), so that such a Π exists without loss of generality. We are now ready to define a locally-provided insurance equilibrium. The first two conditions state feasibility (measurability and market clearing), and the latter two conditions state optimality.

Example. ***(Competitive equilibrium)*** *Given initial endowments* $\boldsymbol{y}_i$ *and measurability constraints* M_i *for* $i = 1, ..., n$*, a state-price* $\Pi^* : \Omega \to \mathbb{R}$ *and a vector of final payoffs* $\mathbf{c}^*$ *form a* ***local insurance equilibrium*** *if:*

1. $\boldsymbol{c}_i^* \in C_i$ *for all* $i \in N$*;*
2. $\sum_{i\in N} [y_i(\omega) - c_i^*(\omega)] = 0$ *for all* $\omega \in \Omega$*;*
3. $\mathrm{E}\, u(\boldsymbol{c}_i) > \mathrm{E}\, u(\boldsymbol{c}_i^*) \implies \sum_{\omega\in\Omega}[\Pi^*(\omega)(y_i(\omega) - c_i(\omega))] < 0$ *for all* $\boldsymbol{c} \in C_i$ *and* $i \in N$*;*
4. $\sum_{\omega\in\Omega}[\Pi^*(\omega)(y_i(\omega) - c_i^*(\omega))] = 0$ *for all* $i \in N$*.*

While the remainder of this paper uses the general setup described in Section 2.1, we refer back to this simple three-agent example throughout the text for intuition. Section 4 reviews our main results through the lens of Application 3 to relate them to well-known graph properties in the long and active literature on risk-sharing networks.

3 Local Insurance

3.1 Missing Markets

Before analyzing the differential access constraints across individuals, we need to understand if and how providing insurance through local instruments is restrictive in itself. Consider therefore a hypothetical agent j who can simultaneously access all local assets $\boldsymbol{A}_{M_i}$ of every agent $i \in N$. Would she be able to trade all joint risks in the economy? And if not, which are the events she can insure against?

While phrased as a hypothetical exercise, this question relates to concrete insurance properties. First, whether the asset span is complete or not matters for equilibrium properties such as the existence of a unique state prices to value payoffs which do not satisfy measurability (LeRoy and Werner, 2014). Second, in terms of policy implications, it is crucial to know whether efficiency can be achieved by making existing local instruments more widely accessible, or whether some markets are missing altogether. Finally, if some instruments are indeed missing, we need to understand how their absence constrains the possibilities for risk sharing, no matter the allocation mechanism.

Given the local constraints $\{M_i\}_{i \in N}$, denote by $\boldsymbol{J}_N$ a payoff matrix which collects all the $(\sum_{i \in N} |\mathcal{L}_{M_i}|)$ columns that appear in the various matrices $\boldsymbol{A}_{M_i}$ across every $i \in N$:

$$\boldsymbol{J}_N = [\ \boldsymbol{A}_{M_i}\ ]_{i \in N}. \tag{7}$$

Notice that $\boldsymbol{J}_N$ defines a meaningful upper bound on the economy's capacity to trade. It combines all trades that any individual has access to. We denote the corresponding payoff space by C_N:

$$C_N := \operatorname{span}(\boldsymbol{J}_N) \subseteq \mathbb{R}^{|\Omega|}. \tag{8}$$

The individual measurability constraints can then be interpreted as differential access constraints in the spirit of Guerdjikova and Quiggin (2019), where each i can only trade a subset of assets (i.e., those corresponding to the columns of $\boldsymbol{J}_N$ which appear in $\boldsymbol{A}_{M_i}$).[7] If $\boldsymbol{J}_N$ is rich enough to span $\mathbb{R}^{|\Omega|}$, we say that the set of instruments is globally complete.

Definition 1. *An economy described by* $\boldsymbol{J}_N$ *satisfies* global completeness *if*

[7] Notice that $\boldsymbol{J}_N$ typically contains redundant assets. This is obvious if one agent can trade strictly more than another agent.

$rank(\boldsymbol{J}_N) = |\Omega|$.

If an analyst had data on the constraints $\{M_i\}_{i \in N}$, she could determine if an economy is complete in a case-by-case basis simply by checking the rank of $\boldsymbol{J}_N$ computationally. However, the goal of this paper is to provide general insights into how the fundamentals of the economy (i.e., the common structure underlying individual constraints) shape the payoff space, and therefore the ability to share risk. We therefore seek a general result that identifies the payoff space C_N–and therefore which economies are complete– as a function of the primitives only (i.e., the sets M_i).

To this end, the next section proposes a novel basis of $\mathbb{R}^{|\Omega|}$ with a key property: for any set $M_i \subseteq Q$, there exists a subset of these basis vectors which spans the resulting constrained payoff space. In other words, each constraint M_i provokes a loss of dimensionality which corresponds to an elimination of the vectors from a common basis. As a result, we can not only interpret individual payoff spaces through a common basis, but we know the extent to which the trading possibilities of i complement those of the remaining agents.

3.2 An Alternative Basis of the State Space

Constructing suitable basis vectors requires two auxiliary objects. First, we fix an arbitrary reference state. Without loss of generality, and for the sake of exposition, let $\omega_0 \in \arg\min_{\omega \in \Omega} \sum_i y_i(\omega)$ be a state which minimizes aggregate income. To ease exposition, once we select ω_0, we simply refer to $\omega_0 = (\underline{z}_1, ..., \underline{z}_Q)$ as *the* worst state and the corresponding $\underline{z}_q$ as *the* worst realization of variable q. Second, for every possible subset $M \subseteq N$, we collect those states that assigns the worst realization $\underline{z}_q$ to all risks q except those belonging to M. That is, the set O_M collects all states in which the joint realization of risks in group M does not contain any worst realization while all remaining risks do obtain their worst realization:

$$O_M \equiv \{\omega \in \Omega \mid \forall q \in M : z_q(\omega) \neq z_q(\omega_0) \text{ and } \forall r \notin M : z_r(\omega) = z_r(\omega_0)\}. \quad (9)$$

Notice that we can associate any state with two unique pieces of information: the set M of variables which do not obtain their worst realization, and their M-local state $L \in \mathcal{L}_M$. The former pins down the realization of the variables in $Q \setminus M$, the latter specifies the realizations in M.[8]

To construct our basis $\boldsymbol{B}$, we collect, for each possible subset $M \subseteq Q$, only those M-local assets $\boldsymbol{a}_{L_M(\omega)}$ for which no member, q of M has the worst realization, $\underline{z}_q$:

$$\boldsymbol{B} = \left[\left[\boldsymbol{a}_{L_M(\omega)} \right]_{\omega \in O_M} \right]_{M \subseteq Q}. \tag{10}$$

To prove that $\boldsymbol{B}$ is indeed a basis for $\mathbb{R}^{|\Omega|}$, we need to show that it consist of $|\Omega|$ linearly independent assets. Lemma 6 in Appendix B.4 shows that $\boldsymbol{B}$ has exactly $|\Omega|$ columns. Intuitively, each ω points to exactly one column $\boldsymbol{a}_{L_M(\omega)}$ where M is defined by the variables whose value is different from $\underline{z}_q$. The following lemma proves that the set of columns is indeed linearly independent.

Lemma 2. *The column vectors in $\boldsymbol{B}$ form a basis of $\mathbb{R}^{|\Omega|}$.*

To see why none of the assets in $\boldsymbol{B}$ are redundant, recall that each of them corresponds to a contingent claim on an event in which all the variables in a set M take a better-than-worst realization. Moreover, notice that only observing a set $K \not\supseteq M$ of variables does not reveal the joint realization of M. Therefore, no linear combination of any such $K \not\supseteq M$-local instruments can ever replicate a finer M-local asset. At the same time, none of the finer $K \supset M$-local assets which belong to $\boldsymbol{B}$ pay in states where any $q \in K$ takes its worst value. However, in order to replicate an M-local asset requires a payment in states that do assign the worst value to the variables $K \setminus M$. As a result, no linear combination of these $K \supset M$-local assets can ever generate a payoff that is measurable with respect to $\mathcal{L}_M$.

Example. *Recall the three-agent examples from Section 2.2, when each $q \in Q$*

[8] In Appendix B.4, we show that each state $\omega \in \Omega$ belongs to an O_M for exactly one group $M \subseteq N$. Notice that equation (9) can be similarly expressed as $O_M = L_{Q \setminus M}(\omega_0) \setminus L_M(\omega_0)$. Given the realizations in state $\omega \in O_M$, an individual knows that reference state ω_0 has not occurred if and only if she observes at least one variable in group $M \subseteq Q$.

corresponds to a binary risk. In this case, the space of state-contingent payoffs corresponds to $\mathbb{R}^8$. Following equation (10), this space has an alternative basis $\boldsymbol{B}$ composed of eight linearly independent M-local assets, one for each subset $M \subseteq Q = \{1, 2, 3\}$. For every subset M, this asset is a contingent claim which only pays 1 in the event where none of the risks in M take the low realization. For $M = \emptyset$, this event trivially includes all states. Since the asset pays 1 in every state, this corresponds to a riskless bond. Similarly, when $M = \{q\}$ for $q = 1, 2, 3$, the M-local asset pays 1 in those four states where the risk q takes the high realization. For any subset $M = \{q, q'\}$ of size two, with $q' \neq q$, the M-local asset pays 1 unit only in those two states where both variables q and $q' \neq q$ simultaneously take the high realization. Finally when $M = \{1, 2, 3\}$ the asset corresponds to an Arrow security for the unique state where all variables take the high realization. Appendix B.5 provides an explicit description of the matrix $\boldsymbol{B}$.

3.3 Characterizing Payoff Spaces

We now show that, for any set of constraints $\{M_i\}_{i \in N}$, there exists a basis $\boldsymbol{B}_n$ for the payoff space C_N which consists of a subset of columns in $\boldsymbol{B}$. Lemma 4 below establishes that $\boldsymbol{B}_N$ contains an M-local asset from $\boldsymbol{B}$ only if there exists an agent i who can trade M, i.e., $M \subseteq M_i$ for some i. For every possible set of individuals $I \subseteq N$, collect the subsets of Q which are traded by at least one $i \in I$

$$\mathcal{M}_I = \{M \subseteq Q : \exists i \in I \text{ such that } M \subseteq M_i\}. \tag{11}$$

Notice that while $\mathcal{M}_{\{i\}}$ collects all subsets of M_i, the set $\mathcal{M}_{\{i,j\}}$ is the union of $\mathcal{M}_i$ and $\mathcal{M}_j$.[9] For $I = N$, the set $\mathcal{M}_N$ collects all subsets contained in any M_i across all $i \in N$.

For any arbitrary set $I \subseteq N$, construct a submatrix of $\boldsymbol{B}$ which only keeps those M-local assets associated with groups M of risks appearing in $\mathcal{M}_I$ from

[9] This is not equivalent to the collection of subsets of $M_i \cup M_j$ (except if M_i and M_j are disjoint)

(11). The resulting matrix $\boldsymbol{B}_I$ removes exactly those local assets from $\boldsymbol{B}$ which are not available to any member of I. Formally, fixing a group $I \subseteq N$, let

$$\boldsymbol{B}_I = \left[\left[\boldsymbol{a}_{L_M(\omega)}\right]_{\omega \in O_M}\right]_{M \in \mathcal{M}_I}. \tag{12}$$

The definition of $\boldsymbol{B}_I$ in (12) is identical to $\boldsymbol{B}$ in (10) except for the outer restriction which selects only the subsets of Q that can be traded by at least one member of I. To fix ideas, consider again the example from section 2.2.

Example. *Equation (23) in Appendix B.5 provides an explicit expression of the matrix $\boldsymbol{B}_I$ for the group of agents $I = \{2,3\}$. Compared to $\boldsymbol{B}$, the matrix $\boldsymbol{B}_I$ removes exactly those M-local which are neither available for $M_2 = \{1,2\}$ nor for $M_3 = \{1,3\}$. That is, $\boldsymbol{B}_I$ eliminates the local assets which pay on a joint high realization of $M = \{2,3\}$ and $M = \{1,2,3\}$. The remaining six M-local asset are therefore induced by the six subsets $M = \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}$.*

Crucially, $\mathcal{M}_I$ can only coincide with the power set 2^Q if at least one individual $i \in I$ can trade every risk (i.e., $M_i = Q$). Therefore, if we can establish that $\boldsymbol{B}_I$ is a basis for $\operatorname{span}(\boldsymbol{J}_I)$, then checking whether $\boldsymbol{B}_I$ and $\boldsymbol{B}$ coincide – that is, whether some member of I can trade every risk – determines whether group I has a rich enough set of instruments to span $\mathbb{R}^{|\Omega|}$. In order to prove the desired result, we begin by proving it for an individual's payoff space, i.e., for $I = \{i\}$.

Lemma 3. *The local assets in $\boldsymbol{B}_{\{i\}}$ form a basis of i's payoff space, $C_i = \operatorname{span}(\boldsymbol{A}_{M_i}) = \operatorname{span}(\boldsymbol{B}_{\{i\}})$.*

We can now construct the payoff space for a group $I \subseteq N$ of any size. Consider how adding trades available to j adds to those available to i. Lemma 3 conveniently allows us to answer this question because it implies that each asset in $\boldsymbol{B}_{\{j\}}$ is either one of the assets in $\boldsymbol{B}_{\{i\}}$ or it is linearly independent from all the assets in $\boldsymbol{B}_{\{i\}}$. Only the latter kinds of assets expand the payoff space, since it adds the possibility to trade a set of joint risks M that i cannot trade.

Proceeding this way, we can identify the joint insurance possibilities for any group $I \subseteq N$. To see this, notice that any local asset in $\boldsymbol{B}_{\{i\}}$ must appear in $\boldsymbol{B}_I$ whenever $i \in I$, and, conversely, for any column in $\boldsymbol{B}_I$, there must exist at least one member $i \in I$ for whom the same asset appears in $\boldsymbol{B}_{\{i\}}$. This leads to the following result.

Lemma 4. *For any group $I \subseteq N$, the assets in $\boldsymbol{B}_I$ form a basis of the payoff space,* $\text{span}(\boldsymbol{J}_I)$.

Our novel basis allows for a convenient interpretation in terms of the primitives of the model. The grand matrix $\boldsymbol{B}$ contains at least one asset for each subset M. As a result, any payoff $\boldsymbol{x} \in \mathbb{R}^{|\Omega|}$ corresponds to a unique set of coordinates which can be interpreted as portfolio weights on particular M-local assets. Whenever an M-local asset gets assigned a non-zero weight, only an individual who can trade M will be able to generate said payoff $\boldsymbol{x}$. Conversely, we can associate to each payoff space C the required groups M of risks in Q that agents must have access to.[10]

By Lemma 4, $\boldsymbol{B}_N$ is a basis for C_N. Checking whether the economy is globally complete therefore amounts to finding necessary and sufficient conditions on the measurability constraints $\{M_i\}_{i \in N}$ such that that $\boldsymbol{B}_N$ coincides with $\boldsymbol{B}$. Following our discussion above, completeness requires that all possible subsets of Q can be traded by at least one agent, hence the following result.

Proposition 1. *The economy $\{M_i\}_{i \in I}$ cannot collectively insure against a joint risk $M \subseteq Q$, unless there exists a member $i \in N$ who can trade it ($M \subseteq M_i$). Moreover, it is globally complete if and only if there exists at least one agent who can trade all risks: $C_N = \mathbb{R}^{|\Omega|}$ if and only if $\exists i \in N$ such that $M_i = Q$.*

[10]Note that the standard basis for $\mathbb{R}^{|\Omega|}$ can also be interpreted as the collection of all M-local assets, each of these assets corresponds to the same group M (i.e.,to the grand set $M = Q$). In contrast to $\boldsymbol{B}$, however, the *standard* coordinates of a payoff $\boldsymbol{x}$ do not reveal a portfolio of necessary M-local assets.

Proposition 1 provides a simple and intuitive condition that is both necessary and sufficient for an economy to be complete. Sufficiency is immediate since $M_i = Q$ means that i alone has access to instruments that already span $\mathbb{R}^{|\Omega|}$. The more striking result, however, is necessity. No matter how many coarse assets we combine – and regardless of how the sets M_i relate to each other – these instruments will never be rich enough to insure against all possible events unless there exists an agent who can trade all risks. In this respect, the economy is only as resilient as its most sophisticated member.

If we did not impose structure on the individual partitions, a result like Proposition 1 would not be available. In general, a set of individually coarse partitions of Ω may complement each other such that combining the resulting assets collectively completes the market. Instead, our partitions are coarsening along a common principle: ignoring a subset of Q.

So far, we have considered economy-wide implications of differential access to insurance. In the next section we consider how each individual can manage her exposure to risk, taking into account that each of the remaining agents may be constrained by a different set M_i. In particular, we study whether others' constraints spill over and limit an agent's access to insurance beyond the limits imposed by her own constraints.

3.4 Resilience and Spill Overs

In any feasible allocation, whenever i trades away from her initial endowment, the remaining individuals must collectively take the opposite position in terms of their net trades. As a result, a feasible net-trade for i not only requires a suitable measurability constraint for herself, but also enough flexibility by others.

More formally, the resource constraint implies that net-trades sum to 0.[11] Any feasible allocation must therefore satisfy $-\boldsymbol{x}_i = \sum_{j \neq i} \boldsymbol{x}_j$, where all indi-

[11] Recall that the previous section on the richness of the asset structure $\boldsymbol{J}_N$ did not require incorporating resource constraints.

viduals involved must satisfy their measurability constraints.[12] Therefore, in order for a net trade $\boldsymbol{x}_i \in \text{span}(\boldsymbol{A}_{M_i})$ to be feasible requires that the remaining agents can jointly meet these insurance demands such that $\boldsymbol{x}_i \in \text{span}(\boldsymbol{J}_{N\setminus i})$, where we used the fact that $-\boldsymbol{x} \in \text{span}(\boldsymbol{J}_{N\setminus i}) \Leftrightarrow \boldsymbol{x} \in \text{span}(\boldsymbol{J}_{N\setminus i})$.

Accordingly, we can define the measurable and feasible payoff profiles available to i as follows:

$$X_i^* = \text{span}(\boldsymbol{A}_{M_i}) \cap \text{span}(\boldsymbol{J}_{N\setminus i}). \tag{13}$$

Formally, $X_i^* \subseteq X_i$ forms another linear subspace of $\mathbb{R}^{|\Omega|}$. Notice that X_i^* allows i to trade a joint risk M if it contains the corresponding contingent claim space $\text{span}(\boldsymbol{A}_M) \subseteq X_i^*$. In this case, we say that i is *resilient* against said joint risk. In particular, if $X_i^* = \text{span}(\boldsymbol{A}_{M_i}) = X_i$, then the remaining agents are flexible enough to match any insurance request that satisfies i's measurability constraints. In this case, we say that i enjoys *local completeness*.

Definition 2. *Agent i is* resilient *against a joint risk $M \subseteq M_i$ if the remaining group $N \setminus i$ can also trade M, i.e.,* $\text{span}(A_M) \subseteq X_i^*$*. If i is resilient against M_i, we say that she enjoys* local completeness*.*

So far, our assumption that $\boldsymbol{y}_i$ satisfies i's measurability constraint has guaranteed that the space of net-trades coincides with the payoff space: i.e., $X_i = C_i$. However, this may no longer be satisfied in terms of the payoff profiles C_i^* that are jointly compatible with all measurability constraints. That is, we may have $X_i^* \neq C_i^*$. To see this note that our assumption does not guarantee that $\boldsymbol{y}_i$ belongs to X_i^*, since X_i^* is potentially restricted further by others' constraints: $X_i^* \subseteq X_i$.

In general, the set of i's payoff profiles C_i^* that are jointly compatible with

[12]We assume that resource constraints hold with equality. Given the monotonicity of preferences, this is without loss of generality in any competitive equilibrium, but also holds in various other assignment mechanisms such as Nash bargaining.

all measurability constraints satisfies

$$C_i^* = \boldsymbol{y}_i + X_i^*. \tag{14}$$

We must therefore investigate whether i is exposed to *uninsurable* endowment risk when others' measurability constraints do not allow her to pass it on.[13]

Notice that for some economies, it is immediate whether i faces additional restrictions beyond her own measurability constraints. For instance, if there exists an individual j who can trade every risk that i can (i.e., $M_i \subseteq M_j$), then all trades available to i must be available to j, such that $\text{span}(\boldsymbol{A}_{M_i}) \subseteq \text{span}(\boldsymbol{A}_{M_j}) \subseteq \text{span}(\boldsymbol{J}_{N\setminus i})$.

In order to characterize X_i^* for any arbitrary set of constraints, we need to proceed like in our characterization of C_i above in order to find a basis for X_i^*. The subsequent analysis shows that we can (again) construct such a basis from the columns of $\boldsymbol{B}$ in (22). To perform the correct selection of assets, define by $\mathcal{M}^*_{\{i\}}$ the subsets $M \subseteq M_i$ of risks that both i and the collective of all remaining individuals can condition on:

$$\mathcal{M}^*_{\{i\}} \equiv \mathcal{M}_{\{i\}} \cap \mathcal{M}_{N\setminus\{i\}}. \tag{15}$$

That is, for each $M \in \mathcal{M}^*_{\{i\}}$, there must exist another individual $j \neq i$ who can trade it, $M \in \mathcal{M}_{\{j\}}$. Based on $\mathcal{M}^*_{\{i\}}$, we now propose a matrix $\boldsymbol{B}^*_{\{i\}}$ which contains all those local assets in $\boldsymbol{B}_{\{i\}}$ which also appear in the basis of $\boldsymbol{B}_{\{j\}}$ for at least one more individual j.

$$\boldsymbol{B}^*_{\{i\}} = \Big[\big[\boldsymbol{a}_{L_M(\omega)}\big]_{\omega\in O_M}\Big]_{M\in\mathcal{M}_i^*} \tag{16}$$

The following result shows that $\boldsymbol{B}^*_{\{i\}}$ defines a basis for the relevant subspace of i's net trades.

Lemma 5. *The matrix $\boldsymbol{B}^*_{\{i\}}$ forms a basis for the payoff space X_i^*. That is*

[13] Formally, if $\boldsymbol{y}_i$ does not belong to X_i^*, then C_i^* becomes an affine space which shifts the subspace X_i^* to pass through $\boldsymbol{y}_i$.

$\text{span}(\boldsymbol{B}^*_{\{i\}}) = X_i^*$.

Lemma 5 shows that the common structure relating individual constraints, M_i, as different subsets of Q yields an intuitive characterization of spill overs. In contrast to X_i, the set of feasible trades, X_i^*, can no longer be expressed in terms of contingent claims on a single partition of Ω. Indeed, if two sets M and M' belong to $\mathcal{M}_i^*$ the agent may be able to generate payoffs which are not measurable with respect to either of the corresponding partitions. On the other hand, and in contrast to the properties of X_i, she will typically be unable to generate payoffs which condition on the joint realization of $M \cup M'$. More formally, this means that combining the M-local and M'-local assets generates a space which is greater than the union of the respective measurable trades but smaller than conditioning on the coarsest common refinement, $\text{span}(\boldsymbol{A}_M) \cup \text{span}(\boldsymbol{A}_{M'}) \subseteq \text{span}([\boldsymbol{A}_M, \boldsymbol{A}_{M'}]) \subseteq \text{span}(\boldsymbol{A}_{M \cup M'})$.

Proposition 2. *Agent i is resilient against the joint risks $M \subseteq M_i$ if and only if there exist an agent $j \neq i$ who can also condition on M, such that $M \subseteq M_j$. Moreover, agent i enjoys local completeness if and only if there exists an individual j who can trade all joint risks that i can trade, such that $M_i \subseteq M_j$.*

Indeed, i can condition on an event which fixes the joint realization of the variables in M if and only if there exists an agent j such that i and j have M as a common set of traded variables $M \subseteq M_i \cap M_j$. Again, the fact that such an agent j is sufficient to condition on M is immediate. But our results surprisingly show that this is also necessary. No matter how rich the possibilities of the collective, their trading possibilities can never compensate for the absence of an individual who can simultaneously trade all the desired risks.

Example. *Consider again the simple example underlying the applications of Section 2.2. Notice that, since agent 1 can trade all risks (i.e., $M_1 = \{1, 2, 3\}$), the non-exclusive subsets $\mathcal{M}^*_{\{1\}}$ simply correspond to those that are jointly tradable by the group of agents $I = \{2, 3\}$. As a result, the matrix $\boldsymbol{B}_{\{2,3\}}$ from* (23) *in Appendix B.5 forms the basis of the feasible payoff space of agent* 1 *such that* $X_1^* = \text{span}(\boldsymbol{B}^*_{\{1\}}) = \text{span}(\boldsymbol{B}_{\{2,3\}})$. *Intuitively,* 1 *can combine* $\{1,2\}$*-local trades with* $\{1,3\}$*-local trades to generate payoffs which are not measurable with respect to either. However, she cannot generate trades which target a particular joint realization of* $\{2,3\}$*, let alone of* $\{1,2,3\}$*, even though her own measurability constraint allows for it.*

Notice that our conditions for global and local completeness are similar. The latter (in Proposition 2) is the local analogue of the condition for global completeness (in Proposition 1). That is, rather than requiring one individual j with $M_j = Q$, it requires an individual j with $M_j \supseteq M_i$. However, the global and local conditions are logically distinct in the sense that one does not imply the other. Indeed, an individual i who completes the market globally always completes the market locally for any individual $j \neq i$ since $M_j \subseteq M_i = Q$. However, this individual i need not enjoy local completeness herself. For instance, in the three-person example in Figure 1, the individual who completes the market globally (individual 1) does not enjoy local completeness herself, since the remaining agents cannot trade $M = \{1,2,3\}$. Conversely, an economy in which every individual enjoys local completeness need not be globally complete. Appendix B.7 provides a six-person example using the network application of Section 4.

Proposition 2 confirms that individuals are typically exposed to uninsurable income risk, even though $\boldsymbol{y}_i$ does not vary with variables outside of M_i. By projecting $\boldsymbol{y}_i$ onto X_i^*, we can use the orthogonal decomposition theorem to uniquely decompose the income risk into its insurable component and an uninsurable residual:

$$\boldsymbol{y}_i = \boldsymbol{y}_i^{\parallel} + \boldsymbol{y}_i^{\perp}, \tag{17}$$

where $\boldsymbol{y}_i^{\parallel} \in X_i^*$ can be traded but the component $\boldsymbol{y}_i^{\perp} \perp \boldsymbol{x}$ is orthogonal to all feasible net-trades $\boldsymbol{x} \in X_i^*$.

While this decomposition can be performed for any instance of our economies, Appendix B.8 illustrates that $\boldsymbol{y}_i^{\perp}$ can be obtained conveniently from an orthogonal basis $\boldsymbol{W}$ of $\mathbb{R}^{|\Omega|}$ with properties similar to $\boldsymbol{B}$. That is, $\boldsymbol{W}$ contains a basis $\boldsymbol{W}_i^*$ for each X_i^* and the columns of $\boldsymbol{W}$ which do not belong to $\boldsymbol{W}_i^*$ conveniently form a basis of the orthogonal complement $(X_i^*)^\perp$, which contains $\boldsymbol{y}_i^\perp$. We can therefore appeal to our methodology of mapping constraint sets $\{M_i\}_{i\in N}$ into a selection criterion for basis vectors. While we delegate further discussion to the Appendix, our results can be illustrated in an example.

Example. *In the example from Section 2.2 with binary risk, consider agent* 1 *with income* $\boldsymbol{y}_1$*. In this case, the orthogonal basis vectors pays* 1 *if there is an even number of bad realizations in group* M*, and* -1 *otherwise. Using the orthogonal decomposition theorem, any uninsurable component must be of the form* $\boldsymbol{y}_1^\perp = (\alpha, -\alpha, -\alpha, \alpha, \beta, -\beta, -\beta, \beta)^T$ *for* $\alpha, \beta \in \mathbb{R}$*, where* $\alpha = 0$ *(resp.* $\beta = 0$*) if, conditional on* $q_1 = b$ *(*$q_1 = g$*),* $\Delta_2\Delta_3 y_1(z_1, z_2, z_3) = 0$*, where* Δ_q *takes the difference with respect to the high and low realizations of* z_q*. That is, agent* 1*'s uninsurable risk arises from her exposure to* q_2 *depending on the realization of* q_3*.*

3.5 Economic Implications

In contrast to the classical model, the possibilities of sharing risk in our setting vary for each individual $i \in N$. However, our results show that those joint risks which are not individually tradable by anyone (i.e., those outside of $\mathcal{M}^*$) cannot be protected against. In particular, no individual or collective can shift resources from the remaining events to the one in which those risks take a joint-worst realization. The economy is therefore vulnerable to events in which negative shocks accumulate.

On an individual level, only appealing to feasibility, our model disciplines the shape of every consumption function c_i. It can be decomposed additively into a set of component functions, each representing a set of variables M for which there exists no other agent who can also trade them. Concretely, denote by $\mathcal{K}_i^*$ the subset of i's non-exclusive groups of risks $\mathcal{M}_i^*$ which are maximal in terms of set inclusion, such that they are not strict subsets of another element in $\mathcal{M}_i^*$. By our results, there exist $|\mathcal{K}_i^*|$ functions k_M with $M \in \mathcal{K}_i^*$ which map from the local states of set M to the real numbers, such that,

$$c_i(z_1, .., z_Q) = \sum_{M \in \mathcal{K}_i^*} k_M(z_{q:q \in M}). \tag{18}$$

In specific examples, like our network in Figure 1, this means $c_1(z_1, z_2, z_3) = k_{\{1,2\}}(z_1, z_2) + k_{\{1,3\}}(z_1, z_3)$ using the the maximal elements $\{1, 2\}$ and $\{1, 3\}$ in the set $\mathcal{M}_1^*$.

On an economy-wide level, this means that the numerous assets represented by the columns $\mathbf{J}_N$ can be reduced without loss of generality. Without the need to restrict attention to specific classes of preferences, feasibility and measurability alone allow for a k-fund separation property (Rubinstein, 1974), where k corresponds to the number of local states in the maximal sets $K \subseteq Q$ observed by at least two individuals. That is, considering only the sets in $M \in \{\mathcal{K}_i^*\}_{i \in N}$ which are maximal by inclusion, it is without loss of generality to assume that only those M-local assets can be traded. For instance, in the networks application, if no two neighbors have another neighbor in common (like in star or ring networks), it is without loss of generality to impose that only assets which pay on the local states of connected pairs are traded.

We can distinguish two potential sources of social costs in this environment. First, for each risk q, there is the direct friction coming from measurability alone, if some members of society cannot trade them. Second, as seen in (18) above, even if an individual is able to trade a set of risks $q \in M$, she is typically constrained in how she can react to their joint realization. The latter effect is relevant under expected utility, except for the special case of constant absolute risk aversion (CARA), as shown in Appendix B.6. In a companion paper (Gierlinger and Milán, 2021) we show that under CARA all risks are optimally shared equally among the individuals who can trade them. In this knife-edge case, equation (18) is not restrictive, and the social cost of local incompleteness vanishes.[14]

[14] The optimal contracts presented in Ambrus et al. (2021) in an i.i.d. normal environment

4 An Application to Risk-Sharing Networks

We now consider what our results imply in the context of Application 3 in Section 2.2, where individual income risks are mutually shared among fixed subsets of individuals. The distinguishing feature of this application is that the sets of risks Q and the set of individuals N coincide. Agent i's neighborhood $N_i \subseteq N$ describes the set of individual's who are sufficiently *close* (in a sense that is relevant for insurance) for her to condition on their income. Therefore, we have $N = Q$ and $M_i = N_i \subseteq N$. Taken together, these neighborhoods generate a *directed risk-sharing network*: a link from i to j implies that i can condition her payoffs to realized values of y_j.[15]

Every such risk-sharing network can be described by a matrix $\boldsymbol{G}$, where $\boldsymbol{G}_{ij} = 1$ if i can condition on the income of $j \neq i$. Following the convention, we set $\boldsymbol{G}_{ii} = 0$. The neighborhood of i (including i) is defined by $N_i = \{j \in N : \boldsymbol{G}_{ij} = 1\} \cup \{i\}$ and $d_i := |N_i|$ is the degree of i (also including i). A path of length ℓ between i and j is a sequence of terms $\boldsymbol{G}_{i,k_1}\boldsymbol{G}_{k_1,k_2}...\boldsymbol{G}_{k_{\ell-1},j} = 1$. Without loss of generality we assume that every pair i and j are indirectly connected through a path in $\boldsymbol{G}$. In some circumstances, i is linked with j if and only if j is linked with i. In these cases we say that the network is undirected, and $\boldsymbol{G}$ is a symmetric matrix.[16] Translating our earlier results to this application, we obtain the following necessary and sufficient conditions on

are consistent with this result.

[15]Risk-sharing networks have been studied widely in numerous contexts, but particularly in the field of economic development. Most existing theories analyze how enforcement is sustained by collateralizing existing relationships (for instance Bramoullé and Kranton (2007); Bloch et al. (2008); Ambrus et al. (2014); Ambrus and Elliott (2021)). These models typically assume that consumption allocations can respond to the economy's full set of income realizations. Instead, the network here summarizes the collection of individual trading restrictions. We do not take a stance on which friction may be keeping two households from trading each other's income risk. Instead, we describe how the resulting network structure shapes the space of feasible payoffs.

[16]Notice that allocations would be further constrained if any bilateral transfer had to be measurable with respect to the pair's joint information partition, as in Ambrus et al. (2021). Apart from tractability, assuming measurability of consumption accommodates centralized institutions such as market exchanges. All our propositions could be suitably adapted under measurability of transfers. The results coincide for *square-free* networks $\boldsymbol{G}$.

network structure that guarantee complete markets in this context,

Proposition 1'. *A risk-sharing network $\boldsymbol{G}$ cannot collectively insure against a joint risk of group $M \subseteq N$, unless there exists a member $i \in N$ who is connected to all of them ($M \subseteq N_i$). Moreover, the network is globally complete if and only if there exists at least one agent who is connected to everyone: $C_N = \mathbb{R}^{|\Omega|}$ if and only if $\exists i \in N$ such that $N_i = N$.*

What characteristics of the risk-sharing network determine the set of joint income shocks that households can trade against? Proposition 2 above can now be reinterpreted as a statement on individuals' resilience to local events.

Proposition 2'. *In any feasible allocation, an agent's consumption may only respond to the joint incomes of those sets of her neighbors who have another neighbor in common. Therefore, agent i enjoys local completeness if and only if there exists another individual j who knows everyone that i knows: $N_j \supseteq N_i$.*

An advantage of this approach is that we can now quantify the extent of households' resilience to joint income shocks directly from observed network structures. In Appendix B.1, we develop a resilience measure $r_i(\boldsymbol{G}) \in [0, 1]$ that captures the share of neighbors' joint shocks households i can insure against.[17] We describe properties of this measure in the context of well-studied families of networks, and we compute $r_i(\boldsymbol{G})$ for each household in a risk-sharing network in rural Malawi.

5 Conclusion

We studied the limits to insurance when some economic risks may only be shared among a subset of the population. We showed that the economy's ability to protect against joint worst scenarios is limited by the least constrained individual. Similarly, due to spill overs effects, an individual income risk may

[17]This measure is reminiscent of the support measure in Jackson et al. (2012) with important differences that we detail in Section B.1

remain uninsurable unless there exists a person who can trade every risk that she can.

In a networks application, we showed that the collection of local instruments (no matter how numerous) cannot replicate complete markets unless at least one individual knows everyone else in the network. Using data from Malawi, we find that individuals are rarely resilient against the joint income shocks of their neighbors. The prevalence of exclusive neighbors suggests an environment in which many risks have to be shared independently, as opposed to pooling them for diversification purposes.

General patterns in constraints, in line with cited applications, translate into specific cross-partial derivatives in consumption functions to zero. This can be tested empirically by checking the co-movements of consumption with the fundamentals of the economy. For instance, in the networks application, given a rich enough panel of income and consumption observations and with a reliable measurement of the underlying network structure, the model's implications, as summarized in equation (18), can be tested directly by regressing consumption on all income levels and their interactions, and checking the statistical significance of the interaction terms that according to the model should be zero.[18] We leave this exercise for future work.

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[18]Provided the correlation of income is not too strong, even if the network is not explicitly measured, it could be constructed from observed dependencies in the consumption and income data. By our definition, individual i's consumption movements can only be explained by fluctuations of j's income if the two are linked.

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A Appendix - Proofs

Lemma 1. *Given initial endowment $\boldsymbol{y}_i$, the vector $\boldsymbol{c}_i$ satisfies i's measurability constraints* (4) *if and only if the net-trade $\boldsymbol{x}_i = (\boldsymbol{c}_i - \boldsymbol{y}_i)$ lies in the column span of $\boldsymbol{A}_{M_i}$, defined in* (3).

Proof. By definition of $\boldsymbol{A}_{M_i}$, its column span consists of all points $\boldsymbol{x} \in \mathbb{R}^{|\Omega|}$ which can be expressed as a linear combination of i 's local assets, defined in (2). To show necessity, we need that any vector $\boldsymbol{\alpha} \in \mathbb{R}^{|\mathcal{L}_{M_i}|}$, results in a consumption vector $\boldsymbol{c}_i = \boldsymbol{y}_i + \boldsymbol{A}_{M_i}\boldsymbol{\alpha}$ that satisfies measurability (4). Fixing any state ω, by definition of $\boldsymbol{A}_{M_i}$, we have $c_i(\omega) = y_i(\omega) + \alpha_{L_{M_i}(\omega)}$, where $\alpha_{L_{M_i}(\omega)}$ is the coordinate assigned to asset $\boldsymbol{a}_{L_{M_i}(\omega)}$. Moreover, for any other $\omega' \in L_{M_i}(\omega)$ (i.e. that assigns the same local state as ω), we have that both endowment $y_i(\omega') = y_i(\omega)$ and the payoff $\alpha_{L_{M_i}(\omega')} = \alpha_{L_{M_i}(\omega)}$ remain unchanged, proving that $\boldsymbol{c}_i = \boldsymbol{y}_i + \boldsymbol{A}_{M_i}\boldsymbol{\alpha}$ satisfies (4). Finally, to show sufficiency, we need to show that, for any measurable $\boldsymbol{c}_i$, a portfolio vector α exists whose payoff replicates the net-trade $\boldsymbol{x}_i = (\boldsymbol{c}_i - \boldsymbol{y}_i) = \boldsymbol{A}_{M_i}\boldsymbol{\alpha}$. Setting weight $\alpha_{L_{M_i}(\omega)} = c_i(\omega) - y_i(\omega)$ equal to the net-trade in local state $L_{M_i}(\omega)$ completes the proof. ■

Lemma 2. *The column vectors in $\boldsymbol{B}$ form a basis of $\mathbb{R}^{|\Omega|}$.*

Proof. Thanks to Lemma 6 in Appendix B.4, what remains to be shown is that all $|\Omega|$ columns of $\boldsymbol{B}$ are linearly independent. That is, for any $|\Omega| \times 1$ vector $\boldsymbol{\alpha}$, and for $\boldsymbol{x} = \boldsymbol{B}\boldsymbol{\alpha}$, we have $\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{\alpha} = \boldsymbol{0}$. Consider any arbitrary state ω and the ω-th row of the basis $\boldsymbol{B}$. By the second part of Lemma 6, ω belongs to a unique set O_M for $M \subseteq Q$. For convenience, from now on, we keep track of the relevant set M by referring to our state by ω_M. Since $\boldsymbol{B}$ consists of local assets, for a column to pay in ω_M, it must be a K-local assets $\boldsymbol{a}_{L_K(\omega_M)}$ which agrees with ω_M for the coordinates in K. However, since $\omega_M \in O_M$, it assigns the worst realization for all $Q \setminus M$ coordinates. Hence, whenever $K \nsubseteq M$, the candidate column $\boldsymbol{a}_{L_K(\omega_M)}$ cannot belong to $\boldsymbol{B}$ because it assigns the worst realization for all coordinates in $K \setminus M$. In contrast, for any remaining set

$K \subseteq M$, the set O_K contains a state $\omega_K \in O_K \cap L_K(\omega_M)$ which agrees with ω_M for all K (while all coordinates outside K take the worst value, by definition of O_K). Therefore, the ω_M-th row of $\boldsymbol{B}$ has 0 in all columns except for assigning a 1 to every K-local asset that agrees with ω_M on K and where K must be a weak subset of M. As a result, the ω_M-th coordinate of the multiplication $\boldsymbol{x} = \boldsymbol{B\alpha}$ equals

$$x_{\omega_M} = \alpha_{L_M(\omega_M)} + \sum_{K \subset M} \alpha_{L_K(\omega_K)} \mathbf{1}_{\omega_K \in \{L_K(\omega_M) \cap O_K\}},$$

where $\alpha_{L_M(\omega)}$ is the weight that $\boldsymbol{\alpha}$ assigns to the column $\boldsymbol{a}_{L_M(\omega)}$ in $\boldsymbol{B}$. We now show that $\boldsymbol{x} = \mathbf{0} \Rightarrow \boldsymbol{\alpha} = \mathbf{0}$ using the above definition and proceeding by induction, starting from $x_{\omega_\emptyset}$ – i.e., starting from the set $M = \emptyset$ with no strict subset. In fact, given our definition for O_M defined in (9), we have that $\omega_\emptyset = \omega_0$ corresponds to the reference state. Notice that for all ω_M, the reference state, ω_0, satisfies $\{\omega_0\} = L_\emptyset(\omega_M) \cap O_\emptyset$. Moreover, since only the column $\boldsymbol{a}_{L_\emptyset(\omega_0)}$ pays on the ω_0-coordinate, we get $x_{\omega_0} = \alpha_{L_\emptyset(\omega_0)} = 0$. Similarly, for the next bigger subset $K = \{q\}_{q \in M}$, only the column $\boldsymbol{a}_{L_{\{q\}}(\omega_{\{q\}}}$ and $\boldsymbol{a}_{L_\emptyset(\omega_0)}$ can pay in the state $\omega_{\{q\}} \in O_{\{q\}}$. And since $\alpha_{L_\emptyset(\omega_0)} = 0$, it follows that also $\alpha_{L_{\{q\}}(\omega_{\{i\}}} = 0$. Finally, since M is the maximal element in this partially ordered set of variables $K \subseteq M$, proceeding analogously yields that $x_{\omega_M} = \alpha_{L_M(\omega_M)} + \sum_{K \subset M} \left(\mathbf{1}_{\{\omega_K \in \{L_K(\omega_M) \cap O_K\}\}}\right) \alpha_{L_K(\omega_K)} = \alpha_{L_M(\omega_M)} = 0$, where the second equality follows from successive application of the previous argument which fixes the coefficients on the local assets for all mentioned sets to 0. Therefore, for all local assets $\alpha_{L_M(\omega_M)}$ in $\boldsymbol{B}$, whenever $\boldsymbol{x} = 0$, the coefficient $\alpha_{L_M(\omega_M)}$ must be 0. ■

Lemma 3. *The local assets in $\boldsymbol{B}_{\{i\}}$ form a basis of i's payoff space, $C_i = \operatorname{span}(\boldsymbol{A}_{M_i}) = \operatorname{span}(\boldsymbol{B}_{\{i\}})$.*

Proof. We know that the columns of $\boldsymbol{B}$, and therefore $\boldsymbol{B}_{\{i\}}$, are linearly independent by Lemma 2. We now need to show that all local assets in $\boldsymbol{A}_{M_i}$ can be expressed as a linear combination of local assets in $\boldsymbol{B}_{\{i\}}$ and vice versa. By definition of $\boldsymbol{A}_{M_i}$ in (3), each column represents a unit claim conditional on a joint realization of M_i. The following Lemma will be useful for the remainder of the proof.

Lemma. *For any $M \subseteq M_i$, an asset $\boldsymbol{a}_{L_M(\omega)}$ belongs to $\boldsymbol{B}_{\{i\}}$ if and only if $L_M(\omega)$ has a nonempty intersection with O_M.*

Proof. If $L_M(\omega) \cap O_M \neq \emptyset$, by its definition, $\boldsymbol{B}_{\{i\}}$ contains column $\boldsymbol{a}_{L_M(\omega)}$ whenever $M \subseteq M_i$. Conversely, if $\boldsymbol{a}_{L_M(\omega)}$ belongs to $\boldsymbol{B}_{\{i\}}$, by definition, a state $\widetilde{\omega} \in L_M(\omega)$ must exist such that $z_q(\widetilde{\omega}) \neq z_q(\omega_0)$ for all $q \in M$ and $z_k(\widetilde{\omega}) = z_k(\omega_0)$ for all $k \in Q \setminus M$. Therefore, $\widetilde{\omega} \in L_M(\omega) \cap O_M \neq \emptyset$. ■

Fix an arbitrary state $\widetilde{\omega} \in \Omega$. We now show that the corresponding local asset $\boldsymbol{a}_{L_{M_i}(\widetilde{\omega})}$ in $\boldsymbol{A}_{M_i}$ can be expressed as a linear combination of columns in $\boldsymbol{B}_{\{i\}}$. Generically, $\widetilde{\omega}$ has $k \geq 0$ risks in M_i which the worst outcome. Consider first the case with $k = 0$ by taking a state $\widetilde{\omega}^0$ that does not assign the worst outcome to any of the risks in M_i. The corresponding local state $L_{M_i}(\widetilde{\omega}^0)$ must contain an element $\omega' \in L_{M_i}(\widetilde{\omega}^0)$ which fixes the outcome of all risks outside of M_i to their worst level: $z_r(\widetilde{\omega}) = z_r(\omega_0)$ for all $r \notin M_i$. By definition of O_{M_i} in (9), $\widetilde{\omega} \in O_{M_i}$. Therefore, by Lemma 3, the local asset for $L_{M_i}(\widetilde{\omega}^0)$ must be represented as a column in both $\boldsymbol{A}_{M_i}$ and $\boldsymbol{B}_{\{i\}}$.

Next, consider the case $k = 1$ where a state $\widetilde{\omega}^1$ assigns the worst outcome to exactly one risk $r \in M_i$. Since O_{M_i} requires that no risk in M_i generate its worst outcome, none of the states in $L_{M_i}(\widetilde{\omega}^1)$ belongs to O_{M_i}, so the column $\boldsymbol{a}_{L_{M_i}(\widetilde{\omega}^1)}$ does not appear directly in $\boldsymbol{B}_{\{i\}}$. Still, this column can be constructed from a linear combination of columns of $\boldsymbol{B}_{\{i\}}$. To see this, note that the coarser event $L_{M_i \setminus \{r\}}(\widetilde{\omega}^1)$ corresponds to the union of all $L_{M_i}(\omega)$ across all ω that fix the outcome of $M_i \setminus \{r\}$ to agree with $\widetilde{\omega}^1$. We now show that by subtracting appropriate events $L_{M_i}(\omega)$ from this coarser event we can construct $L_{M_i}(\widetilde{\omega}^1)$, and that all events used to construct $L_{M_i}(\widetilde{\omega}^1)$ are represented in $\boldsymbol{B}_i$. To see this, consider any state that agrees with $\widetilde{\omega}^1$ for $M_i \setminus \{r\}$ while assigning $z_r \neq z_r(\omega_0)$ to r. Since all outcomes in M_i are different from ω_0 the set O_{M_i} must contain one such state. Therefore the states in $O_{M_i} \cap L_{M_i \setminus \{j\}}(\widetilde{\omega}^1)$

select the appropriate $L_{M_i}(\omega)$ used to construct $L_{M_i}(\widetilde{\omega}^1)$ so that we obtain the following expression:

$$L_{M_i}(\widetilde{\omega}^1) = L_{M_i\setminus\{r\}}(\widetilde{\omega}^1) \setminus \cup_{\omega\in O_{M_i}\cap L_{M_i\setminus\{r\}}(\widetilde{\omega}^1)} L_{M_i}(\omega). \tag{19}$$

Finally, we show that the columns corresponding to the local states on the right-hand side of the above equation must be represented in $\boldsymbol{B}_{\{i\}}$. First, $L_{M_i\setminus\{r\}}(\widetilde{\omega}^1)$ has a nonempty intersection with $O_{M_i\setminus\{r\}}$ by definition of (9). Moreover, the local states $L_{M_i}(\omega)$ for $\omega \in O_{M_i} \cap L_{M_i\setminus r}(\widetilde{\omega}^1)$ have a nonempty intersection with O_{M_i}. Therefore, by the arguments proved in Lemma 3, each of these local states must be represented in $\boldsymbol{B}_{\{i\}}$. Specifically, the column $\boldsymbol{a}_{L_{M_i}(\widetilde{\omega}^1)}$ of $\boldsymbol{A}_{M_i}$ is a linear combination of columns in $\boldsymbol{B}_{\{i\}}$ as follows:

$$\boldsymbol{a}_{L_{M_i}(\widetilde{\omega}^1)} = \boldsymbol{a}_{L_{M_i\setminus r}(\widetilde{\omega})} - \sum_{\omega\in O_{M_i}\cap L_{M_i\setminus r}(\widetilde{\omega})} \boldsymbol{a}_{L_{M_i}(\omega)}.$$

To show the result for any $k \geq 2$, notice that any $\widetilde{\omega} \in \Omega$ can be expressed as an element of a sequence of states $\widetilde{\omega}^0, \widetilde{\omega}^1, \widetilde{\omega}^2, ..., \widetilde{\omega}^{|M_i|}$, indexed by the number of risks in M_i which generate their worst outcome, where $\widetilde{\omega}^{k+1}$ relates to its predecessor $\widetilde{\omega}^k$ by one additional risk taking its worst outcome. We now proceed by induction on this sequence to show that the local asset for any element must be a linear combination of columns in $\boldsymbol{B}_{\{i\}}$.

Let $K \subseteq M_i$ be a set of k risks with worst outcomes. The appropriate $L_{M_i}(\omega)$ events to subtract from $L_{M_i\setminus K}(\widetilde{\omega}^k)$ now must not only account for local states in which no risks in K assign the worst outcome, but also those where only a strict subset $M \subset K$ assign the worst outcome. Proceeding as before for the case $k = 1$, we obtain the following generalization of (19):

$$L_{M_i}(\widetilde{\omega}^k) = L_{M_i\setminus K}(\widetilde{\omega}^k) \setminus \cup_{M\subset K} \left(\cup_{\omega\in O_{M_i\setminus M}\cap L_{M_i\setminus K}(\widetilde{\omega}^k)} L_{M_i}(\omega)\right).$$

The first term $L_{M_i\setminus M}$ has a nonempty intersection with $O_{M_i\setminus M}$ and therefore corresponds to a local state with a column in $\boldsymbol{B}_{\{i\}}$. The remaining $L_{M_i}(\omega)$

terms in the parentheses correspond to local states where $M \subset K$ assign the worst outcome. Assume by induction that, whenever $M \subset K$ assign the worst outcome, $\boldsymbol{a}_{L_{M_i}(\omega)}$ can be expressed from columns in $\boldsymbol{B}_{\{i\}}$. Then, by the above equation and the arguments proved in Lemma 3, $L_{M_i}(\widetilde{\omega}^k)$ can also be expressed from local states that have a corresponding column in $\boldsymbol{B}_{\{i\}}$. In other words, we have shown that, provided that the result holds when any strict subset M of risks in K assign the worst outcome, then the result must also hold whenever all risks in $K \subset M_i$ assign the worst outcome. This proves the inductive step, and together with the result above for $k = 0$ and $k = 1$, we obtain the result for any $k \geq 0$. Converting the local states to their local assets, we can therefore express $\boldsymbol{a}_{L_{M_i}(\omega)}$ of $\boldsymbol{A}_{M_i}$ as follows:

$$\boldsymbol{a}_{L_{M_i}(\widetilde{\omega})} = \boldsymbol{a}_{L_{M_i \setminus K}(\widetilde{\omega})} - \sum_{M \subset K} \left(\sum_{\omega \in O_{M_i \setminus M} \cap L_{M_i \setminus K}(\widetilde{\omega})} \boldsymbol{a}_{L_{M_i}(\omega)} \right).$$

Applying our induction argument for local states analogously to the local assets $\boldsymbol{a}_{L_{M_i}(\omega)}$ for any $\omega \in O_{M_i \setminus M} \cap L_{M_i \setminus K}$, proves that each of them must be a linear combination of the columns in $\boldsymbol{B}_{\{i\}}$ induced by local states $M_i \setminus P$ with $P \subset M$.

Finally, we need to show that each column of $\boldsymbol{B}_{\{i\}}$ can be expressed as a linear combination of the columns in $\boldsymbol{A}_{M_i}$. Notice that, by definition of $\mathcal{M}_{\{i\}}$, for each column $\boldsymbol{a}_{L_M}(\widetilde{\omega})$ in $\boldsymbol{B}_{\{i\}}$, since i can generate any arbitrary payoff as long as it only conditions on the outcomes of risks in M_i, the column $\boldsymbol{a}_{L_M}(\widetilde{\omega})$ must belong to $\mathrm{span}(\boldsymbol{A}_{M_i})$. ■

Lemma 4. *For any group $I \subseteq N$, the assets in $\boldsymbol{B}_I$ form a basis of the payoff space,* $\mathrm{span}(\boldsymbol{J}_I)$.

Proof. The matrix $\boldsymbol{B}_I$ contains every column of $\boldsymbol{B}_{\{i\}}$ for any $i \in M$. Moreover, each column in $\boldsymbol{J}_I$ must belong to $\boldsymbol{A}_{M_i}$ for some $i \in I$. By Lemma 3, each column in any $\boldsymbol{A}_{M_i}$ must belong to $\mathrm{span}(\boldsymbol{B}_{\{i\}})$. Therefore, any column in $\boldsymbol{J}_I$ must belong to $\mathrm{span}(\boldsymbol{B}_I)$. Finally, each column in $\boldsymbol{B}_I$ must belong to $\boldsymbol{B}_{\{i\}}$

for some $i \in I$. Each column in $\boldsymbol{B}_{\{i\}}$ must belong to $\text{span}(\boldsymbol{A}_{M_i})$ by Lemma 3. Therefore, any column in $\boldsymbol{B}_{\{i\}}$ must belong to $\text{span}(\boldsymbol{J}_I)$. Consequently, every column in $\boldsymbol{B}_I$ must belong to $\text{span}(\boldsymbol{J}_I)$. ∎

Proposition 1. *For each joint risk $M \subseteq Q$, unless there exists a member $i \in N$ who can trade it ($M \subseteq M_i$) the group N cannot trade it collectively either. An economy is globally complete if and only if there exists at least one agent who can trade all risks. In other words, $C_N = \mathbb{R}^{|\Omega|}$ if and only if $\exists i \in N$ such that $M_i = Q$.*

Proof. By Lemma 4, it suffices to show that $\text{span}(\boldsymbol{B}_N) = \text{span}(\boldsymbol{B}) = \mathbb{R}^{|\Omega|}$ if and only if there exists an $i \in N$ such that $M_i = Q$. Recall that, by the definitions of the bases, $\boldsymbol{B}_N$ coincides with $\boldsymbol{B}$ if and only if the set $\mathcal{M}_N$ coincides with the power set 2^Q. If there exists an $i \in N$ such that $M_i = Q$, then, $Q \subseteq M_i$, and $\mathcal{M}_N = 2^Q$. Conversely, if $\mathcal{M}_N \subset 2^Q$, then there must exists a subset $M \subset Q$ which is not contained in any M_i. This implies that no $i \in N$ can trade all members of $M \subset Q$. ∎

Lemma 5. *The matrix $\boldsymbol{B}^*_{\{i\}}$ forms a basis for the payoff space X^*_i. That is* $\text{span}(\boldsymbol{B}^*_{\{i\}}) = X^*_i$.

Proof. We need to show that $\text{span}(\boldsymbol{A}_i) \cap \text{span}(\boldsymbol{J}_{N\setminus i}) = \text{span}(\boldsymbol{B}^*_{\{i\}})$. The proof of Lemma 4 shows $\text{span}(\boldsymbol{B}_I) = \text{span}(\boldsymbol{J}_I)$ for any set of $I \subseteq N$. Therefore, by setting $I = \{i\}$, we obtain $\text{span}(\boldsymbol{B}_{\{i\}}) = \text{span}(\boldsymbol{A}_{M_i})$ and setting $I = N \setminus \{i\}$, we obtain $\text{span}(\boldsymbol{B}_{N\setminus\{i\}}) = \text{span}(\boldsymbol{J}_{N\setminus\{i\}})$. Therefore, we have $\text{span}(\boldsymbol{A}_{M_i}) \cap \text{span}(\boldsymbol{J}_{N\setminus\{i\}}) = \text{span}(\boldsymbol{B}_{\{i\}}) \cap \text{span}(\boldsymbol{B}_{N\setminus\{i\}})$. Since $\mathcal{M}^*_i = \mathcal{M}_{\{i\}} \cap \mathcal{M}_{N\setminus\{i\}}$, the matrix $\boldsymbol{B}^*_{\{i\}}$ contains exactly those columns which simultaneously belong to $\boldsymbol{B}_{N\setminus\{i\}}$ and $\boldsymbol{B}_{\{i\}}$. This implies $\text{span}(\boldsymbol{B}^*_{\{i\}}) \subseteq \text{span}(\boldsymbol{B}_{\{i\}}) \cap \text{span}(\boldsymbol{B}_{N\setminus\{i\}})$. What remains to be shown is that any $\boldsymbol{x} \in \text{span}(\boldsymbol{B}_{\{i\}}) \cap \text{span}(\boldsymbol{B}_{N\setminus\{i\}})$ must also belong to $\text{span}(\boldsymbol{B}^*_{\{i\}})$. Since $\boldsymbol{B}$ is a basis of $\mathbb{R}^{|\Omega|}$

and $\boldsymbol{x} \in \big(\text{span}(\boldsymbol{B}_{\{i\}}) \cap \text{span}(\boldsymbol{B}_{N\setminus\{i\}})\big) \subseteq \mathbb{R}^{|\Omega|}$, there is a unique linear combination of $\boldsymbol{B}$'s columns that generate $\boldsymbol{x}$. Moreover, since $\boldsymbol{x} \in \text{span}(\boldsymbol{B}_{N\setminus\{i\}})$, this linear combination can only put positive weight on columns in $\boldsymbol{B}_{N\setminus\{i\}}$. Similarly, since $\boldsymbol{x} \in \text{span}(\boldsymbol{B}_{\{i\}})$, the linear combination can only put positive weight on columns in $\boldsymbol{B}_{\{i\}}$. Since $\boldsymbol{B}_{N\setminus\{i\}}$ and $\boldsymbol{B}_{\{i\}}$ only contain columns from $\boldsymbol{B}$, the unique linear combination of columns in $\boldsymbol{B}$ that generate $\boldsymbol{x} \in \text{span}(\boldsymbol{B}_{\{i\}}) \cap \text{span}(\boldsymbol{B}_{N\setminus\{i\}})$ can only put positive weight on columns in $\boldsymbol{B}^*_{\{i\}}$ such that $\boldsymbol{x} \in \text{span}(\boldsymbol{B}^*_{\{i\}})$. ■

Proposition 2. *Agent i is resilient against the joint risks $M \subseteq M_i$ if and only if there exist an agent $j \neq i$ who can also condition on M, such that $M \subseteq M_j$. In particular, she enjoys local completeness if and only if there exists an individual j who can trade all joint risks that i can trade, such that $M_i \subseteq M_j$.*

Proof. We need to show $X_i^* = \text{span}(\boldsymbol{B}^*_{\{i\}}) \supseteq \text{span}(\boldsymbol{A}_{\{M\}})$ if and only if there exists a j with $M \subseteq M_i \cap M_j$. To show necessity, note that the presence of j with $M_i \cap M_j \supseteq M$ implies that the set M belongs to $\mathcal{M}_i^*$, since $M \subseteq \mathcal{M}_{\{i\}} \cap \mathcal{M}_{\{j\}} \subseteq \mathcal{M}_i^*$, where the last inclusion uses the definition (15). Moreover, Lemma 4 showed that $\text{span}(\boldsymbol{A}_{\{M\}}) = \text{span}(\boldsymbol{B}_{\{k\}}$ for an agent k with $M_k = M$. By definition of $\boldsymbol{B}^*_{\{i\}}$, all columns of the matrix $\boldsymbol{B}_{\{k\}}$ belong to $\boldsymbol{B}^*_{\{i\}}$. Therefore, $X_i^* = \text{span}(\boldsymbol{B}^*_{\{i\}}) \supseteq \text{span}(\boldsymbol{B}_{\{k\}}) = \text{span}(\boldsymbol{A}_{\{M\}})$. To show necessity, note that for $M \in \mathcal{M}^*_{\{i\}}$, there must exist a $j \in N$ such that $M \in \mathcal{M}_{\{j\}}$, which completes the proof. ■

B Supplementary Appendix *For Online Publication Only*

B.1 A Network Measure of Resilience

For each network $\boldsymbol{G}$, we can measure how resilient each individual i is to the joint shocks of her neighbors. Note that her set of non-exclusive groups of neighbors $\mathcal{M}_i^*$ lies in the interval $[2d_i, 2^{d_i}]$, where the lower bound is obtained when no two neighbors of i have another neighbor in common, and where the upper bound corresponds to local completeness. If $d_i \leq 2$, then i's interval shrinks to a point and her local completeness is trivially guaranteed.

Accounting for these bounds, we propose a normalized measure of i's network centrality which we call her *measure of resilience*,

$$r_i(\boldsymbol{G}) = \begin{cases} (|\mathcal{M}_i^*| - 2d_i)/(2^{d_i} - 2d_i) & \text{if } d_i > 2 \\ 1 & \text{otherwise.} \end{cases} \tag{20}$$

This measure takes values between 0 and 1 and it represents the fraction of non-exclusive subsets of neighbors out of the theoretically possible range.[19] An economy with universal local completeness satisfies $r(i) = 1$ for all i.

This measure is related to notion of "supported links" in Jackson et al. (2012) with two important differences. First, rather than checking at the level of a link if both members share a neighbor, we consider neighborhood subsets of all sizes (whose members need not be connected) and check if all of them have another neighbor in common. Second, this additional common neighbor need not belong to the group. Finally, our measure assigns to an individual a centrality based on the proportion of neighborhood subsets that are "supported".

To fix ideas, consider what we know about the support centrality of individ-

[19] Notice that $r(i)$ does not discriminate on the size of subsets.

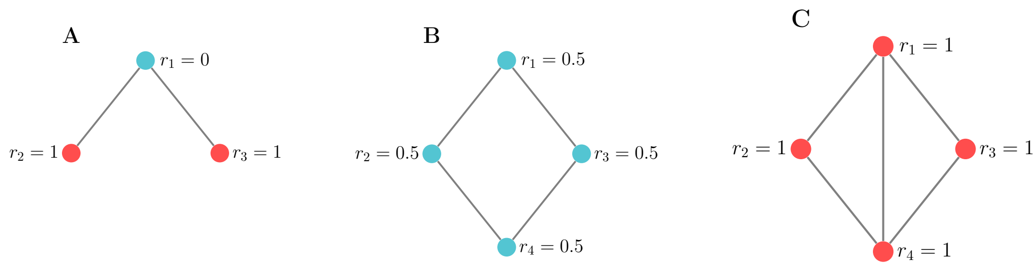


Figure 2: The Support Centrality Measure r_i for Three Simple Networks. Red nodes represent locally complete individuals (i.e., with $r_i = 1$) and blue nodes are not locally complete (i.e. $r_i < 1$).

uals for some simple networks. Figure 2 provides the centrality measure for each individual in three different networks. Notice that agent 1 gets closer to completeness – as captured by her support centrality measure – as we move from network A to network B. This happens because an additional subset of 1's neighborhood is observed by another agent (agent 4). Agent 1 achieves local completeness only in network C (i.e., $r(1) = 1$).

Moreover, beyond particular examples, some well-known classes of networks have a link structure which is sufficiently regular to determine the distribution of r_i, independent of their size. For instance, consider the set of core-periphery networks.[20] The star network in Figure 2 (Panel A) is a special case of such a structure in which the core contains a single node. Notice that a periphery member's neighborhood is always nested by the neighborhood of some core member. Therefore, all periphery members have an $r(i)$ equal to 1 (they enjoy local completeness). Conversely, as soon as $n > 2$, all core members have an $r(j) < 1$. The precise value in the core depends on its size and the number of peripheral nodes attached to each j. For $|g_2| = 1$ we already know that $r(j) = 0$: no two neighbors of j have a neighbor in common. More generally, we provide the following observation.

[20] A core-periphery network partitions the population N into two groups g_1 and g_2, with $n > |g_2|$ and $g_1 \cup g_2 = N$. Individuals in g_1 constitute the *periphery* of the network. They each have a single link with one node in g_2. Individuals in the set g_2 constitute the *core* of the network. They are fully linked with each other and with a subset of nodes in g_1.

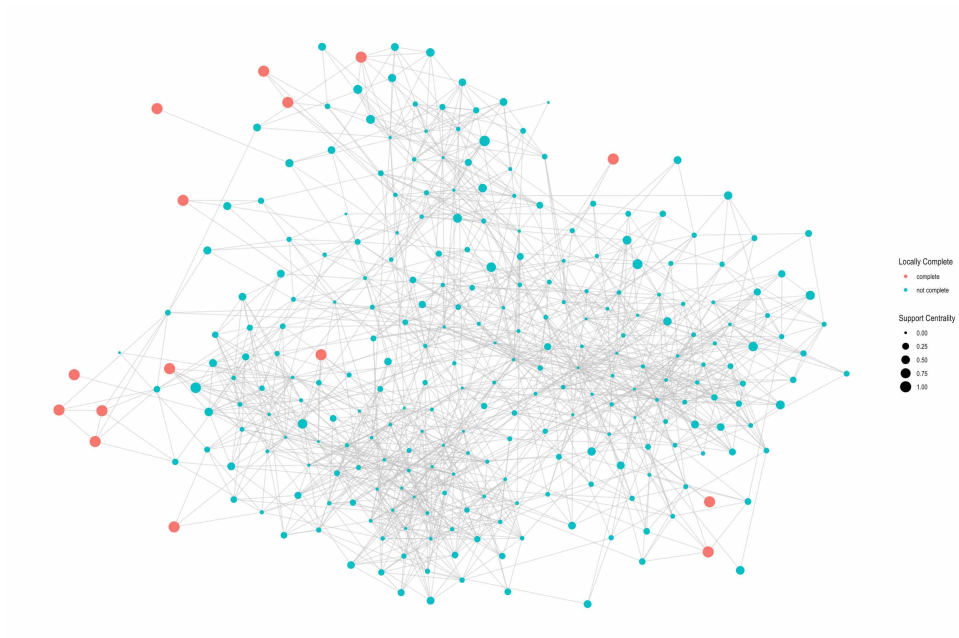


Figure 3: Resilience Centrality and Local Completeness on the Malawi Village Network

Observation 1. *For any network $\boldsymbol{G}$ with a core-periphery structure defined by groups g_1 (the periphery) and g_2 (the core), $r(i) = 1$ for all $i \in g_1$, and for any $j \in g_2$ with degree d_j we have $r(j) = (2^{|g_2|} - 2|g_2|)/(2^{d_j} - 2d_j)$.*[21]

In core-periphery networks with a relatively large periphery, the majority of the population can be locally complete. At the other extreme, some network structures have $r(i) = 0$ for all individuals. This is the case for ring networks of 5 or more members.[22] In such cases, an individual's neighbors cannot possibly share a connection. This leads to the following observation.

Observation 2. *For $n > 4$, any network $\boldsymbol{G}$ with a ring structure has $r(i) = 0$ for all $i \in N$.*

[21] This follows from the observation that for any j in the core with degree d_j, $\mathcal{M}_j^* = 2^{|g_2|} + 2(d_j - |g_2|)$

[22] In contrast, the 4-ring network in Panel B of Figure 2, the maximum distance between any two nodes is 2, so all pairs have friends in common.

For arbitrary network structures, general results on the distribution of $r(i)$ are difficult to obtain. To answer this question, we develop a fast algorithm which computes, for any network, all bilateral neighborhood intersections and tallies all unique subsets of those intersections for every household i. We can assign an $r(i)$ measure to each household: the results are visualized on the graph in Figure 3. We find that 15 out of the 270 households that make up this village (i.e. 5.5%) are locally complete. This is a striking finding: it suggests that local insurance restrictions spill over considerably in real world networks, such that the vast majority of households are affected indirectly by the insurance restrictions of their neighbors. Moreover, our $r(i)$ measure allows us to determine the extent of these spill-over effects.

While those households with largest degree are never locally complete, the relationship between degree and our support measure $r(i)$ is not strong, with a correlation of only -0.6. Figure 5 presents several scatterplots that relate $r(i)$ to well-known centrality measures in the literature. For both the betweenness and eigenvector centrality, the correlation is also negative, but even weaker than for degree (of about -0.45 for both measures).[23] As expected, our support centrality $r(i)$ correlates positively with the clustering centrality (correlation coefficient of about 0.75). Clustering computes, for each i, the proportion of pairs of neighbors that are jointly connected. All else equal, a larger clustering coefficient therefore implies that more subsets (pairs at least) of N_i are jointly observed, thereby increasing $r(i)$. However, $r(i)$ can still be large for households with low clustering. For instance, notice that $r(1)$ increases from network A to network B in Figure 2, while 1's clustering coefficient remains constant and equal to 0.[24]

The distribution and density function of $r(i)$ are plotted in Figure 4. The mean support in the village is $\bar{r} = 0.192$ and the median is $r_{50} = 0.089$. The distribution is bimodal and skewed to the left, with most of its mass centered at an $r(i)$ of about 0.1. Around 90% of the population lack the necessary

[23] For more details on what each of these centralities captures see Jackson (2010).

[24] Similar examples can be found which do not require the presence of individuals outside of N_1.

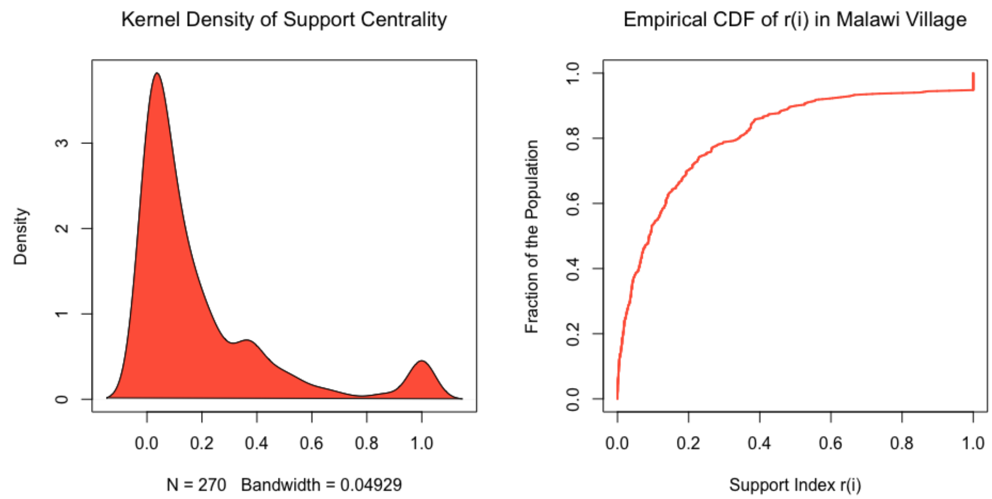


Figure 4: Estimated Density and Empricial CDF of $r_{\boldsymbol{G}}(i)$ in Malawi

network support to condition on even half of the joint shocks arising in their own neighborhoods, i.e., $r(i) \leq 0.5$.[25] The distribution of $r(i)$ in the Malawi network therefore suggests that the local constraints identified in this paper may be pervasive and severe for the majority of households.

Our results suggest that, all else equal, more connections between individuals translate to a richer payoff space. However, the number of links is not a sufficient statistic to rank alternative networks. In order to rank payoff spaces, we could say that an alternative network $\boldsymbol{G}'$ dominates $\boldsymbol{G}$ if, for any i, the new payoff space $C_i^{*\prime}$ in $\boldsymbol{G}'$ contains the payoff space $C_i^* \subseteq C_i^{*\prime}$ in $\boldsymbol{G}$. By our results, this criterion is equivalent to the condition that all non-exclusive groups of neighbors $\mathcal{M}_i^*$ available in $\boldsymbol{G}$ are maintained in $\boldsymbol{G}'$. Thus, for $\boldsymbol{G}'$ to dominate, $N_i' \supseteq N_i$, must hold for all i, and if subgroup $M \subseteq N_i$ is known by at least two agents in $\boldsymbol{G}$, the same must hold in $\boldsymbol{G}'$. Therefore if $\boldsymbol{G}'$ is obtained from $\boldsymbol{G}$ by adding links, the consumption possibilities expand strictly for any individual i who enjoys a new connection. Conversely, if any link $\{i, j\}$ in $\boldsymbol{G}$ is missing in $\boldsymbol{G}'$, this cannot be compensated by any additional links in $\boldsymbol{G}'$, since

[25] More than 60% of households cannot condition on 20% of joint shocks they are subject to.

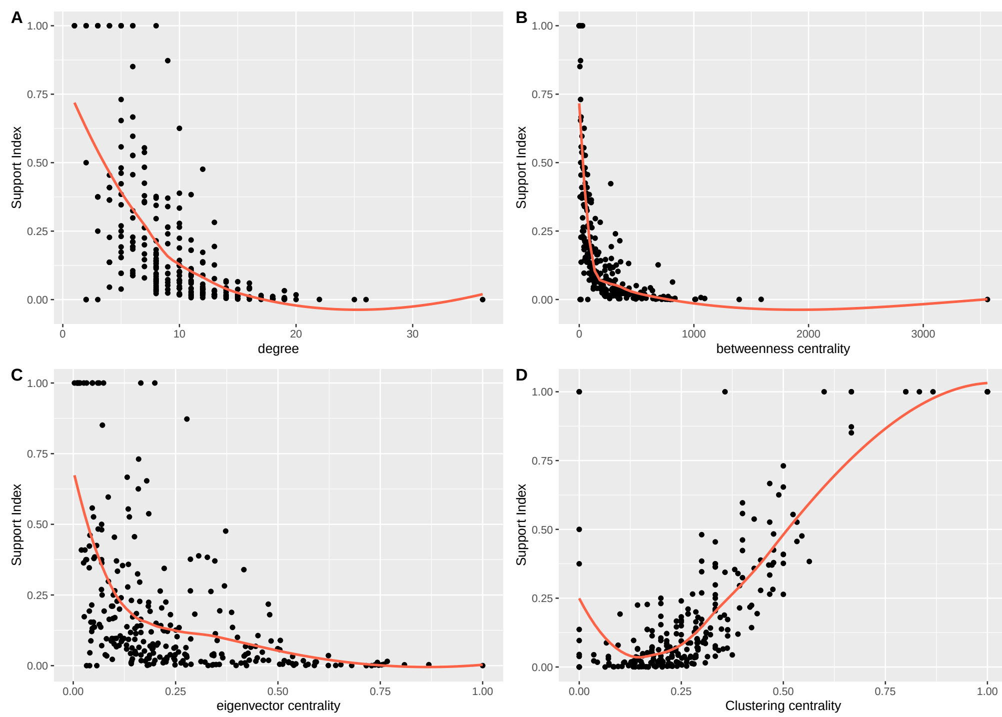


Figure 5: Resilience Measure $r_{\boldsymbol{G}}(i)$ against several Network Centralities. The red curve represents a LOESS fit to the data.

i can no longer condition on j's income. As a result the criterion of expanding payoff spaces is characterized by the condition that $\boldsymbol{G}'$ is obtained from $\boldsymbol{G}$ through the addition of links.

B.2 Generalized State Space

. We will say that two states are indistinguishable according to an attribute $q \in Q$ if they belong to the same cell in $\mathcal{L}_{\{q\}}$. Such an attribute could be observable, like a realized weather event or news about an economic sector. Or it could be abstract, where distinguishing two states may simply require information, expertise, skills, or the legal means to trade across them, which may not be available to all agents.

Our assumption of a rich state space $\Omega = Z_1 \times \cdots Z_{\overline{q}}$ amounts to the joint assumption of two properties. The first requires the set of payoff-relevant variables to be exhaustive, so that the finest partition $\mathcal{L}_Q$ is the trivial partition with singleton cells. In other words, observing the joint realizations of the risks in Q is sufficient to identify the true state. The second property requires that all joint realizations occur with positive probability. This implies that any refinement must be strict, with $M \supset M' \Rightarrow \mathcal{L}_M < \mathcal{L}_{M'}$. In other words, observing the joint realizations of the risks in Q is necessary to identify the true state.

While the first condition merely requires an accurate specification of Q and Ω, the second is indeed restrictive. Without it, even observing a strict subset of variables $M \subset Q$ could reveal some states of the world. Such a situation arises when there is a perfect conditional correlation between some of the variables in Q.

However, our qualitative results on (local) completeness continue to hold if this revelation only occurs for some local states. Specifically, assume alternatively that the state space corresponds to a subset of the Cartesian product $\Omega \subset Z_1 \times \cdots Z_{\overline{q}}$. As long as there exist two realizations $\widehat{Z}_q = \underline{z}_q, \overline{z}_q$ for each variable $q \in Q$ such that all joint realizations occur with positive probability, $\Omega \supseteq \widehat{Z}_1 \times \cdots \widehat{Z}_{\overline{q}}$, our results on completeness follow.

Proposition 3. *If there exist two realizations $\widehat{Z}_q = \underline{z}_q, \overline{z}_q$ for each $q \in Q$ such that $\Omega \supseteq \widehat{Z}_1 \times \cdots \widehat{Z}_{\overline{q}}$, then:*

1. *The economy is globally complete if and only if there exists an agent who can trade all risks, such that $M_i = Q$.*

2. *Agent i enjoys local completeness if and only if there exists an agent $j \neq i$ who can trade all risks that she can, $M_j \subseteq M_i$.*

Proof. Consider a subset $\widehat{\Omega} \subseteq \Omega$ with $\widehat{\Omega} = \widehat{Z}_1 \times \cdots \widehat{Z}_{\overline{q}}$. Applying Proposition 1 to $\widehat{\Omega}$ says that there exists an instrument which separates $\omega \in \widehat{\Omega}$ from a distinct $\omega' \in \widehat{\Omega}$ if and only if there exists an i with $Q_i = Q$. Expanding the state space to Ω would expand the number of columns in $\boldsymbol{A}_i$ for each i, but any new columns would assign 0 to the states in $\widehat{\Omega}$. As a result, any two states $\omega \in \widehat{\Omega} \subseteq \Omega$ and $\omega' \in \widehat{\Omega} \subseteq \Omega$ can still only be separated if and only if there exists an i with $Q_i = Q$, which proves the first statement. Proceeding analogously for isolating a local state $L_{Q_i}(\omega)$ with $\omega \in \widehat{\Omega}$ proves the second statement. ■

B.3 Generalized Submarkets

We presented our main results in terms of direct constraints on the payoff space rather than on the set of traded instruments. That is, we assumed that the individual constraints map directly into a set of risks which can be traded without frictions $M_i \subseteq Q$ and a remaining set of risks which cannot be traded. Guerdjikova and Quiggin (2019) provide sufficient conditions on the richness of assets with arbitrary payoffs to generate a contingent claims space on the elements of partitions.

However, models of heterogeneous access to submarkets, such as the seminal paper by Merton (1987) or the current literature on decentralization (Malamud and Rostek, 2017), model constraints in terms of access to arbitrary assets. In this case, depending on the kinds of assets traded in her submarkets, an agent's may condition on a different set of risks for each asset she has access to.

To see how our setup can be adapted to capture similar constraints, consider a model where K is a set of asset classes. Let the payoffs of asset class $k \in K$ condition on a set $Q_k \subseteq Q$ of risks. Again, since we are not concerned with markets missing by assumption, let the assets in each class $k \in K$ be rich enough to generate any payoff that is measurable with respect to the resulting partition $\mathcal{L}_{Q_k}$. Each asset class k therefore generates a contingent claims space $\operatorname{span}(\boldsymbol{A}_{Q_k})$, where $\boldsymbol{A}_{Q_k}$ is the Q_k-local asset matrix from (3). Not every individual may be able to trade each asset class. For every i, we denote by $K_i \subseteq K$ the asset classes that she has access to. Notice that this agent's constraints no longer need to induce a partition of the state space. Moreover, note that for the purpose of determining feasible payoffs, the composition of the submarkets is not crucial, as it matters whether – not where – specific assets can be traded.

To illustrate, consider Application 2. An asset class $k = 2$ may pay on the joint risks of commodities and the broad market $Q_2 = \{1, 2\}$. Another asset class $k = 3$ may pay on the joint risks of commodities and the exchange rate, $Q_3 = \{1, 3\}$. If agent 1 has access to asset classes $K_1 = \{2, 3\}$, then her payoffs are no longer measurable with respect to any partition. Her simultaneous access to classes 2 and 3 means that she can generate payoffs which differ across all 8 states of Ω.[26] Instead, the linear dependence among the assets she has access to results in a subspace which corresponds to the span of the payoff matrix $\boldsymbol{J}_{K_1}$ which collects all columns that belongs to $\boldsymbol{A}_{Q_1}$ or $\boldsymbol{A}_{Q_2}$, included repetitions. The matrix $\boldsymbol{J}_{K_1}$ corresponds to the matrix (23) in Appendix B.5.

More generally, define by $\boldsymbol{J}_{K_i}$ the asset matrix for individual i,

$$\boldsymbol{J}_{K_i} = [\ \boldsymbol{A}_{Q_k}\ ]_{k \in K_i}. \tag{21}$$

The specification of the main section can be nested by letting $K_i = \{k_i\}$ be singleton sets with $Q_{k_i} = M_i$, so that $\boldsymbol{J}_{K_i} = \boldsymbol{A}_{M_i}$.

Proceeding analogously to section 3, the global payoff space of a (potentially counterfactual) agent with access to all assets would be $\boldsymbol{J}_K$ and the economy satisfies global completeness if and only if the grand payoff space $\text{span}(\boldsymbol{J}_K)$ coincides with $\mathbb{R}^{|\Omega|}$. Any individual i enjoys local completeness if and only if her payoff space $\text{span}(\boldsymbol{J}_{K_i})$ is a subset of the collective payoff space from combining the assets available to the remaining agents $\text{span}([\boldsymbol{J}_{K_i}]_{j \in N \setminus \{i\}})$.

Once we adapt the set $\mathcal{M}_I = \{M \subseteq Q_k : k \in K_i, i \in I\}$, we obtain the usual objects $\boldsymbol{B}_N$ as the basis of the joint payoff space in terms of M-local assets, where $M \subseteq \mathcal{M}_N$. Similarly, we obtain $\boldsymbol{B}_i^*$ as the subset of i's payoff space for

[26] Using asset class 2, she can buy a portfolio which pays a different quantity for all joint realizations of commodities and the broad market index. Using asset class 3 she can add another portfolio which whenever the exchange rate is high, pays a sufficiently large amount to generate a different payoff in all states.

which there exists a counterparty.

Proposition 4. *Consider an economy in which each agent has access to a subset K_i of asset classes K.*

1. *The economy satisfies global completeness if and only if there exists an asset class $k \in K$ which conditions on all joint risks $Q_k = Q$.*

2. *Individual i enjoys local completeness if and only if, for each asset $k_i \in K_i$ that she can trade, there exists an agent j who can trade a finer asset $k_j \in K_j$ with $Q_{k_i} \subseteq Q_{k_j}$.*

3. *The payoff space C_i^* for any individual i is characterized by the basis $\boldsymbol{B}^*_{\{i\}}$ from (16) when $\mathcal{M}_i^*$ is suitably adapted.*

Proof. Given that the payoff space for each individual i is spanned by $\boldsymbol{J}_{K_i} = [\ \boldsymbol{A}_{Q_k}\ ]_{k \in K_i}$, Lemma 4 implies that the individual and global bases from the main section can be derived by redefining $\mathcal{M}_I = \{M \subseteq Q_k : k \in K_i, i \in I\}$ for each i. Proceeding analogously to proving the main propositions, proves the result. ■

Notice that if each submarket is attended by two or more agents, local completeness holds for every agent, since all relevant asset classes can be traded by at least one more agent. Moreover, and in contrast to Section 3, each i's local completeness can be granted collectively by a group of agents: a different one for each submarket.

B.4 Representation of States in Terms of Their Local Effects

We now show that for every group M, the number of M-local states, $|\mathcal{L}_M|$ – each of which defines a particular realization of risks in group M – corresponds to the number of elements in the sets O_K (as defined in (9)) across all groups

K that belong to group M. In particular, since any state ω satisfies the in- and outgroup conditions for at most one set O_K, the collection $\cup_{K \subseteq M} O_K$ consists of disjoint sets. Hence the following result.

Lemma 6. *For any set of risks $M \subseteq Q$, the number of M-local states satisfies $|\mathcal{L}_M| = \sum_{K \subseteq M} |O_K|$. In particular, each state $\omega \in \Omega$ belongs to O_M for exactly one set $M \subseteq Q$, such that $|\Omega| = \sum_{M \subseteq N} |O_M|$.*

Proof. Consider a set of risks $M \subseteq Q$. We need to show that every state in $\cup_{K \subseteq M} O_K$ belongs to exactly one cell $L_M \in \mathcal{L}_M$, and that each cell $L_M \in \mathcal{L}_M$ contains exactly one element of $\cup_{K \subseteq M} O_K$. The former is immediate since $\mathcal{L}_M$ is a partition of the state space $\Omega \supseteq \cup_{K \subseteq M} O_K$. To show the latter, notice that every cell $L_M \in \mathcal{L}_M$ contains any state which assign the L_M-characteristic outcome for all $q \in M$. Among them, there exists exactly one state $\omega \in L_M$ which assigns the worst outcome to the members of the out-group, $z_r(\omega) = z_r(\omega_0)$ for all $r \in Q \setminus M$. This state ω must belong to exactly one set O_K with $K \subseteq M$, where K is the subset of risks in M whose outcome is different from the worst state. And, by definition of (9), all other element of L_M can only belong to O_K if $K \not\subseteq M$. ■

B.5 An Explicit Basis Representation for the Three-Agent Example of Section 2.2

We provide an explicit representation of $\boldsymbol{B}$, as described in Section 3.2. Without loss of generality, order the 8 states as follows: (b, b, b), (b, b, g), (b, g, b), (b, g, g), (g, b, b), (g, b, g), (g, g, b), and (g, g, g). That is, the vector $(0, 0, 0, 0, 0, 0, 0, 1)^T$ corresponds to a unit claim on the state in which all three risks generate g. Following the definition in equation (10) the matrix $\boldsymbol{B}$ corresponds to:

$$
\boldsymbol{B} = \left(\underset{M=\emptyset}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{1\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{2\}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{3\}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{M=\{1,2\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{1,3\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{M=\{2,3\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{M=\{1,2,3\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right). \quad (22)
$$

When interpreted as assets, the first column represent a bond (with constant payoff), the second to fourth columns each represents a contingent claim on the event that $q = 1, 2, 3$ have realization g, respectively. The fourth to sixth columns each represents a contingent claim on the events where both q and $q' \neq q$ simultaneously generate g, for $q, q' \in 1, 2, 3$. Finally, the last column corresponds to the Arrow security that pays in the unique state, ω, where all three variables generate g. The group M defining the local asset is labelled at the bottom of each vector for clarity.

As we argue in Section 3.3, the space of feasible payoffs obtained from the trades available to agents 2 and 3 are described by a basis that takes the columns of $\boldsymbol{B}$ corresponding to the groups M that are in M_2 or M_3. In the current example, the joint insurance possibilities of agents 2 and 3 correspond to the following 6 sets $M = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}\}$. Therefore, the matrix $\boldsymbol{B}_{\{2,3\}}$ generates the same basis vectors as $\boldsymbol{B}$, but without the last two columns:

$$
\boldsymbol{B}_{\{2,3\}} = \left(\underset{M=\emptyset}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{1\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{2\}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{3\}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{M=\{1,2\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{M=\{1,3\}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}} \right) \tag{23}
$$

Finally, as we show in section 3.4, an agent's resilience to joint shocks depends on the constraints of others. In the context of the simple three-agent example, we show that agent 1's feasible payoff space is restricted, despite her access to the full set of Arrow securities. This is because agents 2 and 3 cannot jointly satisfy every trade available to agent 1. It turns out that 1's feasible payoff space is therefore reduced from $\mathbb{R}^8$ to a 6-dimensional subspace, given by $\boldsymbol{B}_{\{2,3\}}$. Formally, we have as a specific instance of Proposition 2 that

$$
\boldsymbol{B}^*_{\{1\}} = \boldsymbol{B}_{\{2,3\}}
$$

B.6 Welfare in the Three-Agent Example of Section 2.2

Denote by Δ_3 agent 1's consumption response to a positive shock in z_3, conditional on $z_1 = z_2 = g$,

$$
\Delta_3 = c_1(g, g, g) - c_1(g, g, b).
$$

In any allocation which respects the measurability constraint of 2 and 3, c_1 must respond to z_3 without conditioning on z_2, such that $\Delta_3 = c_1(g, b, g) - c_1(g, b, b) = c_1(g, g, g) - c_1(g, g, b)$. Similarly, since 2 cannot condition on z_3, this means $c_2(g, g, g) = c_2(g, g, b)$ and $c_2(g, b, g) = c_2(g, b, b)$.

In order to show that conditioning on the joint realization of $q = 2$ and $q = 3$ would improve welfare, it suffices to show that marginal rates of substitution across agents 1 and 2 cannot be simultaneously equated across the four states mentioned above. Note that by definition of Δ_3, marginal rates of substitution for agent 1 satisfy

$$\frac{u'(c_1(g,g,b))}{u'(c_1(g,b,b))} = \frac{u'(c_1(g,g,g) - \Delta_3)}{u'(c_1(g,b,g) - \Delta_3)}.$$

At the same time, by measurability of c_2, we have

$$\frac{u'(c_2(g,g,g))}{u'(c_2(g,b,g))} = \frac{u'(c_2(g,g,b))}{u'(c_2(g,b,b))}.$$

We can now determine whether, at such an allocation, they could gain from sharing 2's risk differently across levels of z_3. Consider any interior point, and pairs of states which differ by z_2 alone. No mutual improvements exist if the ratios of marginal utility across them are equalized. Combining the above, this would, however, require

$$\frac{u'(c_1(g,g,g))}{u'(c_1(g,b,g))} = \frac{u'(c_2(g,g,g))}{u'(c_2(g,b,g))} = \frac{u'(c_2(g,g,b))}{u'(c_2(g,b,b))} = \frac{u'(c_1(g,g,g) - \Delta_3)}{u'(c_1(g,b,g) - \Delta_3)}.$$

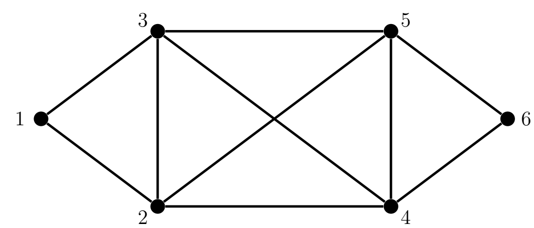
The first ratio of marginal utilities represents the same change in 1's consumption as the fourth ratio of marginal utilities, except at a different level of consumption since $\Delta_3 > 0$ whenever 1 provides (partial) insurance to 3.[27] As a result, unless u exhibits constant absolute risk aversion (where Δ_3 has no effect on ratios of marginal utility) welfare always decreases if c_1 cannot condition on the joint risks between $q = 2$ and $q = 3$.[28]

[27] By strict concavity of u, it is easy to show $\Delta_3 > 0$.

[28] It is easy to show that marginal utility ratios are only invariant to wealth changes if u is exponential: see also Pratt (2000). Note that Ambrus et al. (2021) find that optimal consumption must be a weighted sum of neighbor's i.i.d shocks. Our result shows that this feature relies on CARA preferences.

B.7 Universal Local Completeness and Global Completeness

Using the network application from Section 4, consider the following example with six individuals. Note that each i is connected to another agent j such that $N_j \subseteq N_i$. As a result, each i enjoys local completeness. In particular, $N_1 \subset N_2$, $N_2 = N_3$, $N_4 = N_5$, and $N_6 \subset M_4$. Yet, the economy is not globally complete, since no N_i contains the group $\{1, 6\}$.



B.8 Orthogonal Decomposition

Consider the binary case, where each q takes a value in $\{b, g\}$. Define for each $q \in Q$ and every $\omega \in \Omega$, the following function:

$$h_q^\omega(\omega') = \begin{cases} -1 & \text{if } z_q(\omega') = \underline{z}_q \\ +1 & \text{otherwise,} \end{cases}$$

where we write $z_q(\omega)$ for the realization of variable q in state ω.

Given $h_q^\omega(\omega')$, we can now construct an asset

$$a^o_{L_M(\omega)}(\omega') = \begin{cases} \prod_{q \in M} h_q^\omega(\omega') & \text{if } M \neq \emptyset \\ 1 & \text{if } M = \emptyset. \end{cases}$$

which assigns to every state ω', the product of the $h_q^\omega(\omega')$ values across the risks $q \in M$. If $M = \emptyset$, the asset $a^o_{L_M(\omega)}$ corresponds to the bond which pays 1 in every state. Otherwise, the asset simply pays -1 in states where the number of variables $q \in M$ generating value b is odd, and 1 if it is even. Just like the contingent claims which form our local assets $\boldsymbol{a}_{L_M(\omega)}$, these assets $\boldsymbol{a}^o_{L_M(\omega)}$ are measurable with respect to $\mathcal{L}_M$.

We can now proceed to construct an orthogonal basis which, like $\boldsymbol{B}$, consist of the bond and an M-measurable asset for every possible M-local state in which none of the variables $q \in M$ agree with ω_0,

$$\boldsymbol{W} = \left[\left[\boldsymbol{a}^o_{L_M(\omega)} \right]_{\omega \in O_M} \right]_{M \subseteq Q}. \tag{24}$$

In the case of the three agents examples from Section 2.2, the orthogonal matrix is as follows:

$$\boldsymbol{W} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It can easily be checked that the columns are orthogonal. For each $M \neq \emptyset$, the column assigns to half of its M-local states -1, and $+1$ otherwise, which makes their payoff orthogonal to the bond. Similarly, for any two distinct sets, without loss of generality, refer by M' to a set which is not contained by the other, $M \not\supseteq M'$. Conditional on any $M-$local state, where the first asset pays a constant, the exclusive variables $q \in M' \setminus M$ generate an equal number of entries with -1 and 1 in the second asset. As a result, $\boldsymbol{a}^o_{L_M}$ and $\boldsymbol{a}^o_{L'_M}$ also have a zero inner product.

In the general setting when each q takes an arbitrary number of realizations $|Z_q|$, an orthogonal basis can be constructed on the same principles. Adapt for each $q \in Q$ and every $\omega \in \Omega$, the function:

$$h_q^\omega(\omega') = \begin{cases} 1 - |Z_q| & \text{if } z_q(\omega') = \underline{z}_q \\ 1 + \sqrt{|Z_q|}(|Z_q| - 2) & \text{if } z_q(\omega') = z_q(\omega) \\ 1 - \sqrt{|Z_q|} & \text{otherwise.} \end{cases}$$

If $|Z_q| > 2$, the function h_q^ω takes three values: a positive value for states where q assigns the same realization as in ω, a negative value for states in which q agrees with ω_0, and a less negative value for all other states. In the binary case, the function simplifies to what we had above. The entries are chosen such that they sum to 0 when weighted by the cardinality of the respective event. The orthogonal matrix $\boldsymbol{W}$ can then be obtained from (24).